Vector Bundles

1 Vector Bundles and maps

$\boxed{\text{The study of vector bundles is the study of parameterized linear algebra.}}$

Definition 1. A *vector bundle* is a map $\pi : E \to B$ together with a vector space structure on $\pi^{-1}b$ for each $b \in B$ so that for every $x \in B$ there is a neighborhood U and a $k \in \mathbb{Z}_{\geq 0}$, a homeomorphism $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$ so that there is a commutative diagram

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\phi", "\cong"'] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\ & U & \end{tikzcd}$$

so that the induced bijection $\pi^{-1}b \to \{b\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a vector space isomorphism for all $b \in U$.

B is called the *base space*, E is called the *total space*, and the vector spaces $E_b = \pi^{-1}b$ are called the *fibers*.

A vector bundle is *smooth* if E and B are smooth manifolds, π is a smooth map and if for every $x \in B$ there is a neighborhood U and a smooth chart $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$ as above.

A *map of vector bundles* is a commutative diagram

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d] & E' \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

which induces a linear map on the "fibers" $\pi^{-1}x \to \pi'^{-1}f(x)$.

An example of a vector bundle is the tangent bundle of a manifold. The differential of a smooth map $f : X \to Y$ gives a map of vector bundles $df : TX \to TY$.

If $X \subset \mathbb{R}^n$ then

$$TX = \{(p, v) \in X \times \mathbb{R}^n \mid v \text{ is the tangent vector of a curve in } X \text{ through } p\}$$

If X^k is an abstract smooth manifold with atlas $\mathcal{A} = \{\phi : V \to U \subset \mathbb{R}^k\}$ then the tangent bundle can be defined as a quotient

$$TX = \frac{\coprod U \times \mathbb{R}^k}{\sim}$$

A *map of vector bundles over* B is a commutative diagram

$$\begin{tikzcd} E' \arrow[rr, "\hat{f}"] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$

which induces a linear map on the fibers.

2 Extra structure on vector bundles

Definition 2. An *oriented vector bundle* is a vector bundle $\pi : E \to B$ together with an orientation on each fiber, so that there is an atlas of charts $\{\phi_U : \pi^{-1}U \to U \times \mathbb{R}^k\}$ inducing orientation-preserving isomorphisms $\pi^{-1}b \to \mathbb{R}^k$ for each chart ϕ_U and for each $b \in U$.

An oriented manifold is a manifold X with an orientation on its tangent bundle TX.

Definition 3. An *vector bundle with metric* is a vector bundle $\pi : E \to B$ together with an inner product $\langle\ ,\ \rangle_b : \pi^{-1}b \times \pi^{-1}b \to \mathbb{R}$ on each fiber so that there is an atlas of charts $\{\phi_U : \pi^{-1}U \to U \times \mathbb{R}^k\}$ inducing isometries for each chart ϕ_U and for each $b \in U$.

Every vector bundle over a paracompact space admits a metric.

An Riemannian manifold is a manifold X with a smooth metric on its tangent bundle TX.

3 New vector bundles from old

Definition 4. Given vector bundles $\pi' : E' \to B'$ and $\pi : E \to B$, *the product bundle* is product map $\pi' \times \pi : E' \times E \to B' \times B$.

Definition 5. Given vector bundles $\pi' : E' \to B$ and $\pi : E \to B$, *the Whitney sum* is the bundle $E' \oplus E \to B$ where $E' \oplus E = \{(e', e) \in E' \times E \mid \pi'(e') = \pi(e)\}$. The fiber above b is $\pi'^{-1}b' \oplus \pi^{-1}b$.

Definition 6. A *subbundle* of a bundle $\pi : E \to B$ is a subspace $E' \subset E$ so that $\pi|_{E'} : E' \to B$ is a vector bundle. Given a subbundle, there is the quotient bundle $E/\sim_{E'} \to B$ where $\sim_{E'}$ is the equivalence relation on E given by $e_1 \sim e_2$ if they are both in the same fiber and if $e_1 - e_2 \in E'$.

If $E' \to B$ is a subbundle of a bundle $E \to B$ with a metric, then $E \to B$ is a Whitney sum $E' \oplus E'^{\perp} \to B$, where $E'^{\perp} = \{e \in E \mid \langle e, E'_{\pi(b)} \rangle = 0\}$. Furthermore the obvious map $E'^{\perp} \to E/\sim_{E'}$ gives an isomorphism of vector bundles over B.

As a consequence one sees that a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles over a paracompact B splits.

The *restriction* of a vector bundle $\pi : E \to B$ to $B' \subset B$ is the vector bundle $\pi^{-1}B' \to B'$. We write this as $E|_{B'} \to B'$.

Example 7. Let $X^k \subset Y^l \subset \mathbb{R}^n$ be submanifolds. Let $N(X \subset Y)$ be the orthogonal complement $TX^{\perp}$ of TX in $TY|_X$. Let $\nu(X \subset Y)$ be the quotient bundle $(TY|_X)/TX$ (or rather $TY|_X/\sim_{TX}$ in the previous notation). We call both of these (isomorphic) bundles *the normal bundle of* $X \subset Y$. Note

$$TY|_X = TX \oplus TX^{\perp} = TX \oplus N(X \subset Y)$$

In particular the tangent bundle and normal bundle of $X \subset \mathbb{R}^n$ are Whitney sum inverses.

Definition 8. Given a vector bundle $\pi : E \to B$ and a map $f : B' \to B$, the *pullback bundle* is given by $f^*E \to B$ where $f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$. (One also writes $f^*E = B' \times_B E$.) Use the commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & E \arrow[d] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

which induces a bijection on the fibers to define the vector space structure on the pullback.

As an example, if $i : B' \hookrightarrow B$ is the inclusion then $E|_{B'}$ is the pullback bundle i^*E.

Exercise 9. Suppose

$$\begin{tikzcd}
E' \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

π and π' are vector bundles and f is a continuous map. There is a bijection between vector bundle maps $\hat{f} : E' \to E$ over f and vector bundle maps $E' \to f^*E$ over B. In particular, there is a fiberwise isomorphism covering f if and only if E' and f^*E are isomorphic vector bundles over B.

$$\begin{tikzcd}
E' \arrow[r, dashed] \arrow[dr] & f^*E \arrow[r] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
 & B' \arrow[r, "f"] & B
\end{tikzcd}$$

4 Bundles and transversality

Lemma 10. *Let $f : X \to Y$ be a linear transformation, $Z \subset Y$ be a subspace, and $S = f^{-1}Z$. Then*

$$f(X) + Z = Y \iff \overline{f} : X/S \to Y/Z \text{ is an isomorphism.}$$

Theorem 11. *Let $f : X \to Y$ be smooth map of manifolds and $Z \subset Y$ be a submanifold. Then*

$$f \pitchfork Z \iff$$

1. *$S = f^{-1}Z$ is a manifold.*
2. *$df : \nu(S \subset X) \to \nu(Z \subset Y)$ is a fiberwise isomorphism of vector bundles ($\iff df : \nu(S \subset X) \cong f^*\nu(Z \subset Y)$).*

5 Bundles, orientation, and transversality

An orientation on two of the three vector spaces E', E'', and $E' \oplus E''$ determines a orientation on the third. The same is true with vector spaces replaced by vector bundles over B.

Given a short exact sequence of vectors spaces

$$0 \to E' \to E \to E'' \to 0$$

an orientation on two of the three vector spaces determines an orientation on the third. The same is true with vector spaces replaced by vector bundles over B.

Definition 12. Suppose $f : X \to Y$ with $f \pitchfork Z$. Suppose X, Y, and Z are oriented. Then we orient $S = f^{-1}Z$ (equivalently we oriented TS) using the equations

1. $N(Z \subset Y) \oplus TZ = TY|_Z$
2. $df : N(S \subset X) \xrightarrow{\cong} f^*N(Z \subset Y)$
3. $N(S \subset X) \oplus TS = TX|_S$

Note that (unfortunately) order matters in points 1 and 3 above.

References

[1] Davis–Kirk, **Lecture Notes in Algebraic Topology**.

[2] Milnor–Stasheff, **Characteristic Classes**.