

Minicourse on smoothing theory

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Slogan (Bill Clinton): It depends on what the meaning of the word “is” is.

Our goal is to prove:

Theorem 1. *Let $n \geq 5$. Let Σ^n be an exotic sphere and Ω^n be an exotic torus, smooth manifolds which are homeomorphic, not diffeomorphic, to a sphere and torus respectively. Then $T^n \# \Sigma^n$ is not diffeomorphic to T^n , and $\Omega^n \times T^k$ is not diffeomorphic to T^{n+k} .*

1 Generalities

1. A *topological n -manifold* X is a second countable Hausdorff space locally homeomorphic to \mathbb{R}^k .
2. A *smooth manifold* is a topological manifold with a maximal smooth atlas.
3. Let X be a topological n -manifold. A *smoothing of X* is a maximal smooth atlas on X . A *marking on X* is a homeomorphism $h : M \rightarrow X$ from a smooth manifold to X ; it induces a smoothing X_h on X . Two markings $h : M \rightarrow X$ and $h' : M' \rightarrow X$ induce the same smoothing ($X_h = X_{h'}$) iff there is a diffeomorphism $\phi : M \rightarrow M'$ so that $h' \circ \phi = h$.
4. The *moduli set* $M(X)$ is the set of diffeomorphism classes of smooth manifolds homeomorphic to X .

5. Two smooth structures X_0 and X_1 on X are *concordant* if there is a smooth structure on $X \times I$ which restricts to X_i on $X \times i$, $i = 0, 1$. (A concordance is determined by a marking $H : W \rightarrow X \times I$).
6. Two smooth structures X_0 and X_1 are *isotopic* if there is a smooth manifold M and a level-preserving homeomorphism $H : M \times I \rightarrow X \times I$ ($H(x, t) = (F(x, t), t)$) inducing X_i on $X \times i$, $i = 0, 1$.

Theorem 2 (Concordance implies isotopy). *If $\dim X \geq 5$, then concordant structures are isotopic (and hence diffeomorphic).*

Thus concordant structures are diffeomorphic. This also follows from the s-cobordism theorem.

7. The *structure set* $S(X)$ is the set of concordance classes of smoothing on X . Note

$$S(X) \twoheadrightarrow M(X)$$

Example 3. (a) A point has an infinite number of markings but one smoothing.

(b) S^1 has an infinite number of smoothings, but $S(S^1) = *$

(c) $S(S^7) = \mathbb{Z}/28$, $M(S^7) = \mathbb{Z}/28/(x \sim -x)$ so $\#M(S^7) = 15$

(d) $M(E_8^{4k}) = \emptyset$. (E_8 is a closed, topological manifold with signature 8, so that $E_8 - *$ is smoothable with a trivial tangent bundle.)

(e) Let M and N be smooth manifolds with ∂ and $f : \partial M \rightarrow \partial N$ be a diffeomorphism. Then $M \cup_f N$ is a topological manifold. It doesn't have a unique smoothing, but it has a unique concordance class of smoothings in $S(M \cup_f N)$.

Why is $S(X)$ nicer than $M(X)$? It is in bijection with a *computable* abelian group, and useful for gluing.

8. $\text{Homeo}(X) \curvearrowright S(X) \quad (\alpha, [X_h]) \mapsto [X_{\alpha \circ h}]$. I.e. $M \xrightarrow{h} X \xrightarrow{\alpha} X$.

Lemma 4. $\text{Homeo}(X) \setminus S(X) \rightarrow M(X)$ is a bijection.

Proof. Well-defined and onto are clear.

Injective? If $f : M \rightarrow X, g : M \rightarrow X$ are markings, then $f = (f \circ g^{-1}) \circ g$. □

Let $\text{Homeo}_0(X)$ be the group of homeomorphisms isotopic to the identity. The *mapping class group* $MCG(X)$ is $\text{Homeo}(X)/\text{Homeo}_0(X)$. Two homeomorphisms $\alpha, \beta \in \text{Homeo}(X)$ are *pseudoisotopic* if there is $\gamma \in \text{Homeo}(X \times I)$ so that $\alpha = \gamma|_{X \times 0}$ and $\beta = \gamma|_{X \times 1}$. The *reduced mapping class group* $\widetilde{MCG}(X)$ is $\text{Homeo}(X)/\Psi\text{Homeo}_0(X)$, so two homeos represent the same element iff they are pseudoisotopic.

Note that $\Psi\text{Homeo}_0(X)$ acts trivially on $S(X)$. We conclude

Lemma 5. $\widetilde{MCG}(X) \backslash S(X) \rightarrow M(X)$ is a bijection.

Remark 6. Suppose X_0 is a smoothing of X and suppose $\text{Diffeo}(X) \backslash S(X) \rightarrow M(X)$ is a bijection. Then any smoothing not concordant to X_0 is not diffeomorphic to X_0 .

This is what happens for the torus.

Lemma 7. $\widetilde{MCG}(T^n) \xrightarrow{\cong} GL_n(\mathbb{Z})$

Proof. There is a split surjection $H_1 : \text{Homeo}(T^n) \rightarrow GL_n(\mathbb{Z})$ split by L . We need to show that any $h : T^n \rightarrow T^n$ is Ψ -isotopic to $L \circ H_1(h)$.

Note they are homotopic: (1) use $T^n = K(\mathbb{Z}^n, 1)$ or (2) lift h to $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Take a straightline homotopy between \tilde{h} and $L \circ H_1(h)$.

Now apply the Borel conjecture for $T^n \times I$, to see the homotopy is homotopic, relative to $\partial(T^n \times I)$ to a homeomorphism. \square

2 Bundles

Let $BO(n) = Gr(n, \mathbb{R}^\infty)$. This is a *classifying space for vector bundles over a finite CW complex B*:

There is an n -plane bundle γ over BO so that

$$\begin{aligned} [B, BO(n)] &\rightarrow \{\text{iso classes of } n\text{-plane bundles over } B\} \\ [f] &\mapsto [f^* \gamma] \end{aligned}$$

Example 8. Let $M^n \subset \mathbb{R}^k$ be a smooth submanifold. Then $M \rightarrow Gr(k, \mathbb{R}^n)$, $p \mapsto T_p M \subset \mathbb{R}^n$.

Definition 9. Two vector bundles η and ξ over B are *stably equivalent* if $\eta \oplus \underline{\mathbb{R}}^k \cong \eta \oplus \underline{\mathbb{R}}^l$

Let $BO = \text{colim } BO(n)$. Then $[B, BO]$ classifies stable vector bundles over B . It is an abelian group, computable by the Atiyah-Hirzebruch spectral sequence.

Definition 10. A n -plane *microbundle* over B is a pair of maps

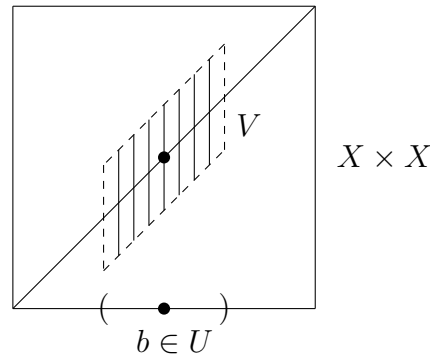
$$B \xrightarrow{i} E \xrightarrow{p} B$$

satisfying $p \circ i = \text{Id}_B$ and the following local triviality condition: for every $b \in B$ there exists open neighborhoods U of b and V of $i(b)$ with $i(U) \subset V$ and a homeomorphism $V \rightarrow U \times \mathbb{R}^n$ so that the following diagram commutes

$$\begin{array}{ccc}
 & V & \\
 i \nearrow & \downarrow \cong & \searrow p \\
 U & & U \\
 \text{Id} \times 0 \searrow & & \nearrow p_1 \\
 & U \times \mathbb{R}^n &
 \end{array}$$

Example 11. The *tangent microbundle* of a topological n -manifold X is

$$X \xrightarrow{\Delta} X \times X \xrightarrow{p_1} X.$$



Most of the machinery of bundle theory applies to microbundles. For example, two bundles $B \xrightarrow{i} E \xrightarrow{p} B$ and $B \xrightarrow{i'} E' \xrightarrow{p'} B$ are *isomorphic* if there are neighborhoods W and W' of $i(B)$ and $i'(B)$ respectively and a commutative diagram

$$\begin{array}{ccc}
 & W & \\
 & \nearrow & \searrow \\
 B & \cong & B \\
 & \searrow & \nearrow \\
 & W' &
 \end{array}$$

An \mathbb{R}^n -*bundle with a zero section* is a fiber bundle $E \rightarrow B$ with fiber \mathbb{R}^n and structure group $TOP(n) = \text{Homeo}(\mathbb{R}^n \text{ rel } 0)$. Every \mathbb{R}^n -bundle with a zero section determines a microbundle. A theorem of Kister and Mazur gives a one-to-one correspondence between isomorphism classes of \mathbb{R}^n -bundles with a zero section and isomorphism class of microbundles.

There are spaces $BTOP(n)$ and $BTOP$ which classify microbundles and stable microbundles. Assume $BO \rightarrow BTOP$ is a fibration by redefining $BO = ETOP/O$ or by replacing the map by a fibration.

3 Fundamental Theorem

Given a diagram

$$\begin{array}{ccc}
 & B & \\
 & \downarrow p & \\
 A & \xrightarrow{\alpha} & C
 \end{array}$$

let $\text{Lift}_p(\alpha)$ be the set of maps $\hat{\alpha} : A \rightarrow B$ so that $p \circ \hat{\alpha} = \alpha$. Let $[\text{Lift}_p(\alpha)]$ be the set of vertical homotopy classes of lifts. A *vertical homotopy* is a map $H : A \times I \rightarrow B$ so that for all $a \in A$ and $t \in I$, $p(H(a, t)) = \alpha(a)$, in which case $H(-, 0)$ and $H(-, 1)$ are vertically homotopic. We will usually leave p out of the notation.

Fundamental Theorem of Smoothing. Let X be a topological manifold with $\dim X \geq 5$. Let $\tau_X : X \rightarrow BTOP$ denote a classifying map of the stable tangent bundle. X admits a smooth structure if and only if there is a lift

$X \rightarrow BO$ such that the following diagram commutes.

$$\begin{array}{ccc} & & BO \\ & \nearrow \text{dashed} & \downarrow \\ X & \xrightarrow{\tau_X} & BTOP \end{array}$$

In fact, the classifying map of the smooth tangent bundle gives a bijection

$$\mathcal{S}(X) \xrightarrow{\cong} [\text{Lift}(\tau_X)]$$

where $[\text{Lift}(\tau_X)]$ denotes vertical homotopy classes of lifts of τ_X (vertical homotopy means a homotopy through lifts).

To illustrate some subtlety, note that exotic spheres are stably parallelizable – why does this not contradict the fundamental theorem?

4 Some homotopy theory

Let X_h be smooth structure on X . We wish to establish a bijection $j_h : [X, TOP/O] \xrightarrow{\cong} \mathcal{S}(X)$.

$$\begin{array}{ccc} TOP/O & \longrightarrow & BO \\ & & \downarrow \\ & & BTOP \end{array}$$

1. TOP/O is 2-connected (showing that it is path-connected is a difficult theorem - the solution to the annulus conjecture.)
2. $TOP/O, O, TOP, BTOP, BO$ are H -spaces (idea: $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$).
3. $TOP/O, O, TOP, BTOP, BO$ are infinite loop spaces: e.g. $\exists A_1, A_2, \dots$ so that $TOP/O \simeq \Omega A_1$, $A_1 \simeq \Omega A_2$, etc., where \simeq means homotopy equivalent.
4. There is a long exact sequence of abelian groups

$$\cdots \rightarrow [\Sigma B, BO] \rightarrow [\Sigma B, BTOP] \rightarrow [B, TOP/O] \rightarrow [B, BO] \rightarrow [B, BTOP]$$
5. $BO \rightarrow BTOP$ is a principal TOP/O -bundle.

TOP/O is an H -space and I am not going to spell out what I mean by a principal H -space bundle. But it does mean that there is a commutative diagram

$$\begin{array}{ccc} BO \times TOP/O & \longrightarrow & BO \\ & \searrow & \downarrow \\ & & BTOP \end{array}$$

and hence a map $[B, BO] \times [B, TOP/O] \rightarrow [B, BO]$.

Corollary 12.

$$\begin{array}{ccc} & & BO \\ & & \downarrow \\ B & \xrightarrow{\tau} & BTOP \end{array}$$

The abelian group $[B, TOP/O]$ acts freely and transitively (on the right) on $[\text{Lift}(\tau)]$.

Thus if we choose a lift $\hat{\tau} : B \rightarrow BO$, then the orbit map gives a bijection $[B, TOP/O] \xrightarrow{\cong} [\text{Lift } \tau]$.

In particular, if M is a smooth manifold of dimension ≥ 5 , there is a bijection

$$j_h : [M, TOP/O] \xrightarrow{\cong} S(M)$$

defined by acting on the tangent bundle $\tau : M \rightarrow BO$ to get a new lift of $p \circ \tau : M \rightarrow BTOP$, and choosing the corresponding smooth structure given by the fundamental theorem.

Furthermore, since TOP/O is an infinite loop space, the Atiyah-Hirzebruch spectral sequence applies to compute $[M, TOP/O]$, which is H^0 of a generalized cohomology theory.

5 Exotic spheres

Let Θ_n be the equivalence classes of smoothing on S^n under orientation-preserving diffeomorphism. This is an abelian monoid under connected sum.

Lemma 13. *Let $n \geq 5$.*

1. Θ_n is a finite abelian group.

2. $S(S^n) \xrightarrow{\sim} \Theta_n$
3. The composite $\pi_n(TOP/O) \xrightarrow{\sim} [\text{Lift}(\tau_{S^n})] \xleftarrow{\sim} S(S^n) \xrightarrow{\sim} \Theta_n$ is an isomorphism of abelian groups.

Discussion of proof. Smale proved that every exotic sphere of dimension ≥ 5 is obtained by gluing $D^n \cup_f D^n$ for some diffeomorphism $f : S^{n-1} \rightarrow S^{n-1}$. It follows that Θ_n is a group. The finiteness is due to Kervaire-Milnor. It is easy to see an epimorphism $S(S^n) \rightarrow \Theta_n$. To show injectivity one uses the Alexander trick. \square

Lemma 14. *Let M and N be smooth manifolds with $\dim M \geq 5$.*

1. If M is closed, connected, let $c : M \rightarrow S^n$ be a degree one map; in general choose an embedded disk $D^n \hookrightarrow M$ and let $c : M \rightarrow M/(M - \text{int } D^n) = S^n$ be the quotient map. The following diagram commutes

$$\begin{array}{ccc} [S^n, TOP/O] & \xrightarrow{c^*} & [M, TOP/O] \\ j \downarrow & & \downarrow j \\ S(S^n) & \longrightarrow & S(M) \end{array}$$

where the bottom horizontal map is $[\Sigma] \mapsto [M \# \Sigma]$.

2. The following diagram commutes

$$\begin{array}{ccc} [M, TOP/O] & \xrightarrow{\text{pr}^*} & [M \times N, TOP/O] \\ j \downarrow & & \downarrow j \\ S(M) & \longrightarrow & S(M \times N) \end{array}$$

where the bottom horizontal map is $[M_h] \mapsto [M_h \times N]$

Proof. 1. The idea is that there is a cobordism W from M to $M \amalg S^n$ and that the corresponding assertion is obvious for $M \amalg S^n$. Indeed, let $\phi : S^0 \times D^n \hookrightarrow M \amalg S^n$ (whose image intersects both M and S^n) and let $W = (M \amalg S^n) \times I \cup_{\phi} D^1 \times D^n$ be the result of adding a 1-handle to M along ϕ . Give W a smooth structure which restricts (up to concordance) to the given smooth structures on the boundary.

There is a commutative diagram

$$\begin{array}{ccccccc}
[S^n, TOP, O] & \longrightarrow & [M \amalg S^n, TOP, O] & \xleftarrow{\cong} & [W, TOP, O] & \longrightarrow & [M, TOP, O] \\
\downarrow j \cong & & \downarrow j \cong & & \downarrow j \cong & & \downarrow j \cong \\
& & S(M \amalg S^n) & \longleftarrow & S(W) & \longrightarrow & S(M) \\
& \nearrow \alpha & & \nearrow \beta & & \nearrow \gamma & \\
S(S^n) & & & & & &
\end{array}$$

The middle and right horizontal arrows are induced by restriction, and the upper left horizontal arrow is induced by inclusion of a summand. For a smooth structure Σ on S^n , let $\alpha[\Sigma] = [M \amalg \Sigma]$ and $\gamma[\Sigma] = [M \# \Sigma]$. For the definition of β choose a point $p \in S^n$ and glue W minus an open tubular neighborhood of $\{p\} \times I$ with $\Sigma \times I$ minus an open tubular neighborhood of $\{p\} \times I$. Note then that $S(W) \rightarrow S(M \amalg S^n)$ is a bijection and Lemma 14.1 follows.

2. We will show that diagram below is commutative

$$\begin{array}{ccc}
[M, TOP/O] & \xrightarrow{\text{pr}^*} & [M \times N, TOP/O] \\
\cong \downarrow & & \downarrow \cong \\
[\text{Lift } \tau_M] & \longrightarrow & [\text{Lift } \tau_{M \times N}] \\
\cong \uparrow & & \uparrow \cong \\
S(M) & \longrightarrow & S(M \times N)
\end{array}$$

where the middle horizontal arrow is defined using the H -space map $BO \times BO \rightarrow BO$ which has that property that if $X_i \rightarrow BO$, $i = 1, 2$ classifies a bundle α_i , $i = 1, 2$ then $X_1 \times X_2 \rightarrow BO \times BO \rightarrow BO$ classifies $\alpha_1 \times \alpha_2$. It follows that the bottom rectangle commutes. The top rectangle commutes because the following diagram commutes:

$$\begin{array}{ccc}
BO \times TOP/O & \longrightarrow & BO \\
\downarrow & & \parallel \\
BO \times BO & \longrightarrow & BO
\end{array}$$

□

6 The torus

Theorem 15. *Let $n \geq 5$. Let Σ^n be an exotic sphere and let Ω^n be an exotic torus.*

1. $[T^n \# \Sigma^n] \neq [T^n] \in S(T^n)$
2. $[T^k \times \Omega^n] \neq [T^{k+n}] \in S(T^{k+n})$.
3. $[T^n \# \Sigma^n] \neq [T^n] \in M(T^n)$ and $[T^k \times \Omega^n] \neq [T^{k+n}] \in M(T^{k+n})$.

Proof. 1. The key fact we need to show is that $c^* : [T^n, TOP/O] \rightarrow [S^n, TOP/O]$ is injective. Since pr is split surjective, pr^* is split injective, so the desired result follows from the above lemma. The key ingredients in showing that c^* is injective (actually split injective) are that TOP/O is 2-connected, that $TOP/O \simeq \Omega A_1$ for some space A_1 , and that $\Sigma T^n \simeq \vee e^j$. We will also use the adjoint correspondence for based homotopy $[\Sigma X, Y]_* \cong [X, \Omega Y]_*$ and that fact that for simply-connected targets, and path-connected domains, the forgetful map from based homotopy to unbased homotopy is a bijection.

Thus $\Sigma c : \Sigma T^n \rightarrow \Sigma S^n$ has a homotopy right inverse, so $(\Sigma c)^* : [\Sigma T^n, A_1] \rightarrow [\Sigma S^n, A_1]$ is injective. Then, by applying the adjoint correspondence, c^* is injective.

2. This follows from Lemma 14 2.

3. This follows from 1. and the fact that $\text{Diffeo}(T^n) \setminus T^n \cong M(T^n)$ which I proved earlier.

□

In fact, the proof above shows

Theorem 16. *For $n \geq 5$, $M(T^n) \cong \oplus_i H^i(T^n; \pi_i(TOP/O)) / GL_n(\mathbb{Z})$.*

The homotopy groups of TOP/O are listed below.

Remark 17. One can show that if X is a stably parallelizable manifold, that X and $X \# \Sigma^n$ are not concordant. The idea is to use the Milnor-Spanier theorem and Spanier-Whitehead duality to show that X stably pinches off the top cell.

7 PL manifolds

Two definitions of PL-manifold

Definition 18. A *PL-manifold* is a topological manifold with a maximal PL-atlas.

Definition 19. A *PL-manifold* is a simplicial complex which is a topological manifold and with the link of every vertex a PL-disk.

The definition of $BPL(n)$ and BPL are a little more complicated, since they are not topological groups. The definition of PL/O is as the homotopy fiber of the map $BPL \rightarrow BO$.

Hirsch-Mazur show smoothing theory PL/O works in every dimension, PL/O is 6-connected.

Kirby-Siebenmann show smoothing theory TOP/PL works in dimension ≥ 5 and that $TOP/PL = K(\mathbb{Z}/2, 3)$.

Thus

$$\pi_i(TOP/O) = \begin{cases} 0 & i = 0, 1, 2, 4, 5, 6 \\ \mathbb{Z}/2 & i = 3 \\ \Theta_i & i \geq 6 \end{cases}$$

Wall proves that $M(T^5)$ has three elements.

8 Nilmanifolds

Using that Θ_* is finite, one can prove

Theorem 20. Let T_h^n be a smooth structure on a torus, $n \geq 5$. Then there is a finite cover diffeomorphic to the T^n .

Theorem 21 (Davis). Let X_h^n be a smoothing of a nilmanifold, $n \geq 5$. Then there is a finite cover diffeomorphic to a nilmanifold.

9 negatively curved manifolds

See my survey on Farrell-Jones.

10 Some references

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