M623: Topics in Geometric Topology
Four Manifolds

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MWF 9:10-10, RH 104
Office Hours, RH 259, W10, F4

This will be periodically updated, see my web page.

Course Announcement

Four dimensional manifolds remain the most mysterious objects in geometric topology: four is on the only dimension where there is an exotic smooth structure on Euclidean space (an exotic $\mathbb{R}^4$), and four is the only dimension where the corresponding question on the sphere is unknown (the 4d Poincaré conjecture). Topological manifolds with some fundamental groups are under control (using very difficult point-set topology and the manifold theory known as surgery). Smooth manifolds are wild (using very difficult analysis motivated by physics - gauge theory). The whole subject is a bit of a mess - which makes it interesting!

This course will give a survey - with the basic text being *The Wild World of 4-manifolds* by Scorpan, but we will dip into Gompf and Stipsicz *4-manifolds and Kirby Calculus* when we need more precision (and less wildness).

Prerequisites: Comfortability with manifolds (e.g. smooth manifolds, bundles, transversality) and comfortability with algebraic topology (e.g. homology - singular and cellular - and fundamental group). Cohomology would help, but maybe you could put some of it in a black box.

Topics

- Topological and smooth manifolds
- Surgery and handles
- Morse Theory (handle decomposition of a manifold)
- Differential topology - transversality - intersection number
- h-cobordism theorem and the generalized Poincare conjecture
- Topological 4-manifolds
- Intersection forms
- Kirby calculus
- Donaldson's theorem and applications - exotic $\mathbb{R}^4$'s
- Modern topics in smooth four-manifolds

Outline

1. Topological manifolds
    (a) Definition/Alternate definitions
    (b) Examples
    (c) 2-manifolds
    (d) Tangent bundle (microbundle)
2. Manifolds with boundary
    (a) Collars
    (b) Submanifold, embedding
    (c) Gluing (connected sum, triads)
3. 3-manifolds

   (a) Loop theorem
   (b) Sphere theorem
   (c) Dehn's lemma

4. Smooth manifolds and smooth maps
   (a) Diffeomorphism vs. homeomorphism
   (b) Making sense of submanifolds, gluing, and products in the smooth category
   (c) Regular values/submersions
   (d) Tangent/normal bundle
   (e) Embeddings and the tubular neighborhood theorem/isotopy
   (f) Immersion/regular homotopy
   (g) Transversality
   (h) Orientability

5. Surgery and handles
   (a) Definitions
   (b) Surgery and cobordism
   (c) Homotopy and homology
   (d) Lickorish-Wallace Theorem
   (e) Dehn surgery and 3-manifolds
   (f) Dehn surgery and 4-manifolds
   (g) The linking matrix of a Kirby diagram is the intersection matrix of the 4-manifold and presents H_1 of the 3-manifold.

6. The Wild World of 4-Manifolds
   (a) Prove that every fp group is the fundamental group of a closed 4-manifold. $\mathbb{Z}^4$ is not the fundamental group of a closed 3-manifold.
   (b) Statement of Freedman's Theorem

(c) Some invariants of intersection forms: signature, rank, even or odd type, definite or indefinite.

(d) Rochlin's theorem

(e) E8 and the Poincare homology sphere.

(f) Donaldson's Theorem, exotic $\mathbb{R}^4$

(g) h-cobordisms, Whitney trick, and Casson Handles

Some Projects

Students will pair up and give a talk on one of the topics below.

- 2-manifolds - Outline the classification of closed 2-manifolds. One source is Massey, *A basic course in algebraic topology. Sergei and Onur*

- Morse Theory I - Prove the Morse Lemma (see eg Milnor, *Morse Theory* Lemma 2.2), and the fact that Morse Functions on a smooth manifold are open and dense (see eg Milnor, *Lectures on the h-cobordism theorem* Theorem 2.5.) *Abhisheck and Jiajun*

- Morse Theory II - Prove that a cobordism without critical points is a product (see eg Milnor, *Lectures on the h-cobordism theorem*, Theorem 3.4 and prove that every cobordism is given by a sequence of elementary cobordisms (see eg Milnor, *A procedure of killing the homotopy groups of differentiable manifolds*, Theorem 1.1 or Milnor, *Lectures on the h-cobordism theorem*, Theorem 3.12. *Ben and Lawford*

- Kirby calculus for 3-manifolds. See Rolfsen, Chapter 9, sections F, G, and H. Zoom with Chuck Livingston and discuss examples *Ramyak and Arijit*

- Kirby calculus for 4-manifolds. See Gompf-Stipsicz, 4-Manifolds and Kirby Calculus, sections 4.4, 4.6, and 5.1. Zoom with Chuck Livingston and discuss examples. *Patrick and Dalton*

- Bilinear forms. Show that every nonsingular symmetric form $Q : V \times V \to \mathbb{R}$ can be represented by a diagonal matrix (hint: first find v_1 so that $Q(v_1, v_1) \neq 0$, then express $V = \mathbb{R}v_1 \oplus v_1^{\perp}$). Define the signature of a symmetric bilinear form over $\mathbb{R}$, giving the proof of Sylvester's law of inertia. Show that two symmetric bilinear forms over $\mathbb{R}$ are isomorphic if and only if they have the same rank and signature. Note that the rank and signature are congruent mod 2. Define the Grothendieck-Witt group $GW(R)$ for a commutative ring R. Compute the Grothendieck Witt group $GW(\mathbb{R})$.

State the basic facts on unimodular forms: (1) Serre's Classification Lemma (S, p238) and corollaries, (2) $GW(\mathbb{Z}) \to GW(\mathbb{R})$ is an isomorphism, (3) van der Blij's Lemma (S, p170) and its corollary that an even definite unimodular form must have signature divisible by 8, and (4) there are only a finite number of definite forms of a given rank S, p239). Assuming Meyer's Lemma, prove Serre's Classification Lemma and van der Blij's Lemma (S, p262-264). Discuss the E8 form. Construct a closed smooth 4-manifold with even type and nonzero signature (the Kummer surface), S 3.3. *Conor and Muhammad*

- Casson Handles. Prove Lemma 1 in Casson's Lectures. Construct a Casson Tower and a Casson Handle. State Freedman's Theorem that Casson Handles are standard. State the corollary that if X is a simply-connected closed manifold with $a, b \in H_2X$ so that $a.b = 1$ and $b.b$ is even, then a can be represented by an embedded S^2. *Arijit and Patrick*

- The Thom Conjecture - Chapter 11 of Scorpan - maybe skip 11.3 if there is not enough time.