Minicourse on smoothing theory
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November 19, 2015

Slogan (Bill Clinton): It depends on what the meaning of the word "is" is.

Our goal is to prove:

Theorem 1. *Let $n \geq 5$. Let Σ^n be an exotic sphere and Ω^n be an exotic torus, smooth manifolds which are homeomorphic, not diffeomorphic, to a sphere and torus respectively. Then $T^n \# \Sigma^n$ is not diffeo to T^n, and $\Omega^n \times T^k$ is not diffeomorphic to T^{n+k}.*

1 Generalities

1. A *topological n-manifold* X is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^k$.

2. A *smooth manifold* is a topological manifold with a maximal smooth atlas.

3. Let X be a topological n-manifold. A *smoothing of* X is a maximal smooth atlas on X. A *marking on* X is a homeomorphism $h : M \to X$ from a smooth manifold to X; it induces a smoothing X_h on X. Two markings $h : M \to X$ and $h' : M' \to X$ induce the same smoothing ($X_h = X_{h'}$) iff there is a diffeomorphism $\phi : M \to M'$ so that $h' \circ \phi = h$.

4. The *moduli set* $M(X)$ is the set of diffeomorphism classes of smooth manifolds homeomorphic to X.

5. Two smooth structures X_0 and X_1 on X are *concordant* if there is a smooth structure on $X \times I$ which restricts to X_i on $X \times i$, $i = 0, 1$. (A concordance is determined by a marking $H : W \to X \times I$).

6. Two smooth structures X_0 and X_1 are *isotopic* if there is a smooth manifold M and a level-preserving homeomorphism $H : M \times I \to X \times I$ ($H(x,t) = (F(x,t), t)$) inducing X_i on $X \times i$, $i = 0, 1$.

   Theorem 2 (Concordance implies isotopy)**.** *If* $\dim X \geq 5$*, then concordant structures are isotopic (and hence diffeomorphic).*

   Thus concordant structures are diffeomorphic. This also follows from the s-cobordism theorem.

7. The *structure set* $S(X)$ is the set of concordance classes of smoothing on X. Note
$$S(X) \twoheadrightarrow M(X)$$

   Example 3. (a) A point has an infinite number of markings but one smoothing.

   (b) S^1 has an infinite number of smoothings, but $S(S^1) = *$

   (c) $S(S^7) = \mathbb{Z}/28$, $M(S^7) = \mathbb{Z}/28/(x \sim -x)$ so $\#M(S^7) = 15$

   (d) $M(E_8^{4k}) = \emptyset$. (E_8 is a closed, topological $4k$-manifold with signature 8. Note that $E_8 - *$ is smoothable with a trivial tangent bundle.)

   (e) Let M and N be a smooth manifolds with ∂ and $f : \partial M \to \partial N$ be a diffeomorphism. Then $M \cup_f N$ is a topological manifold. It doesn't have a unique smoothing, but it has a unique concordance class of smoothings in $S(M \cup_f N)$.

   Why is $S(X)$ nicer than $M(X)$? It is in bijection with a *computable* abelian group, and useful for gluing.

8. $\text{Homeo}(X) \curvearrowright S(X) \quad (\alpha, [X_h]) \mapsto [X_{\alpha \circ h}]$. I.e. $M \xrightarrow{h} X \xrightarrow{\alpha} X$.

   Lemma 4. $\text{Homeo}(X) \backslash S(X) \to M(X)$ *is a bijection.*

   Proof. Well-defined and onto are clear.
   Injective? If $f : M \to X, g : M \to X$ are markings, then $f = (f \circ g^{-1}) \circ g$. □

Let $\mathrm{Homeo}_0(X)$ be the group of homeomorphisms isotopic to the identity. The *mapping class group* $MCG(X)$ is $\mathrm{Homeo}(X)/\mathrm{Homeo}_0(X)$. Two homeomorphisms $\alpha, \beta \in \mathrm{Homeo}(X)$ are *pseudoisotopic* if there is $\gamma \in \mathrm{Homeo}(X \times I)$ so that $\alpha = \gamma|_{X\times 0}$ and $\beta = \gamma|_{X\times 1}$. The *reduced mapping class group* $\widetilde{MCG}(X)$ is $\mathrm{Homeo}(X)/\Psi\mathrm{Homeo}_0(X)$, so two homeos represent the same element iff they are psuedoisotopic.

Note that $\Psi\mathrm{Homeo}_0(X)$ acts trivially on $S(X)$. We conclude

Lemma 5. *$\widetilde{MCG}(X)\backslash S(X) \to M(X)$ is a bijection.*

Remark 6. Suppose X_0 is a smoothing of X and suppose $\mathrm{Diffeo}(X)\backslash S(X) \to M(X)$ is a bijection. Then any smoothing not concordant to X_0 is not diffeomorphic to X_0.

This is what happens for the torus.

Lemma 7. $\widetilde{MCG}(T^n) \xrightarrow{\cong} GL_n(\mathbb{Z})$

Proof. There is a split surjection $H_1 : \mathrm{Homeo}(T^n) \to GL_n(\mathbb{Z})$ split by L. We need to show that any $h : T^n \to T^n$ is Ψ-isotopic to $L \circ H_1(h)$.

Note they are homotopic: (1) use $T^n = K(\mathbb{Z}^n, 1)$ or (2) lift h to $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$. Take a straightline homotopy between $\widetilde{h}$ and $L \circ H_1(h)$.

Now apply the Borel conjecture for $T^n \times I$, to see the homotopy is homotopic, relative to $\partial(T^n \times I)$ to a homeomorphism. □

2 Bundles

Let $BO(n) = Gr(n, \mathbb{R}^\infty)$. This is a *classifying space for vector bundles over a finite CW complex B*:

There is an n-plane bundle γ over $BO(n)$ so that

$$\begin{aligned}[B, BO(n)] &\to \{\text{iso classes of } n\text{-plane bundles over } B\} \\ [f] &\mapsto [f^*\gamma]\end{aligned}$$

Example 8. Let $M^n \subset \mathbb{R}^k$ be a smooth submanifold. Then
$M \to Gr(k, \mathbb{R}^k), \quad p \mapsto T_pM \subset \mathbb{R}^n.$

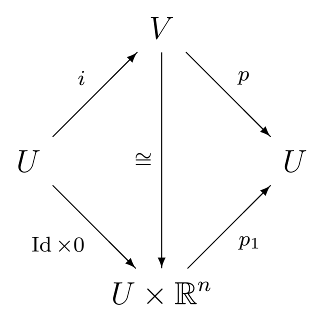
Definition 9. Two vector bundles η and ξ over B are *stably equivalent* if $\eta \oplus \underline{\mathbb{R}^k} \cong \xi \oplus \underline{\mathbb{R}^l}$

Let $BO = \operatorname{colim} BO(n)$. Then $[B, BO]$ classifies stable vector bundles over B. It is an abelian group, computable by the Atiyah-Hirzebruch spectral sequence.

Definition 10. A n-plane *microbundle over B* is a pair of maps

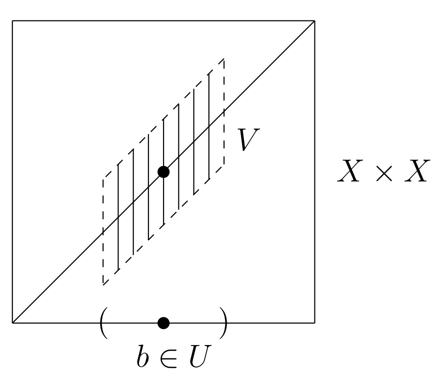
$$B \xrightarrow{i} E \xrightarrow{p} B$$

satisfying $p \circ i = \mathrm{Id}_B$ and the following local triviality condition: for every $b \in B$ there exists open neighborhoods U of b and V of $i(b)$ with $i(U) \subset V$ and a homeomorphism $V \to U \times \mathbb{R}^n$ so that the following diagram commutes



Example 11. The *tangent microbundle of a topological n-manifold X* is

$$X \xrightarrow{\Delta} X \times X \xrightarrow{p_1} X.$$



Most of the machinery of bundle theory applies to microbundles. For example, two bundles $B \xrightarrow{i} E \xrightarrow{p} B$ and $B \xrightarrow{i'} E' \xrightarrow{p'} B$ are *isomorphic* if there are neighborhoods W and W' of $i(B)$ and $i'(B)$ respectively and a commutative diagram

$$\begin{array}{ccccc} & & W & & \\ & \nearrow & & \searrow & \\ B & & \cong \downarrow & & B \\ & \searrow & & \nearrow & \\ & & W' & & \end{array}$$

An $\mathbb{R}^n$*-bundle with a zero section* is a fiber bundle $E \to B$ with fiber $\mathbb{R}^n$ and structure group $TOP(n) = \text{Homeo}\ (\mathbb{R}^n \text{ rel } 0)$. Every $\mathbb{R}^n$-bundle with a zero section determines a microbundle. A theorem of Kister and Mazur gives a one-to-one correspondence between isomorphism classes of $\mathbb{R}^n$-bundles with a zero section and isomorphism class of microbundles.

There are spaces $BTOP(n)$ and $BTOP$ which classify microbundles and stable microbundles. Assume $BO \to BTOP$ is a fibration by redefining $BO = ETOP/O$ or by replacing the map by a fibration.

3 Fundamental Theorem

Given a diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow p \\ A & \xrightarrow{\alpha} & C \end{array}$$

let $\text{Lift}_p(\alpha)$ be the set of maps $\widehat{\alpha} : A \to B$ so that $p \circ \widehat{\alpha} = \alpha$. Let $[\text{Lift}_p(\alpha)]$ be the set of vertical homotopy classes of lifts. A *vertical homotopy* is a map $H : A \times I \to B$ so that for all $a \in A$ and $t \in I$, $p(H(a,t)) = \alpha(a)$, in which case $H(-,0)$ and $H(-,1)$ are vertically homotopic. We will usually leave p out of the notation.

Fundamental Theorem of Smoothing. Let X be a topological manifold with $\dim X \geq 5$. Let $\tau_X : X \to BTOP$ denote a classifying map of the stable tangent bundle. X admits a smooth structure if and only if there is a lift

$X \to BO$ such that the following diagram commutes.

$$\begin{array}{ccc} & & BO \\ & \nearrow & \downarrow \\ X & \xrightarrow{\tau_X} & BTOP \end{array}$$

In fact, the classifying map of the smooth tangent bundle gives a bijection

$$\mathcal{S}(X) \xrightarrow{\simeq} [\mathrm{Lift}(\tau_X)]$$

where $[\mathrm{Lift}(\tau_X)]$ denotes vertical homotopy classes of lifts of τ_X (vertical homotopy means a homotopy through lifts).

To illustrate some subtlety, note that exotic spheres are stably parallelizable – why does this not contradict the fundamental theorem?

4 Some homotopy theory

Let X_h be a smooth structure on X. We wish to establish a bijection $j_h : [X, TOP/O] \xrightarrow{\cong} S(X)$.

$$\begin{array}{ccc} TOP/O & \longrightarrow & BO \\ & & \downarrow \\ & & BTOP \end{array}$$

1. TOP/O is 2-connected (showing that it is path-connected is a difficult theorem - the solution to the annulus conjecture.)

2. $TOP/O, O, TOP, BTOP, BO$ are H-spaces (idea: $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$).

3. $TOP/O, O, TOP, BTOP, BO$ are infinite loop spaces: e.g. $\exists A_1, A_2, \ldots$ so that $TOP/O \simeq \Omega A_1$, $A_1 \simeq \Omega A_2$, etc., where $\simeq$ means homotopy equivalent.

4. There is a long exact sequence of abelian groups

$$\cdots \to [\Sigma B, BO] \to [\Sigma B, BTOP] \to [B, TOP/O] \to [B, BO] \to [B, BTOP]$$

5. $BO \to BTOP$ is a principal TOP/O-bundle.

TOP/O is an H-space and I am not going to spell out what I mean by a principal H-space bundle. But it does mean that there is a commutative diagram

$$\begin{array}{ccc} BO \times TOP/O & \longrightarrow & BO \\ & \searrow & \downarrow \\ & & BTOP \end{array}$$

and hence a map $[B, BO] \times [B, TOP/O] \to [B, BO]$.

Corollary 12.

$$\begin{array}{ccc} & & BO \\ & & \downarrow \\ B & \xrightarrow{\tau} & BTOP \end{array}$$

The abelian group $[B, TOP/O]$ *acts freely and transitively (on the right) on* $[\mathrm{Lift}(\tau)]$.

Thus if we choose a lift $\widehat{\tau} : B \to BO$, there the orbit map gives a bijection $[B, TOP/O] \xrightarrow{\simeq} [\mathrm{Lift}\,\tau]$.

In particular, if M is a smooth manifold of dimension ≥ 5, there is a bijection

$$j_h : [M, TOP/O] \xrightarrow{\simeq} S(M)$$

defined by acting on the tangent bundle $\tau : M \to BO$ to get a new lift of $p \circ \tau : M \to BTOP$, and choosing the corresponding smooth structure given by the fundamental theorem.

Furthermore, since TOP/O is an infinite loop space, the Atiyah-Hirzebruch spectral sequence applies to compute $[M, TOP/O]$, which is H^0 of a generalized cohomology theory.

5 Exotic spheres

Let Θ_n be the equivalence classes of smoothing on S^n under orientation-preserving diffeomorphism. This is an abelian monoid under connected sum.

Lemma 13. *Let* $n \geq 5$.

1. Θ_n *is a finite abelian group.*

2. $S(S^n) \xrightarrow{\sim} \Theta_n$

3. *The composite* $\pi_n(TOP/O) \xrightarrow{\sim} [\mathrm{Lift}(\tau_{S^n})] \xleftarrow{\sim} S(S^n) \xrightarrow{\sim} \Theta_n$ *is an isomorphism of abelian groups.*

   Discussion of proof. Smale proved that every exotic sphere of dimension ≥ 5 is obtained by gluing $D^n \cup_f D^n$ for some diffeomorphism $f : S^{n-1} \to S^{n-1}$. It follows that Θ_n is a group. The finiteness is due to Kervaire-Milnor. It is easy to see an epimorphism $S(S^n) \to \Theta_n$. To show injectivity one uses the Alexander trick. □

Lemma 14. *Let M and N be smooth manifolds with* $\dim M \geq 5$.

1. *If M is closed, connected, let $c : M \to S^n$ be a degree one map; for example, choose an embedded disk $D^n \hookrightarrow M$ and let $c : M \to M/(M - int\ D^n) = S^n$ be the quotient map. The following diagram commutes*

$$\begin{array}{ccc} [S^n, TOP/O] & \xrightarrow{c^*} & [M, TOP/O] \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ S(S^n) & \longrightarrow & S(M) \end{array}$$

   where the bottom horizontal map is $[\Sigma] \mapsto [M \# \Sigma]$.

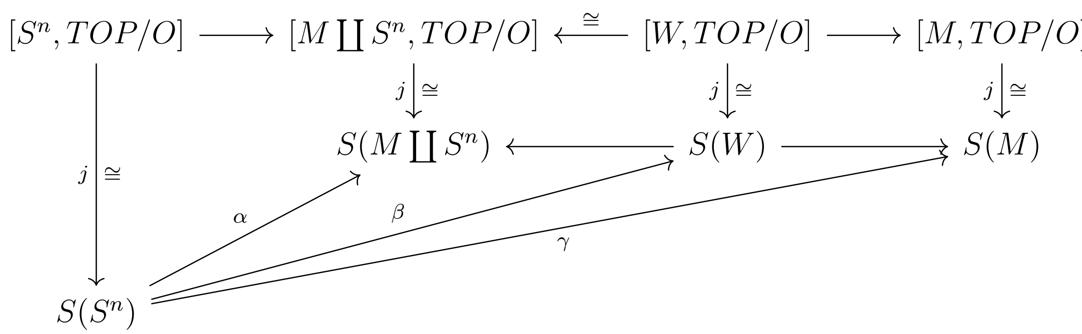
2. *The following diagram commutes*

$$\begin{array}{ccc} [M, TOP/O] & \xrightarrow{\mathrm{pr}^*} & [M \times N, TOP/O] \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ S(M) & \longrightarrow & S(M \times N) \end{array}$$

   where the bottom horizontal map is $[M_h] \mapsto [M_h \times N]$

Proof. 1. The idea is that there is a cobordism W from M to $M \coprod S^n$ and that the corresponding assertion is obvious for $M \coprod S^n$. Indeed, let $\phi : S^0 \times D^n \hookrightarrow M \coprod S^n$ (whose image intersects both M and S^n) and let $W = (M \coprod S^n) \times I \cup_\phi D^1 \times D^n$ be the result of adding a 1-handle to M along ϕ. Give W a smooth structure which restricts (up to concordance) to the given smooth structures on the boundary.

There is a commutative diagram



The middle and right horizontal arrows are induced by restriction, and the upper left horizontal arrow is induced by inclusion of a summand. For a smooth structure Σ on S^n, let $\alpha[\Sigma] = [M \coprod \Sigma]$ and $\gamma[\Sigma] = [M\#\Sigma]$. For the definition of β choose a point $p \in S^n$ and glue W minus an open tubular neighborhood of $\{p\} \times I$ with $\Sigma \times I$ minus an open tubular neighborhood of $\{p\} \times I$. Note then that $S(W) \to S(M \coprod S^n)$ is a bijection and Lemma 14 1 follows.

2. We will show that diagram below is commutative

$$\begin{array}{ccc}
[M, TOP/O] & \xrightarrow{\mathrm{pr}^*} & [M \times N, TOP/O] \\
\cong\downarrow & & \downarrow\cong \\
[\mathrm{Lift}\,\tau_M] & \longrightarrow & [\mathrm{Lift}\,\tau_{M\times N}] \\
\cong\uparrow & & \uparrow\cong \\
S(M) & \longrightarrow & S(M \times N)
\end{array}$$

where the middle horizontal arrow is defined using the H-space map $BO \times BO \to BO$ which has that property that if $X_i \to BO$, $i = 1, 2$ classifies a bundles α_i, $i = 1, 2$ then $X_1 \times X_2 \to BO \times BO \to BO$ classifies $\alpha_1 \times \alpha_2$. It follows that the bottom rectangle commutes. The top rectangle commutes because the following diagram commutes:

$$\begin{array}{ccc}
BO \times TOP/O & \longrightarrow & BO \\
\downarrow & & \| \\
BO \times BO & \longrightarrow & BO
\end{array}$$

□

6 The torus

Theorem 15. *Let $n \geq 5$. Let Σ^n be an exotic sphere and let Ω^n be an exotic torus.*

1. $[T^n \# \Sigma^n] \neq [T^n] \in S(T^n)$
2. $[T^k \times \Omega^n] \neq [T^{k+n}] \in S(T^{k+n})$.
3. $[T^n \# \Sigma^n] \neq [T^n] \in M(T^n)$ *and* $[T^k \times \Omega^n] \neq [T^{k+n}] \in M(T^{k+n})$.

Proof. 1. The key fact we need to show is that $c^* : [T^n, TOP/O] \to [S^n, TOP/O]$ is injective. Since pr is split surjective, pr^* is split injective, so the desired result follows from the above lemma. The key ingredients in showing that c^* is injective (actually split injective) are that TOP/O is 2-connected, that $TOP/O \simeq \Omega A_1$ for some space A_1, and that $\Sigma T^n \simeq \vee e^j$. We will also use the adjoint correspondence for based homotopy $[\Sigma X, Y]_* \cong [X, \Omega Y]_*$ and that fact that for simply-connected targets, and path-connected domains, the forgetful map from based homotopy to unbased homotopy is a bijection.

Thus $\Sigma c : \Sigma T^n \to \Sigma S^n$ has a homotopy right inverse, so $(\Sigma c)^* : [\Sigma T^n, A_1] \to [\Sigma S^n, A_1]$ is injective. Then, by applying the adjoint correspondence, c^* is injective.

2. This follows from Lemma 14 2.

3. This follows from 1. and the fact that $\text{Diffeo}(T^n) \backslash T^n \cong M(T^n)$ which I proved earlier.

□

In fact, the proof above shows

Theorem 16. *For $n \geq 5$, $M(T^n) \cong \oplus_i H^i(T^n; \pi_i(TOP/O))/GL_n(\mathbb{Z})$.*

The homotopy groups of TOP/O are listed below.

Remark 17. One can show that if X is a stably parallelizable manifold, that X and $X \# \Sigma^n$ are not concordant. The idea is to use the Milnor-Spanier theorem and Spanier-Whitehead duality to show that X stably pinches off the top cell.

7 PL manifolds

Two definitions of PL-manifold

Definition 18. A *PL-manifold* is a topological manifold with a maximal PL-atlas.

Definition 19. A *PL-manifold* is a simplicial complex which is a topological manifold and with the link of every vertex a PL-disk.

The definition of $BPL(n)$ and BPL are a little more complicated, since they are not topological groups. The definition of PL/O is as the homotopy fiber of the map $BPL \to BO$.

Hirsch-Mazur show smoothing theory PL/O works in every dimension, PL/O is 6-connected.

Kirby-Siebenmann show smoothing theory TOP/PL works in dimension ≥ 5 and that $TOP/PL = K(\mathbb{Z}/2, 3)$.

Thus

$$\pi_i(TOP/O) = \begin{cases} 0 & i = 0, 1, 2, 4, 5, 6 \\ \mathbb{Z}/2 & i = 3 \\ \Theta_i & i \geq 6 \end{cases}$$

Wall proves that $M(T^5)$ has three elements.

8 Nilmanifolds

Using that Θ_* is finite, on can prove

Theorem 20. *Let T^n_h be a smooth structure on a torus, $n \geq 5$. Then there is a finite cover diffeomorphic to the T^n.*

Theorem 21 (Davis)**.** *Let X^n_h be a smoothing of a nilmanifold, $n \geq 5$. Then there is a finite cover diffeomorphic to a nilmanifold.*

9 negatively curved manifolds

See my survey on Farrell-Jones.

10 Some references

1. Kirby and Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triagulations

2. Hirsch and Mazur, Smoothings of Piecewise Linear Manifolds

3. Davis and Petrosyan, Manifolds and Poincare Complexes `http://www.indiana.edu/~jfdavis/notes/dp.pdf`

4. Davis, The Work of Tom Farrell and Lowell Jones in Topology and Geometry

5. Fisher, Kalinin and Spatzier (Appendix by Davis) "Global Rigidity of Higher Rank Anosov Actions on Tori and Nilmanifolds"

6. Davis and Kirk, Lecture Notes in Algebraic Topology