- Group Theory

Jan 2012 #6
Prove that if G is a nonabelian group, then $G/Z(G)$ is not cyclic.

Aug 2011 #9 (Jan 2010 #5)
Prove that any group of order p^2 is an abelian group.

Jan 2012 #7
G is nonabelian finite group of order p^3, prove $Z(G) = [G, G]$.

Aug 2011 #12
G is a finite group, and $M \subsetneq G$ be a maximal subgroup.
Show that if M is normal subgroup of G, then $|G : M|$ is prime.

Jan 2011 #1(Aug 2010 #8)
Find the element g of order 2 in S_6 with minimal order of the centralizer $C(g) = \{h \in G \mid hg = gh\}$.
(Find the numbers of element in S_5 that commute with $g \in S_5$ where g has order 6.)

Aug 2010 #2
Let G be a finite group and $\Phi : G \to G$ be an automorphism.

1. Show that Φ maps a conjugacy class of G into a conjugacy class of G.
2. Give an example of non-trivial G and Φ such that $\{e\}$ is the only conjugacy class of G that maps into itself. Explain.
3. Show that if $G = S_5$, then g and $\Phi(g)$ must be conjugate for any $g \in G$.

Jan 2010 #6
How many conjugacy classes are there in the symmetric group S_5.

Jan 2010 #4
Suppose G is a group of order 60 that has 5 conjugacy classes of orders 1,15,20,12,12.
Prove that G is a simple group.

Aug 2010 #5
Let G be the group of rigid motions(more specifically, rotations) in $\mathbb{R}^3$ generated by a =a 90 degree rotation about $x-$axis, and b =a 90 degree rotation about $y-$axis.

1. How many elements does G have?
2. Show that the subgroup generated by a^2 and b^2 is a normal subgroup of G.

Jan 2012 #5
Let a, b be elements of a group G. Prove that ab and ba have the same order.

Jan 2012 #8
Determine the group of $Aut(C_2 \times C_2)$, calculating its order and identifying it with a familiar group.

Aug 2011 #11
Find the cardinality of $\text{Hom}(\mathbb{Z}/20\mathbb{Z}, \mathbb{Z}/50\mathbb{Z})$.

Jan 2011 #3
Show that every finite group of order≥ 3 has a non-trivial automorphism.

Jan 2010 #8

1. Show that $\text{Hom}(G, H)$ is an abelian group if H is abelian group.
2. Prove that if G is finite cyclic group, then $\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is isomorphic to G.
3. Find an infinite abelian group G such that $\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is not isomorphic to G.

Aug 2011 #10
Let $a \in G$. Prove that a commutes with each of its conjugates in G iff a belongs to an abelian normal subgroup of G.

Jan 2011 #2
Let G be a group and H_3 and H_5 be normal subgroups of G of index 3 and 5 respectively.
Prove that every element of $g \in G$ can be written in the form $g = h_3 h_5$ with $h_3 \in H_3$ and $h_5 \in H_5$.

- Linear Algebra part 1

Aug 2011 #2
Let V be a finite dimensional real vector space of dimension n. Define an equivalence relation $\sim$ on the set $\mathrm{End}_{\mathbb{R}}(V)$ of $\mathbb{R}$-linear homomorphisms $V \to V$ as follows:
if S, $T \in \mathrm{End}_{\mathbb{R}}(V)$ then $S \sim T$ iff there are invertible maps $A, B : V \to V$ s.t. $S = BTA$.
Determine, as a function of n, the number of equivalence classes.

Aug 2010 #3
Let V and W be real vector spaces, and let $T : V \to W$ be a linear map. If the dimensions of V and W are 3 and 5, respectively, then for any bases B of V and B' of W, we can represent T by a 5×3 matrix $A_{T,B,B'}$. Find a set S of 5×3 matrices as small as possible such that for any $T : V \to W$ there are bases B of V and B' of W such that $A_{T,B,B'} \in S$.

Jan 2011 #5
Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation $T(a, b, c, d) = (a + b - c,\ c + d)$. Find a basis for the null space.

Jan 2010 #7
Let $G = GL_2(\mathbb{F}_5)$. What is the order of G?

Jan 2010 #3
Let A, B be $n \times n$ complex matrices such that $AB = BA$. Prove that there exists a vector $v \neq 0$ in $\mathbb{C}^n$ which is an eigenvector for A and for B.

Jan 2011 #4
The following matrix has four distinct real eigenvalues. Find their sum and their product.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

Jan 2010 #1
Let A be the a $n \times n$ complex matrix which does not have eigenvalue -1. Show that the matrix $A + I_n$ is invertible.

Aug 2011 #3
Let A be the $n \times n$ matrix with zeros on the diagonal and ones everywhere else. Find the characteristic polynomial of A.

- Ring Theory

Aug 2011 #6
Let P be a prime ideal in a commutative ring R with 1, and let $f(x) \in R[x]$ be a polynomial of positive degree. Prove that following statement: if all but the leading coefficient of $f(x)$ are in P and $f(x) = g(x)h(x)$, for some non-constant polynomials $g(x), h(x) \in R[x]$, then the constant term $f(x)$ is in P^2.

Jan 2011 #7
Prove that in a commutative ring with a finite number of elements, prime ideals are maximal.

Jan 2011 #9

1. Give an example of a ring R and a unit $r \in R$ with $r \neq 1$.

2. Give an example of a ring R and a nilpotent element $r \in R$ with $r \neq 0$.

3. Show that for any ring R and for any element $r \in R$, that r is a nilpotent element of R iff $1 - rx$ is a unit in the polynomial ring $R[x]$.

Aug 2010 #6
Let R be a ring with 1. Define $a \in R$ to be periodic of period k if $a, a^2, a^3, ..., a^k$ are all different, but $a^{k+1} = a$.

1. In $R = \mathbb{Z}/76\mathbb{Z}$, find an element $a \neq 0, 1$ of period 1.

2. In the same ring $R = \mathbb{Z}/76\mathbb{Z}$, find an element that is not periodic.

3. In $R = \mathbb{Z}/76\mathbb{Z}$, list the possible periods and the elements of each period.

Jan 2010 #12
Determine the following ideals in $\mathbb{Z}$ by giving generators:
(2)+(3), (4)+(6), $(2) \cap (3)$, $(4) \cap (6)$

Jan 2012 #11
Prove that the rings $\mathbb{F}_{16}$, $\mathbb{F}_4 \times \mathbb{F}_4$, and $\mathbb{Z}/16\mathbb{Z}$ are pairwise non-isomorphic.

Aug 2010 #7
In this problem , R is a finite commutative ring with 1. Let $p(x) \in R[x]$, the ring of polynomials over R.

1. Show that $a \in R$ is a root of $p(x)$ iff $p(x)$can be written as $p(x) = (x - a)g(x)$ with $g(x) \in R[x]$ of degree one less than the degree of $g(x)$.

2. Prove or give a counter example: A polynomial of $p(x) \in R[x]$ of degree n can have at most n distinct roots in R.

Jan 2012 #12(Jan 2010 #9)
Find all the maximal ideals in $\mathbb{R}[x]$. (Describe the prime ideals in $\mathbb{C}[x]$),

Jan 2010 #13
Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n such that f and f'(the derivative of f) have no common roots. Show that the quotient ring $\mathbb{C}[x]/(f)$ is isomorphic to $\mathbb{C} \times \mathbb{C} \times ... \times \mathbb{C}$($n$ times).

Aug 2011 #5
Let $R = K[x, y, z]/(x^2 - yz)$, where K is a field. Show that R is an integral domain, but not a unique factorization domain.

Aug 2010 #12
For which values of n in $\mathbb{Z}$ does the ring $\mathbb{Z}[x]/(x^3 + nx + 3)$ have no zero divisors?

Aug 2010 #11
Let M be the ring of 3×3 matrices with integer entries. Find all maximal two-sided ideals of M.

- Linear Algebra part 2

Jan 2012 #3
Find the eigenvalues and a basis for the eigenspace of the matrix.

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Jan 2010 #2(Jan 2012 #1)
Find invertible matrix P s.t. $P^{-1}AP$ is diagonal where

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad (A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix})$$

Jan 2012 #2

Find the matrix A^{2001} for $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Aug 2011 #4

Find the Jordan canonical form of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix}$.

Jan 2012 #4

Find the matrix $e^C := I + C + \frac{C^2}{2} + \ldots$ where

$C = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

Aug 2010 #9

Let A be a 5×5 real matrix of rank 2 having $\lambda = -i$ as one of its eigenvalues. Show that $A^3 = -A$ and that A is diagonalizable.

Aug 2011 #1

Let A be a matrix in $GL_n(\mathbb{C})$. Show that if A has finite order(i.e. A^k is the identity matrix for some $k \geq 1$), then A is diagonalizable.

Jan 2011 #6

A 5×5 matrix A satisfies the equation $(A - 2I)^3(A + 2I)^2 = 0$. Assume that there are at least two linearly independent vectors v satisfy $Av = 2v$.

What are the possibilities for the Jordan canonical form?

Jan 2011 #10

Let $M_n(\mathbb{C})$ denote the vector space over $\mathbb{C}$ of all $n \times n$ complex matrices. Prove that if M is a complex $n \times n$ matrix, then $C(M) = \{A \in M_n(\mathbb{C}) | \ AM = MA\}$ is a subspace of $M_n(\mathbb{C})$ if dimension at least n.

- Algebraic Number Theory

Jan 2012 #9

Find all irreducible polynomials of degree ≤ 4 in $\mathbb{F}_2[x]$.

Jan 2012 #10

Find the set of polynomials in $\mathbb{F}_2[x]$ which are the minimal polynomials of elements in $\mathbb{F}_{16}$.

Aug 2010 #1

Find all irreducible monic quadratic polynomials in $\mathbb{F}_3[x]$.

Jan 2010 #11

1. Prove that the polynomial $x^2 + x + 1$ is irreducible over the field $\mathbb{F}_2$ with two elements.
2. Factor $x^9 - x$ into irreducible polynomials in $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field with three elements.

Jan 2011 #8

Let $\mathbb{F}_4$ be the finite field with 4 elements. Express $\mathbb{F}_4[x]/(x^4 + x^3 + x^2 + 1)$ as a product of fields. Prove your result.

Aug 2011 #7

Let p be a prime number and denote by $\mathbb{F}_p$ the field with p elements. For a positive integer n, let $\mathbb{F}_{p^n}$ be the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$. Prove that the following are equivalent:

1. k|n
2. $(p^k - 1)|(p^n - 1)$

3. $\mathbb{F}_{p^k} \subset \mathbb{F}_{p^n}$

Aug 2010 #4
Is it possible to find a field F with at most 100 elements so that F has exactly five different proper subfields? If so, find all such fields. If not, prove that no such field F exists.

Aug 2011 #8

1. Show that $x^3 - 2$ and $x^5 - 2$ are irreducible over $\mathbb{Q}$.
2. How many field homomorphism are there from $\mathbb{Q}[\sqrt[3]{2}, \sqrt[5]{2}]$ to $\mathbb{C}$?
3. Prove that the degree of $\sqrt[3]{2} + \sqrt[5]{2}$ over $\mathbb{Q}$ is 15.

Jan 2010 #10
Find the degree of the minimal polynomial of $\alpha = \sqrt{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.

Aug 2010 #10

1. Give an example of an irreducible monic polynomial of degree 4 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$. Explain why your example has the stated property.
2. Show that there are no irreducible monic polynomial of degree 5 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$.

Jan 2012 #13
Let $R = \mathbb{Z}[i]$ and $I \subset R$ be an ideal. If R/I has 4 elements what are the possibilities for I and R/I.