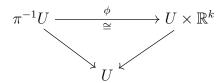
# Vector Bundles

### 1 Vector Bundles and maps

The study of vector bundles is the study of parameterized linear algebra.

**Definition 1.** A vector bundle is a map  $\pi : E \to B$  together with a vector space structure on  $\pi^{-1}b$  for each  $b \in B$  so that for every  $x \in B$  there is a neighborhood U and a  $k \in \mathbb{Z}_{\geq 0}$ , a homeomorphism  $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$  so that there is a commutative diagram



so that the induced bijection  $\pi^{-1}b \to \{b\} \times \mathbb{R}^k \cong \mathbb{R}^k$  is a vector space isomorphism for all  $b \in U$ .

*B* is called the *base space*, *E* is called the *total space*, and the vector spaces  $E_b = \pi^{-1}b$  are called the *fibers*.

A vector bundle is *smooth* if E and B are smooth manifolds,  $\pi$  is a smooth map and if for every  $x \in B$  there is a neighborhood U and a smooth chart  $\phi: \pi^{-1}U \to U \times \mathbb{R}^k$  as above.

A map of vector bundles is a commutative diagram

$$\begin{array}{c} E' \xrightarrow{\hat{f}} E' \\ \downarrow \\ B' \xrightarrow{f} B \end{array}$$

which induces a linear map on the "fibers"  $\pi^{-1}x \to \pi'^{-1}f(x)$ .

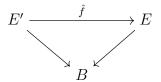
An example of a vector bundle is the tangent bundle of a manifold. The differential of a smooth map  $f : X \to Y$  gives a map of vector bundles  $df : TX \to TY$ .

If  $X \subset \mathbb{R}^n$  then

 $TX = \{(p, v) \in X \times \mathbb{R}^n \mid v \text{ is the tangent vector of a curve in } X \text{ through } p\}$ If  $X^k$  is an abstract smooth manifold with atlas  $\mathcal{A} = \{\phi : V \to U \subset \mathbb{R}^k\}$ then the tangent bundle can be defined as a quotient

$$TX = \frac{\coprod U \times \mathbb{R}^k}{\sim}$$

A map of vector bundles over B is a commutative diagram



which induces a linear map on the fibers.

#### 2 Extra structure on vector bundles

**Definition 2.** An oriented vector bundle is a vector bundle  $\pi : E \to B$ together with an orientation on each fiber, so that there is an atlas of charts  $\{\phi_U : \pi^{-1}U \to U \times \mathbb{R}^k\}$  inducing orientation-preserving isomorphisms  $\pi^{-1}b \to \mathbb{R}^k$  for each chart  $\phi_U$  and for each  $b \in U$ .

An oriented manifold is a manifold X with an orientation on its tangent bundle TX.

**Definition 3.** An vector bundle with metric is a vector bundle  $\pi : E \to B$ together with an inner product  $\langle , \rangle_b : \pi^{-1}b \times \pi^{-1}b \to \mathbb{R}$  on each fiber so that there is an atlas of charts  $\{\phi_U : \pi^{-1}U \to U \times \mathbb{R}^k\}$  inducing isometries for each chart  $\phi_U$  and for each  $b \in U$ .

Every vector bundle over a paracompact space admits a metric.

An Riemannian manifold is a manifold X with a smooth metric on its tangent bundle TX.

#### 3 New vector bundles from old

**Definition 4.** Given vector bundles  $\pi' : E' \to B'$  and  $\pi : E \to B$ , the product bundle is product map  $\pi' \times \pi : E' \times E \to B' \times B$ .

**Definition 5.** Given vector bundles  $\pi' : E' \to B$  and  $\pi : E \to B$ , the Whitney sum is the bundle  $E' \oplus E \to B$  where  $E' \oplus E = \{(e', e) \in E' \times E \mid \pi'(e') = \pi(e)\}$ . The fiber above b is  $\pi'^{-1}b' \oplus \pi^{-1}b$ .

**Definition 6.** A subbundle of a bundle  $\pi : E \to B$  is a subspace  $E' \subset E$ so that  $\pi|_{E'} : E' \to B$  is a vector bundle. Given a subbundle, there is the quotient bundle  $E/\sim_{E'} \to B$  where  $\sim_{E'}$  is the equivalence relation on Egiven by  $e_1 \sim e_2$  if they are both in the same fiber and if  $e_1 - e_2 \in E'$ .

If  $E' \to B$  is a subbundle of a bundle  $E \to B$  with a metric, then  $E \to B$ is a Whitney sum  $E' \oplus E'^{\perp} \to B$ , where  $E'^{\perp} = \{e \in E \mid \langle e, E'_{\pi(b)} \rangle = 0\}$ . Furthermore the obvious map  $E'^{\perp} \to E/\sim_{E'}$  gives an isomorphism of vector bundles over B.

As a consequence one sees that a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles over a paracompact B splits.

The restriction of a vector bundle  $\pi : E \to B$  to  $B' \subset B$  is the vector bundle  $\pi^{-1}B' \to B'$ . We write this as  $E|_{B'} \to B'$ .

**Example 7.** Let  $X^k \subset Y^l \subset \mathbb{R}^n$  be submanifolds. Let  $N(X \subset Y)$  be the orthogonal complement  $TX^{\perp}$  of TX in  $TY|_X$ . Let  $\nu(X \subset Y)$  be the quotient bundle  $(TY|_X)/TX$  (or rather  $TY|_X/\sim_{TX}$  in the previous notation). We call both of these (isomorphic) bundles the normal bundle of  $X \subset Y$ . Note

$$TY|_X = TX \oplus TX^{\perp} = TX \oplus N(X \subset Y)$$

In particular the tangent bundle and normal bundle of  $X \subset \mathbb{R}^n$  are Whitney sum inverses.

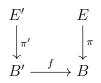
**Definition 8.** Given a vector bundle  $\pi : E \to B$  and a map  $f : B' \to B$ , the *pullback bundle* is given by  $f^*E \to B$  where  $f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$ . (One also writes  $f^*E = B' \times_B E$ .) Use the commutative diagram

$$\begin{array}{cccc}
f^*E & \xrightarrow{\pi_2} & E \\
\downarrow^{\pi_1} & \downarrow \\
B' & \xrightarrow{f} & B
\end{array}$$

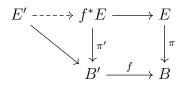
which induces a bijection on the fibers to define the vector space structure on the pullback.

As an example, if  $i: B' \hookrightarrow B$  is the inclusion then  $E|_{B'}$  is the pullback bundle  $i^*E$ .

Exercise 9. Suppose



 $\pi$  and  $\pi'$  are vector bundles and f is a continuous map. There is a bijection between vector bundle maps  $\hat{f} : E' \to E$  over f and vector bundle maps  $E' \to f^*E$  over B. In particular, there is a fiberwise isomorphism covering f if and only if E' and  $f^*E$  are isomorphic vector bundles over B.



#### 4 Bundles and transversality

**Lemma 10.** Let  $f : X \to Y$  be a linear transformation,  $Z \subset Y$  be a subspace, and  $S = f^{-1}Z$ . Then

 $f(X) + Z = Y \iff \overline{f} : X/S \to Y/Z$  is an isomorphism.

**Theorem 11.** Let  $f : X \to Y$  be smooth map of manifolds and  $Z \subset Y$  be a submanifold. Then

$$f \pitchfork Z \iff$$

- 1.  $S = f^{-1}Z$  is a manifold.
- 2.  $df : \nu(S \subset X) \to \nu(Z \subset Y)$  is a fiberwise isomorphism of vector bundles ( $\iff df : \nu(S \subset X) \cong f^*\nu(Z \subset Y)$ ).

#### 5 Bundles, orientation, and transversality

An orientation on two of the three vector spaces E', E'', and  $E' \oplus E''$  determines a orientation on the third. The same is true with vector spaces replaced by vector bundles over B.

Given a short exact sequence of vectors spaces

$$0 \to E' \to E \to E'' \to 0$$

an orientation on two of the three vector spaces determines an orientation on the third. The same is true with vector spaces replaced by vector bundles over B.

**Definition 12.** Suppose  $f: X \to Y$  with  $f \pitchfork Z$ . Suppose X, Y, and Z are oriented. Then we orient  $S = f^{-1}Z$  (equivalently we oriented TS) using the equations

- 1.  $N(Z \subset Y) \oplus TZ = TY|_Z$
- 2.  $df: N(S \subset X) \xrightarrow{\cong} f^*N(Z \subset Y)$
- 3.  $N(S \subset X) \oplus TS = TX|_S$

Note that (unfortunately) order matters in points 1 and 3 above.

## References

- [1] Davis–Kirk, Lecture Notes in Algebraic Topology.
- [2] Milnor–Stasheff, Characteristic Classes.