Smooth structures, the gluing lemma, straightening corners, surgery, and handles

1 Smooth Structures

Let M be a topological manifold. A *smooth structure on* M is a maximal smooth atlas. A smooth atlas $\mathcal{A}$ determines a unique smooth structure $D(\mathcal{A})$. The smooth structure determines which functions $M \to \mathbb{R}$ and which paths $\mathbb{R} \to M$ are smooth.

Exercise 1. *Show that S^1 has more than one smooth structure.*

Definition 2. Two smooth structures $\mathcal{A}_0$ and $\mathcal{A}_1$ are *concordant* if there is a smooth structure $\mathcal{A}$ on $M \times I$ which restricts to $(M, \mathcal{A}_0)$ on $M \times 0$ and to $(M, \mathcal{A}_1)$ on $M \times 1$. There are *isotopic* if, in addition, $M \times t$ is a smooth submanifold for all $t \in I$.

Same $\Longrightarrow$ isotopic $\Longrightarrow$ concordant $\Longrightarrow$ diffeomorphic. This last implication is a non-trivial theorem.

If $\dim M \neq 4$, isotopic $\Longleftrightarrow$ concordant.

2 The Gluing Lemma

[Pictures of gluing: before and after]

Gluing Lemma. *Let W_1 and W_2 be smooth manifolds with boundary and $f : \partial W_1 \to \partial W_2$ a diffeomorphism. Let $W = W_1 \cup_f W_2$. Choose smooth collars $h_i : \partial W_i \times \mathbb{R}_+ \hookrightarrow W_i$ for $i = 1, 2$.*

1. *There is a unique smooth structure* $\mathcal{A}(h_1, h_2)$ *on* W *so that*

$$W_1 - \partial W_1 \hookrightarrow W$$
$$W_2 - \partial W_2 \hookrightarrow W$$
$$h : \partial W_1 \times \mathbb{R} \hookrightarrow W$$

are smooth embeddings where $h(x,t) = \begin{cases} h_1(x,t) & t \geq 0 \\ h_2(f(x), -t) & t \leq 0 \end{cases}.$

2. *If* h_i, h_i' *are smooth collars for* $i = 1, 2$*, then the smooth structures* $\mathcal{A}(h_1, h_2)$ *and* $\mathcal{A}(h_1', h_2')$ *are isotopic.*

3. *Suppose* W *has a smooth structure so that there are smooth embeddings*

$$W_1 - \partial W_1 \hookrightarrow W$$
$$W_2 - \partial W_2 \hookrightarrow W$$
$$h : \partial W_1 \times \mathbb{R} \hookrightarrow W$$

where the first two maps are inclusions and the last map is a bi-collar, i.e. $h(x, 0) = x$ *for all* $x \in \partial W_1$*. Then the smooth structure on* W *equals* $\mathcal{A}(h_1, h_2)$ *for some* h_1, h_2*.*

To prove part 1 note that the images of the three topological embeddings form an open cover and the transition maps are smooth.

Exercise 3. *Formulate and prove a statement along the lines: A smooth structure on a manifold is determined by smooth manifold charts (with the targets smooth manifolds) and smooth transition functions.*

For the proof of part 2 see Milnor, “Differential manifolds which are homotopy spheres”, and then strengthen his conclusion from diffeomorphism to isotopy.

[Picture where $\mathcal{A}(h_1, h_2) \neq \mathcal{A}(h_1', h_2')$]

The proof of part 3 is clear.

Remark 4. There are examples of a $W = W_1 \cup_f W_2$ with a smooth structure so that

$$W_1 - \partial W_1 \hookrightarrow W$$
$$W_2 - \partial W_2 \hookrightarrow W$$

are smooth embeddings, but where the smooth structure is not of the form $\mathcal{A}(h_1, h_2)$. Indeed, take an exotic diffeomorphism $f : S^6 \to S^6$ and pullback the standard smooth structure via homeomorphism $D^7 \cup_f D^7 \to S^7$.

3 Straightening Corners

Let W_1 and W_2 be smooth manifolds with boundary.

Problem: Give $W_1 \times W_2$ a smooth structure.

Issues:

- $\partial(W_1 \times W_2) = (\partial W_1 \times W_2) \cup_{\mathrm{Id}} (W_1 \times \partial W_2)$ is a glued manifold $\Longrightarrow$ need collars.
- $\mathbb{R}_+ \times \mathbb{R}_+$ is not a smooth submanifold of $\mathbb{R}^2$, but can be given a smooth structure with the chart

$$h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$$
$$(r\cos\theta, r\sin\theta) \mapsto (r\cos 2\theta, r\sin 2\theta)$$
$$\pi/2 \geq \theta \geq 0, \quad r \geq 0$$

[Picture]

Lemma 5. *Choose smooth collars $h_i : \partial W_i \times \mathbb{R}_+ \hookrightarrow W_i$ for $i = 1, 2$. There is a unique smooth structure on $W_1 \times W_2$ so that*

$$W_1 \times W_2 - (\partial W_1 \times \partial W_2) \hookrightarrow W_1 \times W_2$$
$$h : \partial W_1 \times \partial W_2 \times \mathbb{R} \times \mathbb{R}_+ \hookrightarrow W_1 \times W_2$$
$$(x, y, r\cos\theta, r\sin\theta) \mapsto (h_1(x, r\cos(\theta/2)), h_2(y, r\sin(\theta/2))$$

are smooth embeddings.

4 Surgery

Smooth $\varphi : S^p \times D^{q+1} \hookrightarrow M^n$ with $n = p + q + 1$.

Claim: $\chi(M, \varphi)$, the surgery on φ, and $\omega(M, \varphi)$, the trace of the surgery on φ, have "canonical" smooth structures.

As a special case, the connected sum $M_1 \# M_2$ of smooth manifolds has a smooth structure.

Shrink the framed sphere and define $\phi : S^p \times D^{q+1} \hookrightarrow M^n$ by $\phi(x, y) = \varphi(x, \frac{1}{2}y)$. We will:

1. Construct a smooth structure on $\chi(M, \phi)$.
2. Construct a homeomorphism from $\chi(M, \varphi)$ to $\chi(M, \phi)$ and then pull-back the smooth structure.

[Picture]

The idea for Step 1 is that

$$\chi(M, \phi) = M - \phi(S^p \times \text{int } D^{q+1}) \cup_{\text{Id}} D^{p+1} \times S^q$$

is a glued manifold and that the disk coordinates gives natural collars (albeit defined for $[0, 1/2)$ rather than $[0, \infty)$, but that is not important.)

Here are the collars which determine the smooth structure on $\chi(M, \phi)$.

$$\begin{aligned} S^p \times S^q \times [0, 1/2) &\hookrightarrow M - \phi(S^p \times \mathring{D}^{q+1}) \\ (x, y, t) &\mapsto \varphi(x, (t + (1/2))y) \\ S^p \times S^q \times [0, 1/2) &\hookrightarrow D^{p+1} \times S^q \\ (x, y, t) &\mapsto ((1 - t)x, y) \end{aligned}$$

We will omit the construction of Step 2.

5 Handles aka Gluing Manifold Triads

A *manifold triad* $(W; M_+, M_-)$ is a manifold with boundary W together with a decomposition of its boundary $\partial W = M_+ \cup M_-$ so that $M_\pm$ are both manifolds with boundary themselves and their boundary is their common intersection $\partial M_+ = M_+ \cap M_- = \partial M_-$.

[picture]

A *special manifold triad* or *cobordism* is a manifold triad with $M_+ \cap M_- = \emptyset$, i.e. $\partial W = M_+ \sqcup M_-$. We say M_+ and M_- are *cobordant*.

Note that if $(W; M_+, M_-)$ and $(X; N_+, N_-)$ are topological manifold triads and $f : M_- \to N_+$ is a homeomorphism then $(M \cup_f X; M_+, N_-)$ is a topological manifold triad. We wish to make this construction in the smooth category.

[picture]

Adding a handle using $\varphi : S^p \times D^{q+1} \hookrightarrow \partial W^{n+1}$ is an example of gluing manifold triads:

$$(W; \partial W - \varphi(S^p \times \mathring{D}^{q+1}), \varphi(S^p \times D^{q+1})) \cup_{\varphi^{-1}}$$

$$(D^{p+1} \times D^{q+1}; S^p \times D^{q+1}, D^{p+1} \times S^q)$$

[Picture of gluing a 2-handle]

We first give a smooth structure in the simplest case:

$$(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+ \times 0, \mathbb{R}_- \times 0) \cup_{\mathrm{Id}} (\mathbb{R} \times \mathbb{R}_-; \mathbb{R}_- \times 0, \mathbb{R}_+ \times 0)$$

A homeomorphism to the smooth triad $(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+ \times 0, \mathbb{R}_- \times 0)$ is given by $(r\cos\theta, r\sin\theta) \mapsto (r\cos(\theta/2), r\sin(\theta/2))$

[Picture]

Gluing Lemma for Manifold Triads. *Let $(W_1; M_{1+}, M_{1-})$ and $(W_2; M_{2+}, M_{2-})$ be smooth manifold triads and $f : M_{1-} \to M_{2+}$ a diffeomorphism. Let $W = W_1 \cup_f W_2$. Choose smooth collars $h_i : \partial W_i \times \mathbb{R}_+ \hookrightarrow W_i$ for $i = 1, 2$ and smooth collars $g_{i\pm} : (M_{i\pm} \cap M_{i\mp}) \times \mathbb{R}_+ \hookrightarrow M_{i\pm}$, using the diffeomorphism f to guarantee $g_{2+} = f \circ g_{1-}$.*

Then there is a unique smooth structure $\mathcal{A}(h_i, g_{i\pm})$ on the manifold with boundary W so that

$$\begin{aligned} W_1 - M_{1-} &\hookrightarrow W \\ W_2 - lM_{2+} &\hookrightarrow W \\ h : (M_{1+} - \partial M_{1+}) \times \mathbb{R} &\hookrightarrow W \\ g : (M_{1+} \cap M_{1-}) \times \mathbb{R} \times \mathbb{R}_+ &\hookrightarrow W \end{aligned}$$

are smooth embeddings, where $h(x,t) = \begin{cases} h_1(x,t) & t \geq 0 \\ h_2(f(x), -t) & t \leq 0 \end{cases}$ *and*

$$g(x, r\cos\theta, r\sin\theta) = \begin{cases} h_1((g_{1+}(x, r\cos 2\theta), r\sin 2\theta) & \theta \in [0, \pi/4] \\ h_1((g_{1-}(x, -r\cos 2\theta), r\sin 2\theta) & \theta \in [\pi/4, \pi/2] \\ h_2((g_{2+}(x, -r\cos 2\theta), -r\sin 2\theta) & \theta \in [\pi/2, 3\pi/4] \\ h_2((g_{2-}(x, r\cos 2\theta), -r\sin 2\theta) & \theta \in [3\pi/4, \pi] \end{cases}$$