INTRODUCTION TO TRANSFORMATION GROUPS

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ABSTRACT. These are the notes from a course that was an introduction to the theory of finite (or compact) transformation groups, including basic definitions and facts, standard examples, algebraic restrictions, fundamental constructions, all with an emphasis on group actions on manifolds in low-dimensional geometric topology. There is a focus on problems and results that have interesting versions in low dimensional topology, including the study of group actions on surfaces, 3-manifolds, and 4-manifolds. Many other deep results, especially in higher dimensions, are also briefly surveyed without full proofs.

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1. Introduction

The subject of *Transformation Groups* is the study of the symmetries of spaces. As such it is both a rich subfield of topology (thriving in both its algebraic and geometric branches) and at the same time a point of view applicable to many areas of mathematics. One often studies a mathematical object by studying its symmetries. A torus has less symmetry than a sphere, and a surface of genus greater than one has even less symmetry. Standard examples are often the objects with the greatest amount of symmetry. One seeks to classify the symmetries of given objects and to prove that other objects under study necessarily have certain symmetries. At the same time one seeks to construct examples that have little or no symmetry. The most natural invariant of a symmetry is its fixed point set. And one tries to classify the possible fixed points. And so on.

We will restrict attention to compact or finite groups of transformations, with the primary emphasis on finite groups. If one drops this, and for example considers actions of the discrete infinite cyclic group, one is in the much, much different realm of topological dynamics.

References on Transformation Groups. The best general introduction is Bredon's book [4]. Its introduction to Smith Theory, its examples, and its introduction to spectral sequence methods are especially recommended. The book by Kawakubo [6] is another more or less elementary text. The seminar notes edited by Borel [3] are mainly of historical interest. It marks a major step forward in the use of spectral sequence methods and in the attention to Cech-type cohomology appropriate for actions on very general spaces. The text by Allday and Puppe [2] is the most recent and exhaustive treatment of cohomological methods.

2. Some basic definitions

2.1. G**-spaces.** Let G denote a finite group (with the discrete topology), or, more generally, a compact topological group, with $e \in G$ its identity element. An *action* of G on a space X is a continuous map

$$\theta : G \times X \to X$$

such that for $g, h \in G$ and $x \in X$ we have

(1) $\theta(gh, x) = \theta(g, \theta(h, x))$, and
(2) $\theta(e, x) = x$.

Strictly speaking this is a *left* action, with the definition being easily adapted to give an analogous notion of right action.

We sometimes view the action as given by a corresponding homomorphism of groups $\hat{\theta} : G \to \text{Homeo}(X)$, where $\hat{\theta}(g)(x) = \theta(g, x)$. When only one group action is under consideration we often suppress the name θ, writing simply gx for $\theta(g, x)$.

When the group G is a cyclic group C_n, with generator g, acting on a space X we will often identify the group action with the action of the generator $g : X \to X$ viewed as a *periodic map* of *period* n. A periodic map of period 2 is called an *involution*.

We will almost exclusively consider actions that are *effective* in the sense that $\hat{\theta}$ is injective. Equivalently, for every non-identity element $g \in G$ there is some $x \in X$ such that $gx \neq x$.

The *orbit* of x is the subspace of X given by

$$G(x) = \{gx : g \in G\}$$

The *isotropy group* of x is the subgroup of G

$$G_x = \{g \in G : gx = x\}$$

The natural map $G \to X$ given by $g \to gx$ gives a bijection (indeed homeomorphism)

$$G/G_x \to G(x)$$

It is a quick calculation to check that

$$G_{g(x)} = gG_xg^{-1}$$

The *fixed point set* of G acting on X is the set

$$\text{Fix}(G, X) = X^G = \{x \in X : gx = x \text{ for all } g \in G\}$$

A group action is *free* if the only group element fixing a point is the identity. In other words, all isotropy groups G_x are trivial. The study of free actions is closely related to covering space theory, and more generally to the study of fiber bundles.

An action is *semifree* if for each point in the space, the point is either fixed by the whole group or fixed only by the identity element. In other words the isotropy groups are either trivial or the whole group, nothing in between. An action of a cyclic group C_p, p prime, is automatically semifree.

An action is *pseudofree* if the set of points with nontrivial isotropy group is discrete.

The *orbit space* of an action of a group G on a space X, denoted X/G is the quotient space of X by the equivalence relation generated by setting $x \sim gx$ for $x \in X$ and $g \in G$. If G is finite and acts freely on X, then the orbit map $X \to X/G$ is a regular covering with G as group of deck transformations.

2.2. G**-maps.** What are the morphisms and equivalences of G-spaces? A map $\varphi : X \to Y$ of G-spaces is *equivariant* (or a "G-map") if $\varphi(gx) = g\varphi(x)$ for all $x \in X$ and $g \in G$.

Lemma 2.1. *If $\varphi : X \to Y$ is a G-map, then $G_x \subset G_{\varphi(x)}$.*

In particular fixed points map to fixed points.

If the action satisfies the stronger condition that $G_x = G_{\varphi(x)}$ for all $x \in X$, then the G-map is said to be *isovariant*.

Note that a G-map $\varphi : X \to Y$ induces H-maps $\varphi|X^H : X^H \to X^H$ for all subgroups $H \subset G$.

A G-homeomorphism is a G-map that is a homeomorphism. One easily checks that the inverse map is also a G-map. Two G-spaces that are G-homeomorphic will sometimes be called *equivalent*.

A G-map $\varphi : X \to Y$ is a G-homotopy equivalence if there exists a G-map $\psi : Y \to X$ such that the compositions $\psi\varphi$ and $\varphi\psi$ are G-homotopic to the identity. That is, there exist G-maps $H : X \times I \to X$ and $K : Y \times I \to Y$ (where G acts trivially on the $I = [0,1]$ factor) such that $H(x,0) = \psi\varphi(x)$, $H(x,0) = x$, $K(y,0) = \varphi\psi(y)$, and $K(y,1) = y$, for all $x \in X$ and $y \in Y$.

Lemma 2.2. *If $\varphi : X \to Y$ is a G-homotopy equivalence, then the restriction*

$$\varphi|X^H : X^H \to Y^H$$

is a homotopy equivalence for all subgroups $H \subset G$.

This is to be contrasted with the weaker notion of a G-map that is a homotopy equivalence.

3. Standard examples

It is always helpful to have some standard familiar examples illustrating definitions and suggesting questions for investigation. Here we consider linear actions on euclidean spaces, spheres, quotients of spheres, and related manifolds.

Throughout, we ask how nearly do general group actions resemble the standard examples?

3.1. **Spheres and Euclidean spaces.** The compact group $\mathrm{O}(n)$ acts on both $\mathbb{R}^n$ and S^{n-1}. More generally, any finite or compact subgroup $G \subset \mathrm{O}(n)$ acts linearly on $\mathbb{R}^n$ and S^{n-1}. Note that the fixed point set of such a linear action on $\mathbb{R}^n$ is a vector subspace, homeomorphic to $\mathbb{R}^k$ for some k in the former case. In the latter case the fixed point set is a sphere S^{k-1}.

The isotropy group of a point is a conjugate of the standard $\mathrm{O}(n-1) \subset \mathrm{O}(n)$. the action of $\mathrm{SO}(n)$ on S^{n-1} is transitive, and the inclusion of an orbit identifies $S^{n-1} \cong \mathrm{O}(n)/\mathrm{SO}(n-1)$.

Question 1. *Is a finite group action on $\mathbb{R}^n$ or S^{n-1} equivalent to a linear action?*

Question 2. *If a finite group acts effectively on $\mathbb{R}^n$ or S^{n-1} is it isomorphic to a subgroup of $SO(n)$?*

Question 3. *If a finite group acts effectively on $\mathbb{R}^n$ or S^{n-1} is the fixed point set homeomorphic to a euclidean space or a sphere of some dimension less than n? In low dimensions? In general? In low codimensions?*

Question 4. *Can an action on $\mathbb{R}^n$ have no fixed points?*

Question 5. *Can an action of a finite group on S^n have just one fixed point? (In the purely topological category this would be equivalent to the preceding question.)*

In general these sorts of questions tend to have simple answers in low dimensions and complicated answers in higher dimensions. One naturally tries to classify or understand the more complicated situations.

3.2. **Projective spaces.** Linear actions on euclidean space induce natural actions, also called "linear", on projective spaces.

3.2.1. *Actions on the complex projective plane.*

Problem 1. *Analyze the actions of a cyclic group of order n on $\mathbb{C}P^2$ for which a group generator g acts by multiplication by powers of a primitive nth root of unity $\zeta = \exp(2\pi/n)$ in the coordinates.*

Problem 2. *Analyze the action of the symmetric group of degree 3 on $\mathbb{C}P^2$ induced by permutations of the homogeneous coordinates.*

Question 6. *How nearly does an arbitrary action on $\mathbb{R}P^n$ or $\mathbb{C}P^n$ resemble a linear action?*

3.3. **Lens spaces.** Consider, for example, a 3-dimensional lens space $L(p,q)$.

Problem 3. *Compare and contrast standard actions on $L(p,q)$ by the group C_p with actions of a group C_n where n is prime to p.*

3.4. **Brieskorn manifolds.** Let integers $a_1, \dots, a_{n+1} \geq 2$ be given, and consider a complex polynomial of the form

$$f(z_1, \dots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

The origin is the only critical point of f, so the intersection of $V = f^{-1}(0)$ with a small $(2n+1)$-sphere S_ε of radius ε centered at the origin in $\mathbb{C}^{n+1}$ is a smooth manifold Σ^{2n-1} of dimension $2n-1$. Often Σ^{2n-1} is a homotopy sphere, and Σ^{2n-1} has some interesting actions obtained by restricting standard linear actions on $\mathbb{C}^{n+1}$.

3.5. **Other homogeneous spaces G/H.** Here we mention a few other examples of standard or geometric group actions. If G is a compact Lie group and H is a closed subgroup, G and its subgroups act on the left coset space G/H by left translation where $g(xH) = (gx)H$. If H is normal (e.g. H is trivial), then G also acts on G/H by conjugation, where $g(xH) = gxg^{-1}H$.

Problem 4. *Interpret some of the earlier examples of standard actions as being of this sort. Find more interesting example that arise this way.*

4. Actions on spheres and tori in dimensions 1 and 2

4.1. **Dimension 1.** We attempt to give a hands-on discussion of actions on the interval, line, and circle.

Proposition 4.1. *An effective finite group action on $[-1,1]$ or $\mathbb{R}$ is equivalent to the C_2 action in which the generator acts by multiplication by -1.*

Proposition 4.2. *An effective finite group action on S^1 is equivalent to an orthogonal action by a subgroup of $O(2)$. In particular such a group is either cyclic or dihedral.*

Proof idea. There is a point that is freely permuted by the group action and one can therefore choose a small interval neighborhood of the point that is moved off itself by all nontrivial group elements. Let J be a maximal interval with the property that its interior is mapped off itself by all nontrivial group elements.This interval and its images under the group action give the circle the structure of an n-gon preserved by the group action. It follows that the group is cyclic of order n or dihedral of order $2n$ and the action is standard. □

4.2. **Surfaces.** We will have more to say about actions on surfaces later.

4.2.1. *The 2-sphere.* Here is a statement of the general classification theorem in this case.

Theorem 4.3. *A compact group action on the 2-sphere is equivalent to the action of a subgroup of $O(3)$.*

Here we content ourselves with a hands-on discussion of involutions on S^2. We also assume the action is simplicial with respect to some triangulation of the 2-sphere.

Proposition 4.4. *A nontrivial action of C_2 on S^2 is equivalent to one of the following three actions (in which we describe the action of a generator):*

(1) *The antipodal map $(x,y,z) \to (-x,-y,-z)$*
(2) *Rotation by 180° about the north-south axis, $(x,y,z) \to (-x,-y,z)$, or*

(3) *Reflection in the equator,* $(x, y, z) \to (x, y, -z)$.

When the fixed point set is empty, one applies covering space theory, arguing that the orbit space must by $\mathbb{R}P^2$.

Applying our analysis of the 1D case locally, we see that fixed points are either isolated or contained in the interior of a fixed interval. Therefore the fixed point set in general is a union of isolated points and simple closed curves.

If there is a simple closed curve, then we may apply the Jordan Curve Theorem to see that there are no other fixed points, and the Schönflies Theorem to see that the action is equivalent to reflection in the equator.

It remains to consider the case of a nonempty fixed point set F consisting of a finite number of isolated points. In this case one can argue that the orbit space S^2/C_2 is a surface, even though there are isolated fixed points .(The quotient of a circle by a free involution is again a circle!) Carefully counting vertices, edges, and faces of the induced triangulation leads to the following formula:

$$\chi(S^2) = 2\chi(S^2/C_2) - |F|$$

where $|F| > 0$ denotes the cardinality of the fixed point set. Otherwise, for now, just visualize as many different kinds of involutions on surfaces as you can. By the classification of surfaces we know that $\chi(S^2/C_2) \leq 2$. From this it follows that $\chi(S^2/C_2) \leq 2$ and $|F| = 2$.

Let x_0 and x_1 be the two fixed points and let D_0 and D_1 be two small, disjoint, invariant disk neighborhoods, obtained by taking simplicial neighborhoods of the two fixed points. The action on each of the disks D_i is a cone and by the analysis of actions on circles is equivalent to a standard rotation. The portion $X = \mathrm{cl}(S^2 - (D_0 \cup D_1))$ is a cylinder homeomorphic to $S^1 \times I$, $I = [0, 1]$.

The action on the cylinder is standard, as one can see from covering space theory and the classification of surfaces with boundary.

A standard action has exactly the same decomposition into three standard pieces.

One can then construct an equivariant homeomorphism to the standard action, piece by piece: one disk, then the cylinder, then the second disk.

This completes our outline.

4.2.2. *Actions on the torus.* The first issue is describe the natural actions on the torus T^2.

In this case T^2 is itself a compact Lie group, which acts on itself by left translation. It also has an automorphism group that acts on T^2. So we have

$$1 \to T^2 \to E_2 \to Aut(T^2) \to 1$$

An automorphism of T^2 lifts to an automorphism of $\mathbb{R}^2$ preserving the integer lattice $\mathbb{Z}^2$ by covering space theory. In this way we can identify $\mathrm{Aut}(T^2) = \mathrm{GL}_2(\mathbb{Z})$, which we can also think of as the automorphism group of the fundamental group, or of the first homology group.

Theorem 4.5. *An action of a compact group on T^2 is equivalent to the action of a compact subgroup of E_2.*

We will content ourselves with a hands-on, informal, discussion of involutions.

Theorem 4.6. *An action of a generator of C_2 on T^2 is equivalent to one of the follwing standard actions on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$:*

- $(x, y) \to (-x, y)$ *where* $F = S^1 \cup S^1$

- $(x, y) \to (y, x)$ *where* $F = S^1$
- $(x, y) \to (-x, -y)$ *where* $F = 4$ *points*
- $(x, y) \to (x + 1/2, y)$ *where* $F = \emptyset$
- $(x, y) \to (x + 1/2, -y)$ *where* $F = \emptyset$

First consider the case that the fixed set is empty. Then the orbit map $T^2 \to T^2/C_2$ is a covering. What are the surfaces whose fundamental group contains $\mathbb{Z}^2$ as a subgroup of index 2. One finds two: the torus itself and the Klein bottle. Up to equivalence there is one 2-fold covering of the torus by the torus; and there is a unique orientable 2-fold covering of any nonorientable manifold. This gives the last two cases above.

Now consider the case that the fixed set consists of isolated points. The Euler characteristic formula $\chi(T^2) = 2\chi(T^2/C_2) - |F|$, combined with the classification of surfaces, shows that $T^2/C_2 = S^2$ and that $|F| = 4$. One can easily show that this action is equivalent to the third one above. Indeed, any two such involutions with four isolated fixed points are equivalent. Any such involution can be built from four disks with standard involutions by adding three pairs of bands equivariantly in a row, with orbit space modelling the diagram $* - * - * - *$ and then capping off with two disks. Using this description one can build an equivariant homeomorphism between any two such involutions.

Finally consider the case that the fixed point set contains a simple closed curve. It follows that the action is orientation-reversing. Therefore one cannot have any isolated fixed points. If the fixed set separates T^2, it must separate into two homeomorphic pieces and the orbit space is homeomorphic to the closure of one the the two pieces. It follows that the orbit space is an annulus and T^2 is expressed as a union of two annuli. This is the first case above.

Finally, if the fixed set is non-separating, we conclude that it is a single simple closed curve (with a neighborhood that is an invariant annulus) with complementary domain a second annulus. This is case two above. The quotient of each annulus is a Möbius band, one of which is a neighborhood of the image of the fixed set in the orbit space.

4.2.3. *Higher genus surfaces.* In these higher genus cases it is more difficult to give a complete and satisfactory account from first principles. These surfaces have distinctly less symmetry. It turns out that any transformation group of any higher genus surface is finite of bounded order. Moreover the group always acts faithfully on homology. We will prove these facts later.

For the moment we will content ourselves with considering various possibilities for examples in the next subsection.

4.2.4. *Problems.* Here are a few problems to ponder:

Problem 5. *What are the possibilities for the number of isolated fixed points of an involution on a surface of genus g that you can find?*

Problem 6. *What are the possibilities for the number of circles of fixed points of an (orientation-reversing) involution on a surface of genus g that you can find?*

Problem 7. *Find an involution on a surface that has both isolated fixed points and circles of fixed points!*

Problem 8. *Visualize a C_3 action with more than 2 fixed points. With any number of fixed points $k \geq 2$? With just 1 fixed point?*

5. Topological and geometric considerations

One can consider continuous group actions on rather general spaces, but meaningful results often require spaces with more structure preserved by the group actions. For consideration of group actions and covering spaces it is natural to assume the spaces involved are locally path-connected and semilocally 1-connected, for example.

It is often nice to be able to assume one's G-spaces are subspaces of linear G-spaces. It is also useful to be able to assume that fixed point sets have nice invariant neighborhoods.

Here we mention several particularly useful categories of G-spaces and their simplest properties.

5.1. **Finitistic G-spaces.** For technical reasons one needs spaces to have some properties. We will assume all spaces under consideration are Hausdorff and paracompact. As suggested above, when issues of covering spaces are concerned, it is generally appropriate to assume spaces are locally path-connected and semilocally 1-connected. Moreover, for algebraic topology purposes it is appropriate to assume spaces are *finitistic* (in the sense of Swan) i.e., every covering has a finite-dimensional refinement. This condition allows one to apply Cech homology and cohomology techniques to extend results that we will only give in detail in the case of simplicial or CW spaces.

5.2. **Simplicial G-spaces.** The main issue here is that the action should be *regular* in the sense that a simplex is only mapped into itself as the identity. This can always be achieved by passing to the barycentric subdivision. From a geometric point of view we consider mainly PL actions on polyhedra, which become simplicial after choosing an invariant (regular) triangulation. We will work in this category when we develop the basic algebraic topology of finite group actions. At that point we'll give more details.

5.3. **G-CW-complexes.** These are CW-complexes on which a finite group G acts by permuting the cells. We require that if a cell is left invariant by an element of the group, then that group element acts as the identity on that cell. One can think of these G-spaces as being built up inductively by equivariantly attaching "G-cells" of the form $G/H \times e^n$. The main advantage over simplicial G-spaces is that G-CW-spaces tend to have much fewer cells.

5.4. **Smooth G-manifolds.** Smooth actions arise naturally in many examples and applications. But at some point technical issues such as exotic spheres intervene in natural conjectures. A key property is that in a neighborhood of any point x the action of the isotropy subgroup G_x is equivalent to the linear action of G_x on the tangent space TM_x. It follows that fixed point sets are smooth submanifolds with equivariant tubular neighborhoods that come from equivariant normal bundles. It is known that any smooth G-manifold admits a compatible G-simplicial structure.

5.5. **Locally linear G-manifolds.** These are actions on topological manifolds M such that in a neighborhood of any point $x \in M$ the action of G_x is equivalent to a linear G action. These have the good local properties of smooth actions but are defined on topological manifolds and do not have the delicate global issues that arise from smooth structures. (If one pursues this too deeply, however, they do have their own delicate global issues: Do fixed submanifolds have equivariant normal bundles? Does the complement of a fixed point set of a group action on a compact manifold have the homotopy type of a finite complex?)

5.5.1. *Examples of wild actions.* All actions on surfaces are equivalent to PL and smooth actions. But in higher dimensions things go "wild".

Example 1. There is an involution on the 3-sphere whose fixed point set is a wildly embedded 2-sphere. Bing 1950s.

Example 2. There is an involution on the 3-sphere whose fixed point set is a wildly embedded knot.

Project 1. Read Bing's proof that the union of two copies of the wild half of the Alexander Horned sphere is homeomorphic to S^3.

Exercise 1. Assuming Bing's result, argue that there are "wild" actions in all higher dimensions.

6. Simplicial complexes

6.1. **Basics about simplicial complexes.** See Spanier [10], Chapter 3, Sections 1-4, for more details.

A simplicial complex on a set V is a collection K of finite nonempty subsets of V such that

(1) If $s \in K$ and $t \subset s$, $t \neq \emptyset$, then $t \in K$, and
(2) For each $v \in V$, the singleton $\{v\} \in K$.

Sometimes for emphasis we might call this an *abstract simplicial complex*.

The singletons $\{v\}$ are the *vertices* of K. The elements $s \in K$ are the *simplices* and the nonempty subsets of s are the *faces* of s. If $|s| = n$, then we call s an n-simplex and also sometimes say $\dim s = n$. A 0-simplex is just a vertex; a 1-simplex is sometimes called an edge.

The set of proper faces of a simplex s is itself a simplicial complex, denoted ∂s or $\dot{s}$. If K is a simplicial complex, then its *q-dimensional skeleton* K^q is the simplicial complex consisting of all p-simplices of K with $p \leq q$.

The topological realization $|K|$ of a simplicial complex K, determined by $|s|$ for all $s \in K$. As a set we define $|K|$ to be the set of all functions $\alpha : V \to I$ such that

(1) For any α, the set
$$\operatorname{supp} \alpha = \{v \in V : \alpha(v) \neq 0\}$$
is a simplex of K. (In particular $\alpha(v) \neq 0$ for only a finite set of vertices.)
(2) for any α, $\sum_{v \in V} \alpha(v) = 1$.

The topology on $|K|$ is induced from the product or compact-open topology on $I^V = \text{Maps}\,(V, I)$. Alternatively define a metric on the set $|K|$ by the formula
$$d(\alpha, \beta) = \sqrt{\sum_{v \in V} [\alpha(v) - \beta(v)]^2}$$
and let $|K|_d$ denote $|K|$ with the metric topology. Either of these choices works fine, and gives the same result, for finite complexes.

To handle infinite simplicial complexes appropriately we look more closely at the topology induced on a q-simplex $s = [v_0, v_1, \ldots, v_q]$. There is a natural one-to-one correspondence between s and
$$\{x \in \mathbb{R}^{q+1} : 0 \leq x_i \leq 1, \sum x_i = 1\}$$
We assign to $|s|$ the topology $|s|_d$ making this correspondence a homeomorphism. We then give $|K|$ the weak topology generated by the topologies on all the simplices, in which a subset $X \subset |K|$ is defined to be closed if and only if $X \cap |s|$ is closed in $|s|$ for all simplices $s \in K$.

In particular, it follows that a function $f : |K| \to X$, where X is a topological space, is continuous if and only the restriction $f||s| : |s| \to X$ is continuous for every $s \in K$.

For technical reasons it is useful to define certain subdivisions of a simplicial complex, which become more flexible in a certain way, but still have homeomorphic topological realizations. The *barycentric subdivision* K' of K is defined as follows. The vertices of K' are the simplices of K and a set $[s_0, s_1, \dots, s_q]$ of simplices of K form a simplex of K' if and only (in some ordering) we have a string of proper inclusions

$$s_0 \subset s_1 \subset \cdots \subset s_q$$

Our goal is to show that there is a natural homeomorphism $|K'| \cong |K|$.

We need to look more closely at the geometric q-simplex

$$\{x \in \mathbb{R}^{q+1} : 0 \leq x_i \leq 1, \sum x_i = 1\}$$

and more generally any geometric q-simplex, defined as the convex hull Δ^q of any set of $q+1$ affinely independent points $x_0, x_1, \dots, x_q$ in some $\mathbb{R}^n$ ($n \geq q$). In this geometric situation, the barycenter of Δ^q is defined to be the point

$$b\Delta^q = \hat{\Delta}^q = \frac{1}{q+1} \sum_{i=0}^{q} x_i$$

We then define $|K'| \to |K|$ by defining $|s'| \to |s|$ for each $s \in K$ as follows. For a face $t \subset s$ assign $t \to \hat{t}$ in $|K|$. This defines a function Vert $K' \to \mathbb{R}^{q+1}$ which extends by linearity on simplices to a continuous function $|s'| \to \mathbb{R}^{q+1}$ that is an embedding onto the image of $|s|$.

6.2. **Basics of Regular Neighborhood Theory.** If L is a subcomplex of a simplicial complex K, define the simplicial neighborhood of L in K to be

$$N = N(L, K) = \{s \in K : \text{there exists } r \in L, t \in K, \text{ with } r < t, s < t\}$$

In words, N consists of a simplices that touch L, together with all the faces of such simplices. Without more care this neighborhood may not adequately reflect the nature of the subcomplex L. One can improve the situation by passing to the second derived subdivision.

Lemma 6.1. *$|N(L'', K'')|$ strong deformation retracts to $|L''|$.*

One constructs the deformation simplex by simplex, mapping vertices of $N(L'', K'') - L''$ into vertices of L'' in such a way that if vertices of N span a simplex, then their images in L'' span a simplex. Then just extend linearly over simplices. Note that the corresponding retraction is homotopic to the identity via a straight line homotopy. With more work one can arrange that the deformation is an *isotopy*, that is a homotopy through homeomorphisms.

There is the following uniqueness statement: If (K_1, L_1) and (K_2, L_2) are simplicial complexes that triangulate a polyhedral pair (X, Y), then there is a PL homeomorphism fixing Y and moving $N(L_1'', K_1'')$ to $N(L_2'', K_2'')$.

See Rourke and Sanderson [8] for me details about the geometric theory of simplicial complexes.

6.3. **Simplicial complexes with group action.** A simplicial G-complex is a simplicial complex K on a set V with an action of the group G on V such that the induced action on subsets of V preserves K. For many purposes this is not completely satisfactory. For example we want K^G to be a simplicial complex. We also want the fixed set $|K|^G = |K^G|$ and we want K/G to be a simplicial complex in a natural way. Consider the action of C_3 on a 2-simplex, cyclically permuting the vertices, to get a sense of the problem.

A simplicial G-complex K is *regular* of for each subgroup $H \subset G$ the following property holds:

If $h_0, h_1, \ldots, h_n \in H$ and both $(v_0, v_1, \ldots, v_n)$ and $(h_0 v_0, h_1 v_1, \ldots, h_n v_n)$ are simplices of K, then there exists an element $h \in H$ such that $h(v_i) = h_i(v_i)$ for all i.

Theorem 6.2 (Bredon)**.** *If K is a G-simplicial complex, then the second derived complex K'' is a regular G-simplicial complex.*

6.4. **Equivariant Regular Neighborhoods.** The basic construction of regular neighborhoods can be arranged to respect a group action. The main case we need is that if K is a regular G-simplicial complex, then the simplicial neighborhood $N = N(K''^G, K'')$ is invariant under the G action and the strong deformation of $|N|$ to $|K''^G|$ may be chosen to be equivariant. More generally, if L is any G-invariant subcomplex of K, then the simplicial neighborhood $N = N(L'', K'')$ is invariant under the G action and the strong deformation of $|N|$ to $|L''|$ may be chosen to be equivariant.

7. Homology of Simplicial Complexes

7.1. **Definition of Simplicial Homology.** We compute the simplicial homology of a simplicial complex K, by using the simplicial chain complex $C_*(K) = \{C_n(K)\}$, where each $C_n(K)$ is the free module (using some unstated choice of commutative coefficient ring) on the set of n-simplices of K. We have the standard boundary operator $\partial = \partial_n : C_n(K) \to C_{n-1}(K)$ given by

$$\partial_n[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

Then one defines $H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

The uniqueness theorem for ordinary homology shows that this agrees with ordinary singular homology of $|K|$ (at least for finite complexes).

7.2. **Euler characteristic.** The Euler characteristic of a finite complex K is defined to be

$$\chi(K) = \sum_i \# \, i\text{-simplices of } K$$

In a similar manner one defines the Euler characteristic of any finitely generated free chain complex over a field or domain by

$$\chi(C_*) = \sum_i (-1)^i \dim C_i$$

If C_* were not free, but we are in a situation where torsion submodule makes sense and where torsion free modules are free, then we can interpret $\dim C_i$ to be $\dim C_i/\text{torsion}$.

7.3. **Homological Invariance.**

Theorem 7.1. *If C_* is a finitely generated free chain complex over a field, then*

$$\chi(C_*) = \chi(H_*(C_*))$$

We have the following families of short exact sequences

$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$

$$0 \to B_{i+1} \to Z_i \to H_i \to 0$$

It follows easily that

$$\dim C_i = \dim Z_i + \dim B_{i-1}$$

and

$$\dim Z_i = \dim H_i + \dim B_i$$

so that

$$\dim C_i = \dim H_i + \dim B_i + \dim B_{i-1}$$

The terms $\dim B_j$ then all cancel in the alternating sum.

It follows that $\chi(K)$ is a homotopy invariant of the underlying polyhedron $|K|$.

7.4. **Sum Theorem.** If a polyhedron Z is the union of two subpolyhedra X and Y, with polyhedral intersection Z, then

$$\chi(Z) = \chi(X) + \chi(Y) - \chi(Z)$$

This is a triviality if one assumes, as we do, that one has a simplicial complex N that is the union of two subcomplexes K and L, with intersection M, since then

$$\#i\text{-simplices in N} = \#i\text{-simplices in K} + \#i\text{-simplices in L} - \#i\text{-simplices in M}$$

7.5. **Euler characteristic of a fixed point set.**

Theorem 7.2. *Let G be the cyclic group of prime order p and let K be a finite regular G-simplicial complex. Then*

$$\chi(K) = p\chi(K/G) - (p-1)\chi(K^G)$$

The proof is a simple counting argument, that shows that

$$\dim C_i(K) = p \dim C_i(K/G) - (p-1) \dim C_i(K^G)$$

for each i: Over each simplex of K/G that does not come from a fixed simplex there are exactly p simplices in K. But there is only one simplex over a simplex corresponding to a fixed simplex in K.

For more complicated groups there would in general be more possibilities for terms in such a formula. Reducing $\mod p$ yields a simple and memorable formula, valid for all p-groups.

Corollary 7.3. *If G is a finite p-group and K is a regular G-simplicial complex, then*

$$\chi(K^G) \equiv \chi(K) \mod p$$

Proof. If $G = C_p$, then this follows immediately from the theorem. therefore we proceed by induction on $|G|$, the order of the group. Assume G is a p-group of order greater than p and that the result is true for p-groups of smaller order. By elementary finite group theory there is a normal subgroup $H < G$ with $G/H \approx C_p$. By induction $\chi(K^H) \equiv \chi(K) \mod p$. Now G/H acts on K^H with $(K^H)^{G/H} = K^G$. Therefore $\chi(K^G) \equiv \chi(K^H) \mod p$ and we are done. □

When we study the Smith Theory of p-group actions we will generalize this to the case of noncompact spaces with finitely generated homology, for which the Euler characteristic is therefore defined. We will show that the Euler characteristic of the fixed point set is well-defined with $\mathbb{Z}_p$ coefficients in such a situation and that a similar formula holds in that case as well.

8. Lefschetz Fixed Point Formula

Here we give a different calculation of the Euler characteristic of a fixed point set of a cyclic action.

8.1. **Traces and Lefschetz Numbers.** If $f : V \to V$ is a homomorphism of a finitely generated module over a field or domain, then one may describe f by a matrix with respect to a basis (of V/torsion) and then define trace f to be the sum of the diagonal entries in the matrix. It is a standard fact of linear algebra that trace f is independent of the choice of basis.

If $f : C_* \to C_*$ is a chain homomorphism of a finitely generated free chain complex, then f induces homomorphisms $Z(f) : Z_*(f) \to Z_*(f)$, $B(f) : B_*(f) \to B_*(f)$, and $H(f) : H_*(f) \to H_*(f)$. Easy examples show that these traces are *not* homotopy invariants. It turns out, however, that suitable alternating sums of traces are homotopy invariants.

Theorem 8.1 (Homological invariance)**.** *If $f = \{f_i\} : C_* \to C_*$ is a chain self-map of finitely generated, free chain complex, then $\sum_i (-1)^i \text{trace } f_i = \sum_i (-1)^i \text{trace } H(f_i)$*

The proof follows the lines of the proof of homological invariance of the Euler characteristic. By choosing suitable bases one shows that

$$\text{trace } C(f)_i = \text{trace } Z(f)_i + \text{trace } B(f)_{i-1}$$

and

$$\text{trace } Z(f)_i = \text{trace } H(f)_i + \text{trace } B(f)_i$$

so that

$$\text{trace } C(f)_i = \text{trace } H(f)_i + \text{trace } B(f)_i + \text{trace } B(f)_{i-1}$$

Once again the extra terms cancel out in the required alternating sum.

Suppose X is a compact polyhedron, or other reasonably nice compact space. If $f : X \to X$, define the *Lefschetz number* of f to be

$$\Lambda(f) = \sum_i (-1)^i \text{ trace}[f_{i*} : H_i(X; \mathbb{Q})) \to H_i(X; \mathbb{Q})]$$

More generally for any chain map $f : C_* \to C_*$ (of a finitely generated, free chain complex) we can define its Lefschetz number in the same way as the alternating sum of its traces.

Theorem 8.2 (Sum Theorem)**.** *If a polyhedron W is the union of two subpolyhedra X and Y, with polyhedral intersection Z, and $f : W \to W$, with $f(X) \subset X$, $f(Y) \subset Y$, and $f(Z) \subset Z$, then*

$$\Lambda(f) = \Lambda(f|X) + \Lambda(f|Y) - \Lambda(f|Z)$$

Assuming that W is the underlying space of a finite simplicial complex K, with X, Y, Z corresponding to subcomplexes L, M, P, then it there is a short exact sequence of simplicial chain complexes of the form

$$0 \to C_*(P) \to C_*(L) \oplus C_*(M) \to C_*(K) \to 0$$

It follows that

$$\text{trace } C_i(f|L) + \text{trace } C_i(f|M) = \text{trace } C_i(f|P) + \text{trace } C_i(f|K)$$

The result follows by taking alternating sums of these equalities.

8.2. **The Fixed Point Formula.** The grandmother of all fixed point theorems is that due to Brouwer.

Theorem 8.3 (Brouwer Fixed Point Theorem)**.** *Let $f : D^n \to D^n$. Then there is a point $x \in D^n$ such that $f(x) = x$.*

It follows that a cyclic group action on a disk has a nonempty fixed point set.

Question 7. *Does* any *finite group action on a disk have a nonempty fixed point set?*

The BFPT has gone through many generalizations in many different directions. One of the best generalizations is due to Lefschetz.

Theorem 8.4. *If $f : X \to X$ is a map of a compact polyhedron to itself without fixed points, then $\Lambda(f) = 0$. Alternatively, if $\Lambda(f) \neq 0$, then f has a fixed point.*

For a full proof (which depends upon subdivision and simplicial approximation) see Spanier. In the case of a simplicial map $f : K \to K$ that maps no simplex to itself (such as arises from a free, regular C_n-simplicial complex), the proof is clear. Such a map induces a chain map $C_*(K) \to C_*(K)$ that has trace zero in each degree. It follows that $\Lambda(C_*(f)) = 0$, hence that $\Lambda(H_*(f)) = 0$.

In the case of periodic maps there is the following stronger statement.

Theorem 8.5 (LFPT)**.** *Let $g : X \to X$ be a piecewise linear periodic map on a compact polyhedron with fixed point set F. Then $\Lambda(g) = \chi(F)$.*

Addendum: $\sum_{g\in G} \Lambda(g) = |G|\chi(K/G)$, which follows from elementary representation theory. If V is a finite dimensional, linear G-space, then $\sum_{g\in G} \operatorname{trace}(g) = |G| \dim V^G$.

If the map is not PL (i.e., simplicial in some triangulation), or some related hypothesis, then the theorem becomes false. We will eventually construct a periodic map of a high-dimensional sphere with exactly one fixed point.

To reduce to the more familiar version, what we need is a nice compact g-invariant polyhedral neighborhood N of the fixed point set F. Set $Y = \overline{X - N}$, and $Z = Y \cap N$. Then

$$\begin{aligned} \Lambda(g) &= \Lambda(g|N) + \Lambda(g|Y) - \Lambda(g|Z) \\ &= \Lambda(g|F) + 0 - 0 \\ &= \chi(F) \end{aligned}$$

Exercise 2. Compute the Lefschetz numbers of the five standard involutions on 2-tori and compare with the Euler characteristic of the fixed point set in each case.

9. Transfer

Suppose that a finite group G acts on a space X with orbit space X/G and orbit map $\pi : X \to X/G$. We seek to understand $H_*(X/G)$, given $H_*(X)$ with its induced action of G. We succeed nicely in the case of rational coefficients. Dealing with torsion of order dividing $|G|$, however, is much more delicate.

Theorem 9.1. *There is a homomorphism (of groups) $\tau_* : H_k(X/G) \to H_k(X)$, such that $\pi_*\tau_* = |G|Id$ and $\tau_*\pi_* = \sum_{g\in G} g_*$.*

The transfer τ_* does not come from a map of spaces, but rather comes from a homomorphism $\tau : C_*(X/G) \to C_*(X)$ of chain complexes. It is nonetheless a natural transformation. The transfer theorem remains valid for any choice of (ordinary, untwisted) coefficients.

Here is the idea in the simple case of a covering map, that is, the case when action of G on X is free. Then the transfer on the chain level is intuitively clear: to any k-simplex σ in X/G assign the sum of the k-simplices in X over σ. The required identities clearly hold at the chain level. One verifies that this τ is in fact a chain map, commuting with boundary operators. It then determines a homomorphism on homology and a fortiori satisfies the required identities. We note that the transfer exists for any finite covering projection, regular or not. Of course in general one then only has the relation $\pi_*\tau_* = |G|Id$.

In the case of rational coefficients the situation is particularly simple. The equation $\pi_*\tau_* = |G|Id$ shows that π_* is surjective. Then the second equation shows that the image of τ_* is fixed, i.e., in $H_*(X;\mathbb{Q})^G$. Then the first equation shows that π_* is surjective when restricted to $H_*(X;\mathbb{Q})^G$. But on $H_*(X;\mathbb{Q})^G$, $\sum_{g\in G} g_*$ is the same as $|G|Id$. We therefore have the following result.

Corollary 9.2. *The orbit map induces an isomorphism $H_*(X;\mathbb{Q})^G \to H_*(X/G;\mathbb{Q})$.*

Question 8. *If G acts freely on a closed oriented n-manifold M, then $H_n(M) = \mathbb{Z} = H_n(M/G)$. Find different examples of what π_* and τ_* can be as homomorphisms from the integers to the integers.*

There is a *relative* version for relative homology, where one has a G-space X and invariant subspace Y and considers relative homology $H_*(X,Y)$.

There is a similar cohomology transfer. The cohomology transfer is not in general a ring homomorphism.

There also is a more subtle transfer for actions of compact Lie groups in general.

Proof in the simplicial case. We now give a proof of the existence of the transfer under the simplifying assumption that the action is PL, i.e, simplicial with respect to some triangulation, which, by subdivision may be assumed to be regular. Therefore, suppose now that K is a regular G-simplicial complex, with well-defined orbist space simplicial complex K/G. Then the simplicial chain groups $C_*(K)$ form a chain complex of modules over the integral group ring $\mathbb{Z}[G]$.

We need to understand the homomorphism

$$\pi_* : C_*(K) \to C_*(K/G)$$

induced by the orbit map. Note that π_* is onto, since by the definition of regular G-simplicial complex, every simplex in the orbit space comes from a simplex in K.

Consider the element $\sigma = \sum_{g\in G} g \in \mathbb{Z}[G]$ and its action on the chain groups $C_*(K)$

$$\sigma_* : C_*(K) \to C_*(K)$$

On simplices we will define $\tau_*(s) = \sigma s_1$, where $\pi s_1 = s$. But there are details to check.

Lemma 9.3. $\ker \sigma_* = \ker \pi_*$.

To see this, we must show that for any chain $c \in C_q(K)$, $\sigma_*(c) = 0$ if and only if $\pi_*(c) = 0$. Now a typical chain has the form $\sum n_i s_i$, where the s_i are q-simplices of K. For checking when π_* or σ_* vanishes it suffices to assume the simplices $s_1, \dots, s_n$ form a single G-orbit of q-simplices (although some of the coefficients n_i might be 0. In particular we may assume all s_i map to the same q-simplex $s \in K/G$. Then $\pi_* \sum n_i s_i = \sum n_i s$. It follows that $\pi_* \sum n_i s_i = 0$ if and only if $\sum n_i = 0$.

On the other hand $\sigma_* c = \sum_{g\in G} g_* \sum_{i=1}^n n_i s_i = \sum_{i=1}^n n_i \sum_{g\in G} g_* s_i = \sum_{i=1}^n n_i \sigma_* s_i = \sum_{i=1}^n n_i \sigma_* s_1 = (\sum_{i=1}^n n_i)\sigma_* s_1 = (\sum_{i=1}^n n_i)\frac{|G|}{n}(s_1 + \cdots + s_n)$.

It follows that $\sigma(\sum n_i s_i) = 0$ if and only if $\sum n_i = 0$, as required. We note that this argument would be valid working over any commutative ring other than $\mathbb{Z}$ for which $\frac{|G|}{n}$ is not a zero divisor.

Note that the image of σ_*, $\sigma C_*(K)$ is a G-invariant subcomplex of $C_*(K)$.

Corollary 9.4. *The orbit map induces an isomorphism of chain complexes*

$$\sigma C_*(K) \to C_*(K/G)$$

The inverse of this isomorphism, followed by inclusion, defines the transfer on the chain level.

We will check that τ then satisfies the required identities on the chain level, hence also on the homology level. An equivalent way of describing τ is the following. If $s \in K/G$, then there are simplices $s_1, \ldots, s_n$ in K that map to s. Then $\tau(s) = \frac{|G|}{n}(s_1 + \cdots + s_n)$. Then it is clear that $\pi_* \tau(s) = \frac{|G|}{n}(ns) = |G|s$.

On the other hand, suppose $s_1 \in K$ with $\pi(s_1) = s \in K/G$. Let $s_1, \ldots, s_n$ be the full orbit of s_1 under the action of G. The $\tau(s) = \frac{|G|}{n}(s_1 + \cdots + s_n) = \sum_{g \in G} s_1$, as required.

$\square$

10. Smith theory

We seek deeper relationships between the homology of a G-space and the homology of its fixed point set and its orbit space. We follow Bredon [4], chapter 3, in part.

10.1. Main Conclusions.

Here we briefly state some of the main consequences of the theory to be developed.

Theorem 10.1. *If a p-group acts on a $\mathbb{Z}_p$-acyclic space X (i.e. $H_*(X;\mathbb{Z}_p) = H_*(pt;\mathbb{Z}_p)$), then X^G is also $\mathbb{Z}_p$-acyclic.*

More generally,

Theorem 10.2. *If a p-group acts on a "finitistic" space X with finitely generated $\mathbb{Z}_p$ homology, then*

$$\sum_{i\geq n} \dim H_i(X^G;\mathbb{Z}_p) \leq \sum_{i\geq n} \dim H_i(X;\mathbb{Z}_p)$$

and

Theorem 10.3. *If a p-group G acts on a "finitistic" space X with $\chi(X;\mathbb{Z}_p)$ defined, then $\chi(X^G;\mathbb{Z}_p)$ is defined and $\chi(X^G;\mathbb{Z}_p) \equiv \chi(X;\mathbb{Z}_p) \mod p$.*

In particular, the preceding result eliminates the finiteness (compactness) assumption when we previously derived a similar Euler characteristic formul by more elementary means.

As a consequence we have the following.

Theorem 10.4. *If a p-group G acts on a space X with $H_*(X;\mathbb{Z}_p) \approx H_*(S^n;\mathbb{Z}_p)$, then $H_*(X^G;\mathbb{Z}_p) \approx H_*(S^k;\mathbb{Z}_p)$ for some $k \leq n$. Moreover, if p is odd, then $n-k$ is even.*

Some minimal hypotheses are necessary on the spaces and/or the action. In very general cases one must use Cech homology, or something of that sort. We will prove the results in the case of G-simplicial complexes.

10.2. Smith homology groups (simplicial case).

Unless stated otherwise, we let $G = C_p = \langle g \rangle$, a cyclic group of prime order p, with a chosen generator g. (More general p-groups will be handled later by induction on order of the group.) Let K denote a regular G-simplical complex. We seek relations among the homology groups $H_*(K)$, $H_*(K/G)$, and $H_*(K^G)$. Here and throughout this section we understand all homology as well as chain complexes to be using $\mathbb{Z}_p$ coefficients. Some parts of what follows have suitable analogues for $\mathbb{Z}$ coefficients, but the crucial parts use $\mathbb{Z}_p$ in an unavoidable way. It can be an interesting exercise to find those parts that do have integral analogues.

Consider the simplicial chain groups, $C_*(K)$, $C_*(K^G)$, and $C_*(K/G)$. These consist of free $\mathbb{Z}_p$-modules, with bases in one-to-one correspondence with the relevant simplices. But $C_*(K)$ also has an action of G, and thus is a module over the group ring $\mathbb{Z}_p[G]$. As such in each dimension it is a direct sum of modules of two possible types

- $\mathbb{Z}_p$, corresponding to a fixed simplex, and
- $\mathbb{Z}_p[G]$, corresponding to a free orbit of simplices.

In the group ring $\mathbb{Z}_p[G]$ there are two important elements that will play a special role.

$$\sigma = 1 + g + \cdots + g^{p-1}, \text{ the sum element}$$

and

$$\delta = 1 - g, \text{ the difference element}$$

More generally we will consider the elements δ^j, $j = 1, \ldots, p-1$.

We may view σ and δ as homomorphisms $\mathbb{Z}_p[G] \to \mathbb{Z}_p[G]$ (by multiplication).

Lemma 10.5. $\ker \delta = \operatorname{im} \sigma$ *and* $\ker \sigma = \operatorname{im} \delta$*.*

The preceding lemma actually holds with coefficients in $\mathbb{Z}$ rather than in $\mathbb{Z}_p$, with the same proof in this case.

Lemma 10.6. *In* $\mathbb{Z}_p[G]$ *we have* $\delta^p = 0$ *and* $\delta^{p-1} = \sigma$*.*

Proof. Expand the powers of $\delta = 1 - g$ using the binomial formula, noting that

$$\binom{p}{j} \equiv 0 \mod p, 0 \le j \le p-1$$

and

$$\binom{p-1}{j} \equiv (-1)^j \mod p, 1 \le j \le p-1$$

□

Corollary 10.7. *For* $j = 1, \ldots, p-1$*, the sequence*

$$0 \to \sigma\mathbb{Z}_p[G] \xrightarrow{i} \delta^{j-1}\mathbb{Z}_p[G] \xrightarrow{\delta} \delta^j\mathbb{Z}_p[G] \to 0$$

in which i denotes inclusion is exact.

The key observation is that since $\sigma = \delta^{p-1} = \delta^{j-1}\delta^{p-j}$, $\operatorname{im} \sigma \subseteq \operatorname{im} \delta^{j-1}$.

Now $\sigma\mathbb{Z}_p[G]$ is generated over $\mathbb{Z}_p$ by the single element σ, since $\sigma \sum a_i g^i = (\sum a_i)\,\sigma$. Thus $\dim_{\mathbb{Z}_p} \sigma\mathbb{Z}_p[G] = 1$. Since $\dim_{\mathbb{Z}_p} \mathbb{Z}_p[G] = p$, it follows that $\dim_{\mathbb{Z}_p} \delta\mathbb{Z}_p[G] = p-1$, and, by induction, that $\dim_{\mathbb{Z}_p} \delta^j\mathbb{Z}_p[G] = p - j$. Then the following result follows by dimension considerations, since we are working over the field $\mathbb{Z}_p$.

Corollary 10.8. *For* $j = 1, \ldots, p-1$*, the sequence*

$$0 \to \delta^{p-j}\mathbb{Z}_p[G] \xrightarrow{i} \mathbb{Z}_p[G] \xrightarrow{\delta^j} \delta^j\mathbb{Z}_p[G] \to 0$$

is exact.

Now consider the chain groups $C_*(K) = C_*(K; \mathbb{Z}_p)$. Each element $\rho = \delta^j$ determines a subchaincomplex (over the group ring $\mathbb{Z}_p[G]$)

$$\rho C_*(K) \subseteq C_*(K)$$

Indeed we obtain a filtration of $C_*(K)$

$$C_*(K) \supset \delta C_*(K) \supset \delta^2 C_*(K) \supset \cdots \supset \delta^{p-1} C_*(K) = \sigma C_*(K)$$

Such a filtered chain complex cries out to be turned into a spectral sequence. But for now we'll avoid that by following a more direct, hands-on approach.

Definition 1 (Smith Special Homology Groups)**.** $H_*^\rho(K) := H_*(\rho C_*(K))$

One of these special homology groups has a concrete interpretation in terms of more familiar objects. For simplicity of notation identify K^G with its isomorphic copy K^G/G in K/G.

Proposition 10.9. $H_*^\sigma(K) \approx H_*(K/G, K^G)$.

Proof. The chain map $\pi_* C_*(K) \to C_*(K/G)$ induces a chain isomorphism

$$\sigma C_*(K) \to C_*(K/G, K^G) = C_*(K/G)/C_*(K^G)$$

and the result follows upon passage to homology groups. □

The other special homology groups are useful mainly for interpolating among $H_*(K)$, $H_*(K^G)$, and $H_*(K/G)$. Toward that end we have the following result. Here $\rho = \delta^j$ and $\overline{\rho} = \delta^{p-j}$.

Theorem 10.10. *For each $\rho = \delta^j$, $j = 1, \dots, p-1$,*

$$0 \to \overline{\rho} C(K) \oplus C(K^G) \xrightarrow{i} C(K) \xrightarrow{\rho} \rho C(K) \to 0$$

is an exact sequence of chain complexes of $\mathbb{Z}_p$-vector spaces and of $\mathbb{Z}_p[G]$-modules.

Proof. Each of the chain complexes splits into the portions arising from orbits of single simplices of K or K^G. Note that $\overline{\rho}C(K)$ involves no fixed simplices. It follows that i is injective. In addition we note that ρ is certainly onto and $\rho i = 0$. It remains to prove exactness at $C(K)$, in particular that $\ker \rho \subset \operatorname{im} i$.

Now there are two kinds of orbits of simplices: $G(s) = \{s\}$ ($s \in K^G$) and $G(s) = \{s_1, \dots, s_p\}$, where $s_1 = s$ and $g(s_i) = s_{i+1}$ (where subscripts are understood mod p).

In the case $G(s) = \{s\}$, the sequence in question reduces to

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to 0 \to 0$$

which is clearly exact. In the case $G(s) = \{s_1, \dots, s_p\}$ the sequence becomes

$$0 \to \overline{\rho}\mathbb{Z}_p[G] \to \mathbb{Z}_p[G] \to \rho\mathbb{Z}_p[G] \to 0$$

and this we proved is exact above in Corollary 10.8. □

As a consequence of the fact that any short exact sequence of chain complexes determines a long exact sequence of homology groups, we obtain long exact sequences involving special Smith groups.

There are also short exact sequences

$$0 \to \sigma C(K) \to \delta^j C(K) \to \delta^{j+1} C(K) \to 0$$

yielding long exact sequences

$$\cdots H_n^{\delta^j}(K) \to H_n^{\delta^{j+1}}(K) \to H_n^\sigma(K) \to H_{n-1}^{\delta^j}(K) \to \cdots$$

We also mention a few other remarks about this construction:

- The Smith groups and the Smith sequences are natural in the sense that simplicial G-maps induce homomorphisms of the Smith groups and of the Smith exact sequences.
- Moreover, the homomorphisms induced by "contiguous" G-maps are equal.
- There is an analogous construction of Smith cohomology groups with similar properties.

10.3. **Ranks.** Here we prove the fundamental fact that the total dimension of the homology of the fixed point set is bounded above by the total dimension of the homology of the space.

Theorem 10.11. *If $G = C_p$ is a cyclic group of prime order p and K is a finite-dimensional, regular, G-simplicial complex, then*
$$\sum_{i\geq n} \dim H_i(K^G;\mathbb{Z}_p) \leq \sum_{i\geq n} \dim H_i(K;\mathbb{Z}_p)$$

Proof. We consider the dimensions (over $\mathbb{Z}_p$) of the groups in the long exact sequence
$$\cdots \to H^{\rho}_{i+1}(K;\mathbb{Z}_p) \to H^{\overline{\rho}}_i(K;\mathbb{Z}_p) \oplus H_i(K^G;\mathbb{Z}_p) \to H_i(K;\mathbb{Z}_p) \to H^{\rho}_i(K;\mathbb{Z}_p) \to \cdots$$
Let $b_i = \dim H_i(K;\mathbb{Z}_p)$, let $f_i = \dim H_i(K;\mathbb{Z}_p)$, and let $r_i = \dim H^{\rho}_i(K;\mathbb{Z}_p)$ and $\overline{r}_i = \dim H^{\overline{\rho}}_i(K;\mathbb{Z}_p)$. Then the exact sequence implies that
$$\begin{gathered}\overline{r}_n + f_n \leq r_{n+1} + b_n\\ \overline{r}_{n+1} + f_{n+1} \leq r_{n+2} + b_{n+1}\\ \overline{r}_{n+2} + f_{n+2} \leq r_{n+3} + b_{n+2}\\ \overline{r}_{n+3} + f_{n+3} \leq r_{n+4} + b_{n+3}\\ \text{etc.}\end{gathered}$$
We would like to add up these inqualities. But we must arrange more cancellation. We can reverse the roles of ρ and $\overline{\rho}$ in the long exact sequence and still have a long exact sequence. We therefore reverse the roles of ρ and $\overline{\rho}$ in every other one of these inequalities, yielding
$$\begin{gathered}\overline{r}_n + f_n \leq r_{n+1} + b_n\\ r_{n+1} + f_{n+1} \leq \overline{r}_{n+2} + b_{n+1}\\ \overline{r}_{n+2} + f_{n+2} \leq r_{n+3} + b_{n+2}\\ r_{n+3} + f_{n+3} \leq \overline{r}_{n+4} + b_{n+3}\\ \text{etc.}\end{gathered}$$
Only finitely many of these inqualities are nonzero. Therefore, adding them up we obtain
$$\overline{r}_n + \sum_{i\geq n} f_i \leq \sum_{i\geq n} b_i$$
Again we may reverse the roles of ρ and $\overline{\rho}$, so that we obtain
$$r_n + \sum_{i\geq n} f_i \leq \sum_{i\geq n} b_i$$
which yields the required inequality. $\square$

Inspection of the end of the proof shows that we actually obtain the inequality
$$\dim_{\mathbb{Z}_p} H^{\rho}_n(K;\mathbb{Z}_p) + \sum_{i\geq n} \dim H_i(K^G;\mathbb{Z}_p) \leq \sum_{i\geq n} \dim H_i(K;\mathbb{Z}_p)$$
and the case where $\rho = \sigma = \delta^{p-1}$ yields
$$\dim_{\mathbb{Z}_p} H_n(K/G, K^G;\mathbb{Z}_p) + \sum_{i\geq n} \dim H_i(K^G;\mathbb{Z}_p) \leq \sum_{i\geq n} \dim H_i(K;\mathbb{Z}_p)$$

Theorem 10.12. *If a p-group G acts on a finite-dimensional G-simplicial complex K, then*
$$\sum_{i\geq n} \dim H_i(K^G;\mathbb{Z}_p) \leq \sum_{i\geq n} \dim H_i(K;\mathbb{Z}_p)$$

Proof. For $G = C_p$, this follows immediately from the preceding theorem. For more general groups we proceed inductively. Assume H is a proper normal subgroup of G and that the result is true for p-groups of order less than $|G|$. Then G/H acts on K^H with fixed point set $\left(K^H\right)^{G/H} = K^G$. The result follows. ☐

10.4. **Euler characteristics.** Using Smith homology groups we can extend our Euler characteristic formula, derived by completely elementary means for finite complexes, to the case of finite-dimensional, but not necessarily finite, complexes.

Theorem 10.13. *If $G = C_p$ is a cyclic group of prime order p and K is a finite-dimensional, regular, G-simplicial complex, for which $\chi(K;\mathbb{Z}_p)$ is defined, then $\chi(K^G;\mathbb{Z}_p)$ and $\chi(K/G;\mathbb{Z}_p)$ are defined, and*

$$\chi(K) = p\chi(K/G;\mathbb{Z}_p) - (p-1)\chi(K^G;\mathbb{Z}_p).$$

Proof. The preceding inequality shows that $\chi(K^G;\mathbb{Z}_p)$ and $\chi(K/G;\mathbb{Z}_p)$ are defined. Standard facts about Euler characteristics and long exact sequences, setting $\rho = \delta$, show that

$$\chi(K;\mathbb{Z}_p) = \chi(K^G;\mathbb{Z}_p) + \chi(H_*^\sigma) + \chi(H_*^\delta)$$

The long exact sequences

$$\cdots H_n^{\delta j}(K) \to H_n^{\delta j+1}(K) \to H_n^\sigma(K) \to H_{n-1}^{\delta j}(K) \to \cdots$$

imply (using the obvious abbreviation $\chi(\delta^j) = \chi(H_*^{\delta j}(K))$ that

$$\begin{aligned}
\chi(\delta) &= \chi(\delta^2) + \chi(\sigma)\\
\chi(\delta^2) &= \chi(\delta^3) + \chi(\sigma)\\
&\vdots\\
\chi(\delta^{p-2}) &= \chi(\delta^{p-1}) + \chi(\sigma)\\
\chi(\delta^{p-1}) &= \chi(\sigma)
\end{aligned}$$

Adding up all of these equalitties and canceling yields

$$\chi(K) = p\chi(\sigma) + \chi(K^G)$$

The long exact sequence of the pair $(K/G, K^G)$ shows that $\chi(K/G) = \chi(K^G) + \chi(K/G, K^G)$, and Proposition 10.9 shows that $\chi(H_*^\sigma(K)) = \chi(K/G, K^G) = \chi(K/G) - \chi(K^G)$. The result follows. ☐

Theorem 10.14. *If a p-group G acts on a finite-dimensional G-simplicial complex K forwhich $\chi(K)$ is defined, then $\chi(K^G)$ is defined, and $\chi(K^G) \equiv \chi(K) \mod p$.*

Proof. For $G = C_p$, this follows immediately from the preceding theorem. For more general groups we proceed inductively. Assume H is a proper normal subgroup of G and that the result is true for p-groups of order less than $|G|$. Then G/H acts on K^H with fixed point set $\left(K^H\right)^{G/H} = K^G$. The result follows. ☐

10.5. **Actions on disks and spheres.**

Theorem 10.15. *Let $G = C_p$ be a p-group, p prime, acting simplicially and regularly on a finite $\mathbb{Z}/p$-acyclic simplicial complex K. Then the fixed point set K^G is also $\mathbb{Z}/p$-acyclic. In particular $K^G \neq \emptyset$.*

Proof. We know
$$\dim H_*(K^G) \le \dim H_*(K) = 1$$
and
$$\chi(K^G) \equiv \chi(K) = 1 \mod p$$
The first condition says that either K^G is empty or has the homology of a point. The second condition guarantees that $K^G \neq \emptyset$. □

Theorem 10.16. *Let $G = C_p$ be a p-group, p prime, acting simplicially and regularly on a finite simplicial complex K with the $\mathbb{Z}/p$-homology of an n-sphere. Then the fixed point set X^G has the $\mathbb{Z}/p$-homology of a k-sphere, for some $k \le n$. If p is odd, then $n - k$ is even.*

Proof. We know
$$\dim H_*(K^G) \le \dim H_*(K) = 2$$
and
$$\chi(K^G) \equiv \chi(K) = 1 + (-1)^n \mod p$$
Thus K^G has $\mathbb{Z}_p$ homology of rank 1 in degrees 0 and $k \le n$ (It is possible that $k = 0$, in which case we understand that $H_0(K^G) = \mathbb{Z}_p^2$. It is also possible that $k = -1$, in which case we understand that $K^G = \emptyset$, which requires that $\chi(K) = 0 \mod p$.) In any case K^G is a $\mathbb{Z}_p$-homology k-sphere, $k \le n$, and $1 + (-1)^k \equiv 1 + (-1)^n \mod p$. Thus if p is odd, then k and n have the same parity, as required. □

Theorem 10.17. *Let $G = C_p$ be a cyclic group of prime order p acting freely and simplicially on a finite dimensional, regular G-simplicial complex K with the $\mathbb{Z}/p$-homology of an n-sphere. Then*
$$H_i(K/G;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p, & 0 \le i \le n \\ 0, & \text{otherwise} \end{cases}$$

Proof. This generalizes the familiar computation of the mod p homology of a lens space. In that case one often proceeds by finding an explicit cell structure and analyzing the boundary (attaching) maps. In this case we try not to make such explicit assumptions about the cell structure.

If $n = 0$, then K consists of two $\mathbb{Z}_p$-acyclic components, which must be interchanged by G. We conclude that $p = 2$ and that the orbit space can be identified with one of the two acyclic components. The result follows. Henceforth we assume $n > 0$.

Then our basic inequality (with $\rho = \sigma$) shows that for $m \ge 0$
$$\dim H_m^\sigma(K) = \dim H_m(K/G) \le \sum_{i \ge m} \dim H_i(K;\mathbb{Z}_p)$$
Thus $H_0(K/G) = \mathbb{Z}_p$ (since K/G is connected), $H_i(K/G) \subset \mathbb{Z}_p$ ($i = 1, \dots, n$), and $H_i(K/G) = 0$ ($i > n$).

Since $K^G = \emptyset$, we have the following two short exact sequences of chain groups.
$$0 \to \sigma C \to C \to \delta C \to 0$$
and
$$0 \to \delta C \to C \to \sigma C \to 0$$
where $C = C_*(K;\mathbb{Z}_p)$. Consider the long exact sequence
$$0 \to H_n^\sigma \to H_n \to H_n^\delta \to H_{n-1}^\sigma \to H_{n-1} \to \cdots$$
Now $H_n \to H_n^\delta$ is induced by multiplication by $\delta = 1 - g$, and g acts trivially on $H_n = \mathbb{Z}_p$ (since aut $\mathbb{Z}_p \approx \mathbb{Z}_{p-1}$), hence $H_n \to H_n^\delta$ is the zero homomorphism.

Since we know $H_n^\sigma \subset \mathbb{Z}_p$, it follows that $H_n^\sigma \approx H_n = \mathbb{Z}_p$, and that $H_q^\delta \approx H_{q-1}^\sigma$, for $q = n, n-1, \ldots, 1$. In particular $H_q^\delta \subset \mathbb{Z}_p$.

We also have the long exact sequence

$$0 \to H_n^\delta \to H_n \to H_n^\sigma \to H_{n-1}^\delta \to H_{n-1} \to \cdots$$

Now $H_n \to H_n^\sigma$ is induced by multiplication by $\sigma = 1 + g + g^2 + \cdots + g^{p-1}$, and g acts trivially on $H_n = \mathbb{Z}_p$, hence $H_n \to H_n^\sigma$ is the zero homomorphism (multiplication by p). It follows that $H_q^\sigma \approx H_{q-1}^\delta$, for $q = n, n-1, \ldots, 1$. Moreover, $H_n^\sigma \approx H_n$.

Alternately combining these two sets of equalities (with their shifts in opposite directions) yields

$$H_n^\sigma \approx H_{n-1}^\delta \approx H_{n-2}^\sigma \approx H_{n-3}^\delta \cdots \quad \text{and} \quad H_n^\delta \approx H_{n-1}^\sigma \approx H_{n-2}^\delta \approx H_{n-3}^\sigma \cdots$$

But at the top we have

$$H_n^\sigma \approx H_n \approx H_n^\delta$$

So all these groups are isomorphic to $\mathbb{Z}_p$, as needed. □

A somewhat similar proof yields the following extension in which the model is a $(k+1)$-fold suspension of a free action on S^{n-k-1}, the computation yields what would correspond to the $(k+1)$-fold suspension of an $(n-k-1)$-dimensional lens space

Theorem 10.18. *Let $G = C_p$ be a cyclic group of prime order p acting simplicially on a finite dimensional, regular G-simplicial complex K with the $\mathbb{Z}/p$-homology of an n-sphere, and with K^G having the $\mathbb{Z}/p$-homology of a k-sphere, $k \leq n$. Then*

$$H_i(K/G; \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p, & i = 0, \text{ or } k+1 \leq i \leq n \\ 0, & \text{otherwise} \end{cases}$$

10.6. **Comments on extensions to more general spaces.** The question here is one of how important it is that we used finite dimensional simplicial complexes with simplicial maps. Certainly not all interesting and important group actions are piecewise linear, although finite smooth actions on smooth manifolds are equivalent to PL actions. Even if the space is polyhedral, the fixed point sets and/or orbits spaces might not be polyhedral. One would like a theory that would be strong enough to say that an arbitrary action of C_p on the a sphere, or even on a space with the homology of a sphere, has fixed point set with the mod p homology of a sphere, for example. It turns out that for more general spaces and general group actions a less familiar homology theory than simplicial or singular theory is required.

One needs a sense in which arbitrary spaces are "approximated by simplicial complexes" and a homology theory that "commutes with limits of spaces." Of course any such theory ought to agree with the simplicial theory when applied to simplicial actions on simplicial complexes.

We may return to these issues later. . . or we may not.

10.7. **Prime power order is necessary.** In general the hypotheses of prime power order and finite-dimensionality are necessary for Smith theory, and a compact polyhedron the Lefschetz fixed point theorem.

Let $G = C_{pq}$, a cyclic group of order pq where p and q are distinct primes (or more generally relatively prime).

We construct a contractible (infinite, locally finite, 4-dimensional) complex on which G acts without fixed points. This construction also leads to such actions on $\mathbb{R}^n$ for n sufficiently large, like $n \geq 8$. See Bredon [4], chapter 1, section 8, for more information.

We will at the same time construct an action of G without fixed points on a compact, 3-dimensional acyclic space.

Both constructions start with a certain standard linear action on S^3 and a not-so-standard G-map of degree 0.

View S^3 as the unit sphere in $\mathbb{C}^2$ and let a generator $T \in G$ act on S^3 by $T(z, w) = (\omega_p z, \omega_q w)$, where in general $\omega_r = \exp(2\pi i/r)$.

Proposition 10.19. *There is a G-map $h : S^3 \to S^3$ of degree* 0.

Let X denote the infinite mapping telescope associated with $h : S^3 \to S^3$.

Proposition 10.20. *The space X is the topological realization of a contractible, four-dimensional, locally finite C_{pq}-simplicial complex with empty fixed point set.*

One can in fact replace the complex X by a euclidean space of suitably high dimension, by embedding X equivariantly and piecewise linearly in $\mathbb{R}^n$ and taking the interior of an invariant regular neighborhood, and perhaps crossing with $\mathbb{R}$.

Such actions exist for any group that is not a p-group. One can pass to one-point compactification to obtain one-fixed point actions on spheres. These actions are not nice near the fixed point. It is harder to construct smooth actions on spheres having just one fixed point, or piecewise linear actions on compact disks with no fixed points. These constructions exist, however, for sufficiently complicated groups in sufficiently high dimensions.

Alternatively, let Y denote the inverse limit space associated with $h : S^3 \to S^3$.

Proposition 10.21. *The space Y is a* 3*-dimensional, compact C_{pq}-space with the Cech cohomology of a point, but with empty fixed point set.*

EXERCISES

Feel free to *assume* all spaces with group actions discussed here are triangulable as regular G-simplicial complexes.

Exercise 3. If $\chi(K;\mathbb{Z}_p)$ is defined and finite for a finite-dimensional simplicial complex K, using $\mathbb{Z}_p$ coefficients, then $\chi(K;\mathbb{Q})$ is defined with $\mathbb{Q}$ coefficients, and $\chi(K;\mathbb{Q}) = \chi(K;\mathbb{Z}_p)$. Give an example for which $\chi(K;\mathbb{Q})$ is defined, but $\chi(K;\mathbb{Z}_p)$ is not.

Exercise 4. Let G be a finite group. Suppose K is a finite-dimensional regular G-simplicial complex with $H_*(K;\mathbb{Z})$ finitely generated (i.e. all $H_i(K)$ are f.g. and there is an n such that $H_i(K) = 0$ for $i > n$). Prove that $H_*(K/G;\mathbb{Z})$ is finitely generated.

Exercise 5. If a finite group G acts on $\mathbb{R}P^{2n}$, then the action is *not* free. Find a free action on $\mathbb{R}P^{2n+1}$.

Exercise 6. If a finite group G acts on a regular G-simplicial complex K and induces a trivial action on $H_*(K;R)$, where R is a field of characteristic 0 or prime to $|G|$, then $\chi(K/G) = \chi(K)$, provided $\chi(K)$ is defined.

Exercise 7. Let $G = C_p$ be a finite cyclic group of prime order p. Suppose K is a finite-dimensional regular G-simplicial complex with the *integral* homology of S^n. Suppose also that K^G is a mod p homology r-sphere, with $n - r$ even. Prove that

$$H_i(K/G;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \text{ and } i = n \\ \mathbb{Z}_p & \text{for } i = r+2, r+4, \dots, n-2 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 8. If a nontrivial finite group G acts freely on a finite G-simplicial complex K, inducing the trivial action on $H_*(K;\mathbb{Q})$, then $\chi(K) = 0$. Find an example where $G = C_2$ acts freely on K, but $\chi(K) \neq 0$. Find a similar action with $G = C_p$, p odd.

Exercise 9. Can you find a free C_p action on $\mathbb{C}P^n$?

Exercise 10. Consider the action of C_3 action on $\mathbb{C}P^2$ coming from cyclically permuting the homogeneous coordinates. Compute $\chi(\mathbb{C}P^2/C_3)$.

Exercise 11. Consider the action of C_2 on $\mathbb{C}P^2$ coming from complex conjugation. Compute $\chi(\mathbb{C}P^2/C_2)$.

Exercise 12. Consider an arbitrary action of C_2 on $\mathbb{C}P^2$ acting by (-1) on $H_2 = \mathbb{Z}$. What are the possibilities for the fixed point set?

11. Lifting in coverings

We discuss the process, where possible, of lifting a group action from the base space to the total space of a covering map. These results are especially useful for applications to group actions on spaces covered by spheres or euclidean spaces, where one can lift the group action, apply basic Smith theory in the covering space, and then interpret what that implies about the fixed point set of the original group action on the base. The spaces considered here are assumed to be path connected, locally path connected, and semi-locally one-connected, so that covering space theory applies, and to be paracompact Hausdorff and compact or finitistic, so that Smith theory applies, using appropriate Cech cohomology, as required. For most purposes one may safely assume the spaces are realizations of finite dimensional regular G-simplicial complexes.

Theorem 11.1. *Let a finite group G act on a (nice) space X with a fixed point x_0. Let $pr : \widetilde{X} \to X$ be the universal covering space and let $\tilde{x}_0 \in pr^{-1}(x_0)$. Then there is an action of G on $\widetilde{X}$ such that the action fixes $\tilde{x}_0$ and $pr : \widetilde{X} \to X$ is a G-map.*

Example 3. Consider the action of $G = C_2$ on the circle S^1 by complex conjugation, aka reflection in the equator, with two fixed points. The universal covering is the exponential map $\mathbb{R} \to S^1$. The action lifts to a C_2 action on $\mathbb{R}$ given by reflection in any chosen point lying over either of the two fixed points in $\mathbb{R}$.

Proof. Let $\phi : G \times X \to X$ be the given action. Recall that $\hat{\phi} : G \to \mathrm{Homeo}(X)$ denotes the corresponding homomorphism into the homeomorphism group. Apply the standard lifting criterion of basic covering space theory to define an action $\psi : G \times \widetilde{X} \to \widetilde{X}$ as follows. For $g \in G$, let $\hat{\psi}(g) : \widetilde{X} \to \widetilde{X}$ be the unique lift of $\hat{\phi}(g) : X \to X$ that fixes $\tilde{x}_0$. The identity $\psi(g)\psi(h) = \psi(gh)$ follows from uniqueness of liftings and shows that the lifting defines the required action. Alternatively one can use path lifting to define the action, as follows: For $y \in \widetilde{X}$, choose a path λ from $\tilde{x}_0$ to y. Project λ to μ in X, apply $g \in G$ and lift $g\mu$ to $\widetilde{X}$, starting at $\tilde{x}_0$. Then gy is the other end of this (projected, transformed, and) lifted path. □

Note that the argument does not require that G be finite. More generally we have the following result about lifting group actions to other covering spaces.

Theorem 11.2. *Let a finite group G act on a (nice) space X with a fixed point x_0. Let $pr : Y \to X$ be a covering space and let $y_0 \in pr^{-1}(x_0)$. Then there is a (unique) action of G on Y such that the action fixes y_0 and $pr : Y \to X$ is a G-map if and only if the induced action on $\pi_1(X, x_0)$ preserves the subgroup $pr_*\pi_1(Y, y_0)$.*

We now examine the relationship between the fixed point set of the action downstairs on X and the fixed point set of the action upstairs on Y. If Z is any space with G-action and $z \in Z^G$, we let Z^G_z denote the path component of the fixed point set Z^G containing z.

Theorem 11.3. *Let a finite group G act on a (nice) space X with a fixed point x_0. Let $pr : Y \to X$ be a covering space projection of degree $d \leq \infty$ and let $y_0 \in pr^{-1}(x_0)$. Suppose there is an action of G on Y such that the action fixes y_0 and $pr : Y \to X$ is a G-map. Then the restriction $Y^G_{y_0} \to X^G_{x_0}$ is a covering projection of degree at most d.*

Proof. First observe that $Y^G_{y_0} \to X^G_{x_0}$ is surjective. This follows by consideration of the lifts of a path lying entirely in $X^G_{x_0}$. Now consider of a G-invariant evenly covered neighborhood U of any point $w \in X^G_{x_0}$ (obtained by intersecting the G-translates of any evenly covered neighborhood). Let $pr^{-1}(U) = \cup V_i$, where each V_i maps homeomorphically to U. For any V_i, either $g(V_i) \cap V_i = \emptyset$

for some $g \in G$, whence $V_i \cap Y^G = \emptyset$, or G leaves V_i invariant and pr maps $V_i^G \to U$ homeomorphically. It follows that $Y^G_{y_0} \to X^G_{x_0}$ is a covering projection. The statement about the degree is immediate. □

Theorem 11.4. *Let a finite group G act on a (nice) space X with a fixed point x_0. Let $pr : Y \to X$ be the universal covering space projection and let $y_0 \in pr^{-1}(x_0)$. Consider the action of G on Y such that the action fixes y_0 and $pr : Y \to X$ is a G-map. Then $pr^{-1}(x_0) \subset Y^G$ if and only if G acts trivially on $\pi_1(X, x_0)$.*

Proof. The idea is simply that with the choice of base point y_0 the fiber $pr^{-1}(x_0)$ is identified with $\pi_1(x, x_0)$ (standard) and the action of G on $\pi_1(X, x_0)$ gets identified with the action on the fiber $pr^{-1}(x_0)$. Here are a few more details.

First assume that G acts trivially on $\pi_1(X, x_0)$ and show that $pr^{-1}(x_0) \subset Y^G$.

Let $z \in pr^{-1}(x_0)$, and $g \in G$. Choose a path $\lambda : (I, 0, 1) \to (Y, y_0, z)$. Let $\mu = pr \circ \lambda : (I, 0, 1) \to (X, x_0)$. Then $g(z) = \widetilde{g\mu}(1)$, where putting a tilde over a loop at x_0 indicates the unique lift of the loop to a path starting at y_0. But since $g\mu \simeq \mu$ rel $\{0, 1\}$, homotopy lifting implies that $gz = \widetilde{g\mu}(1) = \widetilde{\mu}(1) = \lambda(1) = z$, as required.

For the converse, we assume that $pr^{-1}(x_0) \subset Y^G$. and show that G acts trivially on $\pi_1(X, x_0)$. Let μ be a loop based at x_0 in X. Then $\widetilde{\mu}(1) = g\widetilde{\mu}(1) = \widetilde{g\mu}(1)$. since $Y \to X$ is the universal covering, we conclude that $g\mu \simeq \mu$ rel $\{0, 1\}$, as needed. □

Example 4. Consider the action of $G = C_2$ on the real projective plane $\mathbb{R}P^2$ given in homogeneous coordinates by

$$T[x, y, z] = [-x, y, z]$$

This involution fixes a point and a circle, which we should think of as $\mathbb{R}P^0 \cup \mathbb{R}P^1$. There are two very different lifts of the action to S^2, one fixing a circle and the other fixing two points. Explicitly we have $T_1(x, y, z) = (-x, y, z)$ or $T_2(x, y, z) = (x, -y, -z)$.

Theorem 11.5. *Let p be a prime and $G = C_p$ act on a lens space L with $\pi_1(L) = \mathbb{Z}_p$ (or a real projective space when $p = 2$). Then each path component of the fixed point set L^G has the $\mathbb{Z}_p$ homology of a lens space with $\pi_1 = \mathbb{Z}_p$ of dimension at most that of L.*

Proof. The universal covering of L is a sphere S, with deck transformation group $\pi \approx C_p$. The action of G on L lifts to an action of G on S, fixing a given point y over any chosen fixed point $x \in L^G$. Then $S^G_y \to L^G_x$ is a covering. If S^G is discrete, then L^G_x is a single point, which is a homology lens space, of course. Otherwise, by regular Smith Theory S^G is a connected $\mathbb{Z}_p$-homology k-sphere and $S^G \to L^G_x$ is the quotient map for the free π-action on S^G. Smith theory applied to the action of π on S^G shows that $S^G/\pi = L^G_x$ has the $\mathbb{Z}_p$ homology of a lens space, with homology groups $H_i(L^G_x; \mathbb{Z}_p) \approx \mathbb{Z}_p$ for $i = 0, \ldots, k$ and 0 otherwise. □

The space L can be replaced with any finitistic space with the mod p homology of a $\mathbb{Z}_p$ lens space, whose $\mathbb{Z}_p$-cover has the $\mathbb{Z}_p$ homology of a sphere.

Theorem 11.6. *Let p be a prime and $G = C_p$ act on the n-torus T^n. Then each path component of the fixed point set ${T^n}^G$ has the $\mathbb{Z}_p$ homology of a torus of dimension at most n.*

Proof. The universal covering of T^n is $\mathbb{R}^n$. Choose a point $x \in {T^n}^G$. and a point $y \in \mathbb{R}^n$ lying over x. Lift the G action to an action on $\mathbb{R}^n$ fixing y. By basic Smith theory the fixed point set ${\mathbb{R}^n}^G$ is

$\mathbb{Z}_p$-acyclic. In particular it is nonempty and path-connected. Then $\mathbb{R}^{nG} \to T^{n}{}_x^G$ is a covering map. The group of deck transformations π is a subgroup of $\mathbb{Z}^n$, hence isomorphic to $\mathbb{Z}^k$ for some $k \leq n$. The situation is that we have a free action of a group $\pi = \mathbb{Z}^k$ on a $\mathbb{Z}_p$-acyclic space $Z = \mathbb{R}^{nG}$. The standard model of that situation is $\mathbb{Z}^k$ acting on $\mathbb{R}^k$ with quotient T^k. By standard algebraic topology (comparison theorem for spectral sequences with $\mathbb{Z}_p$ coefficients) we can conclude that $H_*(T^{n}{}_x^G; \mathbb{Z}_p) \approx H_*(T^k; \mathbb{Z}_p)$. □

Exercise 13. Construct as many interesting examples of actions on lens spaces and on tori as you can.

Problem 9. *Show that if a group G acts on the torus T^n with a fixed point x and induces the trivial action on $\pi_1(T^n, x)$, then the action is trivial.*

It suffices to assume $G = C_p$. Show that in this case the covering $\widetilde{F} \to F_x$ has the property that $\widetilde{F} = \mathrm{pr}^{-1}(F_x)$, with full deck transformation group $\mathbb{Z}^n$. Cohomology considerations then show that $H_n(F_x; \mathbb{Z}_p) \neq 0$. The result then follows from basic standard facts about the homology of manifolds, since no proper subset of a connected n-manifold has nontrivial nth homology.

Problem 10. *What happens when you try to lift a group action without fixed points (or without attention to fixed points) to the universal covering?*

Answer (or at least the beginning of an answer): Considering all possible lifts of all elements of G acting on X to homeomorphisms of the universal covering $\widetilde{X}$ yields a short exact sequence of groups

$$1 \to \pi \to E \to G \to 1$$

If the action of G on X has a fixed point x_0 then a choice of a fixed point $\tilde{x}_0$ over x_0 yields a splitting of this sequence. Question: If the sequence splits abstractly is there a fixed point?

Problem 11. *What can you say about the fixed point set in the universal covering when the action on $\pi_1(X, x_0)$ is nontrivial? What about the case of other (regular?) coverings?*

12. More group actions on surfaces

It can be proved that a group action on a surface is equivalent to a PL or smooth action. There are no "wild" actions in dimension 2. For convenience, we often assume actions are PL or smooth, or at least locally linear. In dimension 2 this is no loss of generality.

Proposition 12.1. *Any finite group acts effectively, even freely, on some surface.*

Proof. Let G be a finite group and let $g_1, \ldots, g_r \in G$ be a set of generators. Let F be a surface of genus r, with standard presentation for the fundamental group with generators $a_1, b_1, \ldots, a_r, b_r$. Define a surjective homomorphism $\pi_1(F, x) \to G$ by $a_i \to g_i$, $b_i \to 1$. Then the corresponding regular covering space $\widetilde{F} \to F$ with deck transformation group G gives the required action. □

Note that the Euler characteristic of $\widetilde{F}$ satisfies $\chi(\widetilde{F}) = |G|(2 - 2r)$ so that $\widetilde{F}$ has genus $|G|(r - 1) + 1$.

Exercise 14. One may define and explore the "genus" of a finite group G: the smallest g such that G acts on the surface of genus g. Work out the genus of a few groups. As variations one can investigate the "orientation-preserving" genus or the "free genus", etc.

Proposition 12.2. *A finite group acting effectively on a surface of genus ≥ 2 must act faithfully on rational homology.*

Proof. Let G be such a group acting on such a surface F and suppose it does not act faithfully on $H_1(F;\mathbb{Q})$. Without loss of generality we may suppose that $G = C_p$, p prime, acting trivially on H_1. Then

$$\chi(F) = p\chi(F^*) - k(p-1)$$

where k is the number of isolated fixed points. (If $p = 2$ and the action is orientation-reversing, then one might have circles of fixed points, but they contribute 0 to the Euler characteristic calculation.) But since the action is trival on rational homology a standard transfer argument implies that $\chi(F^*) = \chi(F)$. Thus we may conclude that $\chi(F) = k$. But $\chi(F) < 0$ while $k \geq 0$. This contradiction completes the proof. $\square$

It follows that if G is a finite group acting effectively on a closed orientable surface of genus n greater than one, then G is isomorphic to a subgroup of $GL(n,\mathbb{Z})$. One can argue that $GL(n,\mathbb{Z})$ contains only finitely many finite subgroups

Corollary 12.3. *A compact group acting on a surface of genus* ≥ 2 *must be finite.*

Alternatively we will derive this finiteness statement from the following.

Theorem 12.4 (Serre). *If* $A \in SL(k,\mathbb{Z})$ *has finite order* $m > 1$ *and* $A \equiv I \mod n$*, then* $m = n = 2$*.*

Proof. (Argument attributed to C. J. Earle, adapted from Farkas and Kra, pp. 275-276.) Let p be a prime factor of n. Then $A \equiv I \mod p^t$ for some largest $t \geq 1$. It suffices to show that $p = m = 2$ with $t = 1$.

Step 1. If $A \neq I$, but $A \equiv I \mod p^t$, where t is as large as possible, then $t = 1$. Write the order $m = qr$, where q is prime. Then $B = A^r$ has prime order q and $B \equiv I \mod p^t$, Write $B = I + p^s X$ where X is integral and $X \not\equiv 0 \mod p$, and hence $s \geq t$. Then

$$I = (I + p^s X)^q = I + qp^s X + p^{2s} Y$$

Therefore $qX = -p^s Y$, from which it follows that $q = p$ and $X = -p^{s-1}Y$. Since p does not divide X, we conclude that $s = 1$, and since $s \geq t \geq 1$, we also have $t = 1$. It also follows that A has order p^u for some u since all prime factors of m have to be p.

Step 2. A has order exactly p. Consider A^p, which we show must equal I. Clearly A^p has finite order. Write $A = I + pX$. Then

$$A^p = (I + pX)^p = I + p^2 X + p^2 Y = I + p^2 Z$$

so $A^p \equiv I \mod p^2$. This contradicts Step 1 unless $A^p = I$.

Step 3. $p = 2$. If not p is an odd prime, and

$$I = (I + pX)^p = I + p^2 X + \frac{p(p-1)}{2} p^2 X^2 + p^3 Y$$

from which it follows that

$$-X = \frac{p(p-1)}{2} X^2 + pY$$

which would imply that $X \equiv 0 \mod p$, since p is an odd prime, contradicting the choice of $X \not\equiv 0 \mod p$. $\square$

Exercise 15. Identifying $\mathrm{Aut} H_1(F_g) = \mathrm{Sp}_{2g}(\mathbb{Z})$, find an element of order 2 in $\mathrm{Sp}_{2g}(\mathbb{Z})$ that does not come from a periodic homeomorphism of period 2 on F_g.

Project 2. How can one characterize which elements of $\mathrm{Sp}_{2g}(\mathbb{Z})$ are realized by periodic homeomorphisms? (Recent work on this.)

Question 9. *Is a group action on a closed orientable surface of genus greater than or equal to two determined up to equivalence by its action on homology?*

I am pretty sure the answer is "no", but I don't have an example at hand.

12.1. **Bounds on group order.**

Theorem 12.5 (Hurwitz's $84(g-1)$ Theorem)**.** *Let a finite group G act effectively and orientation-preservingly on a closed orientable surface S of genus $g \geq 2$. Then $|G| \leq 84(g-1)$.*

We will content ourselves with proving a simpler statement, although the general case is not too much harder, having just more cases to consider.

Theorem 12.6. *Let a finite cyclic group G of prime order p act effectively and orientation-preservingly on a closed orientable surface S of genus $g \geq 2$. Then $p \leq 2g+1$.*

Proof. Suppose that $p \geq 2g$. We show that in this case $p = 2g+1$. Let the orbit surface S^* have genus g^* and suppose there are $k \geq 0$ fixed points. Then the usual Euler characteristic formula says

$$2g-2 = p(2g^*-2) + k(p-1)$$

Suppose first that $g^* \geq 2$. Then $2g-2 \geq p(2) \geq 4g$, a contradiction.

If $g^* = 1$, then $2g-2 = k(p-1) \geq k(2g-1)$ so $(k-1)(2g-1) \leq -1$. This implies that $k = 0$, in which case the orbit map $S \to S^*$ is a covering and $g = 1$, contradicting the assumption that $g \geq 2$.

Finally, if $g^* = 0$, then we have $2g-2 = -2p + k(p-1)$. It follows that $2g = (k-2)(p-1)$ and, hence, that $2g \geq (k-2)(2g-1)$. This implies $k \leq 3$. the cases $k < 2$ contradicts the original assumption that $g \geq 2$. Thus $k = 3$, and hence $2g = p-1$, or $p = 2g+1$, as required. $\square$

12.2. **Nielsen realization.**

The question is whether a "homotopy action" on a surface comes from an "honest action."

Theorem 12.7 (Nielsen Realization)**.** *If $f : F_g \to F_g$ such that $f \not\simeq id$ but $f^p \simeq id$ for some prime p, then $f \simeq f_1$ such that $f_1^p = id$.*

This is moderately deep and can be generalized somewhat to finite solvable groups using somewhat the same methods. More generally, more recently, and more deeply, Steven Kerkhoff showed that if G is any finite group of homotopy classes of homotopy equivalences of a surface, then G can be realized as a finite group of homeomorphisms of the surface.

13. Groups acting freely on spheres

What groups act freely on some sphere?

13.1. **The p^2 condition.**

We begin by looking at the case of finite groups acting freely on a space of the homotopy or homology type of a sphere. It is crucial to assume some sort of finite-dimensionality. After all, any group G acts freely on any homotopy type X by expressing it as $X \times E_G$, where E_G is a contractible space on which G acts freely (with orbit space of type $K(G,1)$).

Theorem 13.1 (Smith)**.** *The group $G = C_p \times C_p$ does not act freely on a finitistic space with the Cech cohomology of a sphere.*

Alternatively put, if a finite group G acts freely on a finitistic space with the Cech cohomology of a sphere, then it contains no subgroup isomorphic to $C_p \times C_p$.

Bredon gives a complete proof of Smith's theorem in the spirit of basic Smith theory. We concentrate instead on special cases that convey the idea of what's going on.

We say that a finite group satisfies the p^2 conditions if every subgroup of order p^2 is cyclic. Since a group of order p^2 is either C_{p^2} or $C_p \times C_p$, a group satisfies the p^2 condition if and only if it contains no $C_p \times C_p$.

Lemma 13.2. *A finite group satisfies the p^2 condition if and only if the Sylow p-subgroup is cyclic or (when $p = 2$) generalized quaternion.*

The ordinary quaternion group is $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ inside the unit quaternions. Note that -1 has order 2 and the other nontrivial elements all have order 4. More generally the generalized quaternion group of order 2^k ($k \geq 3$) is the group with presentation

$$\left\langle x, y : x^{2^{k-1}} = 1, yxy^{-1} = x^{-1}, y^2 = x^{2^{k-2}} \right\rangle$$

Lemma 13.3. *A finite group G satisfies the p^2 condition if and only if its (co)homology $H_k(G; \mathbb{Z}_p)$ is periodic in k. A finite group G satisfies the p^2 conditions for all p if and only if its (co)homology $H_k(G; \mathbb{Z})$ is periodic in k.*

Equivalently, the cohomology of G is periodic, or G contains no copy of $C_p \times C_p$. It will become clear here that periodic cohomology implies no $C_p \times C_p$. For the converse, which we don't really need, consult *Cohomology of Groups,* by Ken Brown.

Now we give direct proofs of a couple of special cases of Smith's theorem.

Proposition 13.4. *The group $G = C_p \times C_p$ does not act freely on S^n (or any n-manifold with the mod p homology of S^n).*

Proof. If G did so act, then the space $X = S^n/G$ has $\pi_1 = G$ and $\pi_i = 0$ for $2 \leq i \leq n-1$. It follows that $H_i(G; \mathbb{Z}_p) = H_i(X; \mathbb{Z}_p)$ for $i \leq n-1$. But by the Künneth formula $H_{n-1}(G) = \mathbb{Z}_p^{\binom{n-1}{2}}$. On the other hand X is an n-manifold, orientable if p is odd and orientable over $\mathbb{Z}_p$ in any case. Thus X satisfies Poincaré duality over $\mathbb{Z}_p$. In particular $H_{n-1}(G; \mathbb{Z}_p) \approx H_1(G; \mathbb{Z}_p)$. It follows that $\binom{n-1}{2} = 2$, or $n = 4$. But C_p does not act freely on S^4 by the Lefschetz Fixed Point Theorem. And if $p = 2$, we note that C_2 acts freely on S^4 only if it is orientation preserving. But if $C_2 \times C_2$ acts freely, then it has two orientation reversing involutions, whose product is the third involution and would then not act freely. QED $\square$

Proposition 13.5. *The group $G = C_p \times C_p$ does not act freely and cellularly on an n-dimensional CW complex with the homology of S^n.*

Proof. If G did so act, then the cellular chain complex would yield an exact sequence of the form

$$0 \to \mathbb{Z} \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

where the C_i are free over $\mathbb{Z}[G]$. One can formally splice together copies of such sequences to produce a periodic resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module. It follows that the cohomology of G is periodic, while the size of the cohomology H_k of $C_p \times C_p$ grows quadratically in k. $\square$

13.2. **Swan's theorem.** The question then becomes whether a finite group with periodic cohomology acts freely on some sphere. Swan answered that such a group acts freely and cellularly on a finite CW complex of the homotopy type of a sphere.

Theorem 13.6. *A finite group that satisfies the p^2 conditions for all primes p acts freely on a finite CW complex of the homotopy type of a sphere (of some dimension).*

The main challenge in the proof of Swan's theorem is to show that there is a finitely generated free, partial resolution of $\mathbb{Z}$ as a(trivial) $\mathbb{Z}[G]$-modules, i.e. an exact sequence

$$0 \to \mathbb{Z} \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

where the C_i are free over $\mathbb{Z}[G]$. The possible values of n are of the form $dk-1$, where k is the period of the cohomology of G. In general it is not possible to choose $d = 1$, but Swan does show that $d = \gcd(k, \varphi(k))$ does suffice. It turns out that one can always find such a sequence where the C_i are *projective* over $\mathbb{Z}[G]$. There is an obstruction then found in the projective class group $\tilde{K}_0(\mathbb{Z}[G])$ (isomorphism classes of projectives modulo summing with frees, under direct sum). A finiteness theorem shows that one can kill the obstruction by passing to higher multiples of the periodicity degree.

The second smaller challenge is to be sure that one can realize the chain complex as the cellular chain complex of the universal covering of a CW complex with fundamental group G. One needs to take a little care to be sure that the part in degrees ≤ 2 corresponds to the chain complex of the canonical 2-complex associated with a presentation of the group. That can be done.

It is an interesting problem known as "Steenrod's Problem" to realize any finitely generated chain complex of free $\mathbb{Z}[G]$-modules as the cellular chain complex of the universal covering of a CW complex with $\pi_1 = G$. There are counterexamples for so-called "Moore spaces," i.e., chain complexes whose homology in positive dimensions consists of a single *torsion* module, such as $\mathbb{Z}_k$.

Instead of proving Swan's theorem, we concentrate on the simplest nontrivial case, that of the dihedral group $G = D_3$ of order 6, isomorphic to the symmetric group S_3 of degree 3. In this case Swan's obstruction vanishes because it lies in a zero group. We avoid all that theory by a direct construction. The discussion is an elaboration of the last section of Swan's fundamental paper [11].

Theorem 13.7. *The group D_3 acts freely and cellularly on a finite 3-dimensional CW complex $X \simeq S^3$.*

The group G has a standard presentation as a dihedral group

$$\left\langle x, y : x^3, y^2, yxy = x^2 \right\rangle$$

In this case the $x^3 = 1$ relation is redundant: $x = yx^2y = (yxy)^2 = x^4$, from which it follows that $x^3 = 1$. Therefore we work with the presentation

$$\left\langle x, y : y^2, yxyx^{-2} \right\rangle$$

In what follows we let $R = \mathbb{Z}[D_3]$, the group ring of the dihedral group. A typical element of R can be written as a polynomial in x and y, and, because of the special relations, can be written in the form $f(x) + g(x)y$.

In what follows we will need the following two identities in the group ring $\mathbb{Z}[D_3]$:

$$(1 + x - y)(1 + yx) = (1 + yx)$$
$$(y + 1)x(y + 1)(x - 1) = -(y + 1)(x - 1)$$

Now let

$$Y^2 = e^0_a \cup e^1_b \cup e^1_{b'} \cup e^2_b \cup e^2_{b'}$$

be the 2-complex associated with the latter presentation above. The symbols b and b' correspond to x and y, respectively. The symbols c and c' correspond to the two relations y^2 and $yxyx^{-2}$, respectively. The cellular chain complex of the universal covering $\widetilde{Y}^2$ has the form

$$C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \overset{\varepsilon}{\to} \mathbb{Z} \to 0$$

We use the symbols a, b, b', c, c' for corresponding basis elements of these chain modules. In particular a prescribes a base point in $\widetilde{Y}^2$ over the 0-cell of Y^2. And b and b' correspond to choices of 1-cells over the two 1-cells in Y^2, emanating from a.

See Figure 1, where the 1-skeleton of $\widetilde{Y}^2$ is depicted. The group element x is understood to rotate $2\pi/3$ in the counterclockwise direction. The group element y reflects in a vertical line through vertex a and interchanges the inner and outer circles. Edges b and b' are chosen emanating from the chosen vertex a to represent the two attached 1-cells. They are to be oriented pointing counterclockwise. All other vertices and edges are labelled and oriented by equivariance, as shown.

The homomorphisms in this chain complex may be described in terms of these bases as follows:

$$\begin{aligned} \varepsilon(a) &= 1 \\ \partial_1(b) &= (x-1)a \\ \partial_1(b') &= (y-1)a \\ \partial_2(c) &= -(1+x-y)b + (1+xy)b' \\ \partial_2(c') &= -(1+y)b' \end{aligned}$$

In particular, we understand that these are homomorphisms of left $\mathbb{Z}[D_3]$-modules. For example

$$\partial_2(s(x,y)c + s'(x,y)c') = -s(x,y)(1+x-y)b + s'(x,y)(1+xy)b'$$

The plan is to describe $\ker \partial_2$ explicitly and show that it is generated over $\mathbb{Z}[D_3]$ by a single element. We will then extend the chain complex by adding $C_3 \approx \mathbb{Z}[D_3]$ in such a way as to kill this single generator. It will be done so that the new kernel is just $\mathbb{Z}$! Then we will realize the extended chain complex by attaching a single 3-cell to Y^2.

Suppose, then, that $u = sc + s'c' \in \ker \partial_2 \subset C_2(\widetilde{Y}^2)$.

It follows that

$$\begin{aligned} s(x,y)(1+x-y) &= 0 \quad \text{and} \\ s(x,y)(1+xy) + s'(x,y)(1+y) &= 0 \end{aligned}$$

But if $s\dot{(}1+x-y) = 0$, then

$$s\dot{(}1+xy) = s\dot{(}1+x-y)(1+xy) = 0\dot{(}1+xy) = 0$$

by one of the group ring identities mentioned above. Therefore the two conditions unlink to become

$$\begin{aligned} s(x,y)(1+x-y) &= 0 \quad \text{and} \\ s'(x,y)(1+y) &= 0 \end{aligned}$$

First analyze $s'(x,y)$. Write $s'(x,y) = f(x) + g(x)y$. Then multiplying and collecting terms we find that $g(x) = -f(x)$ and $s'(x,y)$ has the general form

$$s'(x,y) = f(x)(1-y)$$

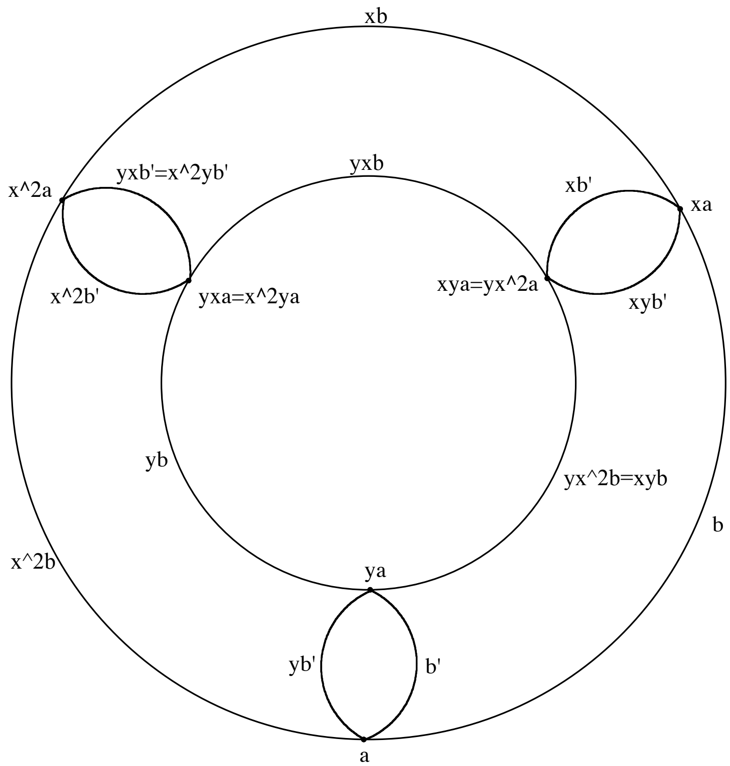

FIGURE 1. The 1-skeleton of the universal covering

Now work on $s(x,y)$. Again, set $s(x,y) = f(x) + g(x)y$, multiply terms out and collect, to find that

$$g(x) = f(x) + f(x)x \quad \text{and}$$
$$f(x) = g(x) + g(x)x^2$$

it follows that $f(x)(1 + x + x^2) = 0$, hence that

$$f(x) = h(x)(1 - x) \quad \text{and}$$
$$g(x) = h(x)(1 - x) + h(x)(1 - x)x$$

Thus $s(x)$ has the form $h(x)(1 - x) + h(x)(1 - x) + h(x)(1 - x)xy$ or

$$s(x,y) = h(x)(1 + y)(1 - x)$$

then the typical element $u = sc + s'c' \in \ker \partial_2$ has the form

$$u = h(x)(1 + y)(1 - x)c + f(x)(1 - y)c'$$

In particular $\ker \partial_2$ is generated by the two elements

$$(1+y)(1-x)c \text{ and } (1-y)c'$$

Our next goal is to show that in fact $\ker \partial_2$ is generated by a single element. Indeed we claim that the single element

$$v = x(y+1)(x-1)c + (y-1)c'$$

generates $\ker \partial_2$. To see this we use the identity

$$(y+1)x(y+1)(x-1) = (y+1)(x-1)$$

mentioned at the beginning of the proof.

We need to show that $(1+y)(1-x)c$ and $(y-1)c'$ can be obtained as multiples of v.

Note that $(1+y)v = (1+y)x(y+1)(x-1)c = (y+1)(x-1)c$, by the identity, since $(y+1)(y+1) = 0$.

Finally, given v and $(1+y)(1-x)c$, as above, we obtain $(y-1)c' = v - (1+y)v = -yv$, as required.

Now we define $\partial_3 : \mathbb{Z}[D_3] \to C_2(\widetilde{Y}^2)$ by

$$\partial_3(e) = x(y+1)(x-1)c + (y-1)c'$$

First of all, we claim that $\ker \partial_3 \approx \mathbb{Z}$ (with trivial D_3 action). In fact, $\Sigma = 1 + x + x^2 + y + xy + x^2y$, the sum of the group elements. We will show that $\ker \partial_3 = \langle \Sigma \rangle$. Certainly Σ has trivial D_3 action.

Next, note that $\partial_3 \Sigma = 0$. For in general $f(x,y)\Sigma = \varepsilon(f)\Sigma = f(x,y)\Sigma f(x,y)$. Therefore

$$\begin{aligned} \partial_3(\Sigma \cdot e) &= \Sigma \cdot x(y+1)(x-1)c + \Sigma \cdot (y-1)c' \\ &= \Sigma \cdot c + \Sigma \cdot c' & = 0 \end{aligned}$$

Finally, suppose $\partial_3 f(x,y)e = 0$. We show that $f(x,y) = m\Sigma$ for some integer m. Then the hypothesis that $\partial_3 f(x,y)e = 0$ implies that

$$f(x,y)x(y+1)(x-1) = 0 \quad \text{and} \quad f(x,y)(y-1) = 0$$

As usual, let

$$f(x,y) = a(x) + b(x)y$$

The second condition becomes

$$(a(x) + b((x)y)(y-1) = 0$$

or

$$b(x) + a(x)y = a(x) + b(x)y$$

Thus $a(x) = b(x)$ and $f(x,y) = a(x)(1+y)$.

Now the first condition above says that

$$a(x)(y+1)(x-1) = 0$$

One of the group ring identities we mentioned before now implies that

$$-a(x)(y+1)(x-1) = 0$$

In particular,

$$a(x)(x^2-1)(y+1) = 0$$

Hence

$$a(x)(x^2-1) = 0$$

which means that

$$a(x) = m(1+x+x^2) \text{ in the group ring } \mathbb{Z}[C_3]$$

Therefore

$$f(x,y) = m(1+x+x^2)(1+y) = m\Sigma$$

as required.

Now attach a 3-cell also denoted by $e = e^3$ to Y^2 using this element $\partial_3(e)$of $\pi_2(Y) = \pi_2(\widetilde{Y}^2) = H_2(\widetilde{Y}^2) = \ker\partial_2$, forming a 3-complex $Y^3 = Y^2 \cup e^3_e$. The universal covering is obtained from $\widetilde{Y}^2$ by equivariantly attaching six 3-cells. The calculations above show that the CW complex $\widetilde{Y}^3$ is a simply connected homology 3-sphere, with free action of D_3, as required. QED

13.3. **The $2p$ condition.** The question then becomes whether a group of periodic cohomology can act freely on the actual sphere itself.

Theorem 13.8. *If finite group G acts freely on a closed manifold with the mod 2 homology of a sphere then every element of order 2 in G lies in the center of G.*

Corollary 13.9. *If a finite group G acts freely on S^n, then for all primes p every subgroup of order $2p$ is abelian.*

We will give a proof valid for actions on the actual sphere S^n. The general version stated above requires slightly more sophisticated algebraic topology, for which we substitute a direct geometric argument, following Bredon.

The theorem follows from the following result.

Theorem 13.10. *Let $g : S^n \to S^n$ be an involution without fixed points and suppose $f : S^n \to S^n$ is a map such that $gf(x) \neq fg(x)$ for all $x \in S^n$. Then $\deg f$ is even.*

Proof of Theorem 13.8 from Theorem 13.10. Let $g \in G$ have order 2. And suppose we also have $f \in G$ such that $fg \neq gf$. Since G acts freely, it follows that $fgx \neq gfx$ for all $x \in S^n$. ($f^{-1}g^{-1}fg \neq e$ implies $f^{-1}g^{-1}fgx \neq x$ for all x.) By Theorem 13.10 $\deg f$ is even, and hence f cannot be a homeomorphism This contradiction completes the proof. □

Before plunging into the proof of Theorem 13.10 we recall a couple of facts about Smith theory for actions of $G = C_2$ on finitistic spaces, using Cech cohomology with $\mathbb{Z}_2$ coefficients.

First recall that $\sigma = 1 + g = 1 - g = \delta$ in $\mathbb{Z}_2[G]$. Then $H^n_\sigma(X) \approx H_n(X/G, X^G)$. Also the Smith inequalities take the following relative form

$$\dim H^n(X/G, X^G) + \sum_{i\geq n} \dim H_i(X^G, X^G) \leq \sum_{i \geq n} \dim H_i(X, X^G)$$

so that

$$\dim H^n(X/G, X^G) \leq \sum_{i\geq n} \dim H_i(X, X^G)$$

We also have the basic Smith sequence in the case when $X^G = \emptyset$ and using $\sigma = \delta$

$$H^{n-1}_\sigma(X) \to H^n_\sigma(X) \to H^n(X) \to H^n_\sigma(X) \to H^{n+1}_\sigma(X)$$

Proof of Theorem 13.10. Let $G = \{e, g\}$. We aim to factor the induced homomorphism $f^* : H^n(S^n) \to H^n(S^n)$ through $H^n(S^n/G)$, factoring f^* as $\pi^*\psi$, where $\pi : S^n \to S^n/G$ is the orbit map, and $\psi : H^n(S^n) \to H^n(S^n/G)$ is a homomorphism yet to be determined.

To find such a factorization we will use an auxiliary space withG action. Set $M = S^n \times S^n - \Delta$, where Δ denotes the diagonal subspace. Then G acts on M by the formula

$$g\dot{(}x,y) = (gy, gx)$$

and we can define a map $\phi : S^n \to M$ by the formula

$$\phi(x) = (f(x), gfg(x))$$

It is completely straightforward to check that this defines a G-map.

We thus obtain the following commutative diagram

$$\begin{array}{ccccc} S^n/G & \xleftarrow{\pi_S} & S^n & \xrightarrow{f} & S^n \\ \overline{\phi}\big\downarrow & & \phi\big\downarrow & & \big\| \\ M/G & \xleftarrow{\pi_M} & M & \xrightarrow{\mathrm{pr}} & S^n \end{array}$$

Applying (Cech) cohomology with $\mathbb{Z}_2$ coefficients, we have

$$\begin{array}{ccccc} H^n(S^n/G;\mathbb{Z}_2) & \xrightarrow{\pi_S^*} & H^n(S^n;\mathbb{Z}_2) & \xleftarrow{f^*} & H^n(S^n;\mathbb{Z}_2) \\ \overline{\phi}^*\big\uparrow & & \phi^*\big\uparrow & & \big\| \\ H^n(M/G;\mathbb{Z}_2) & \xrightarrow{\pi_M^*} & H^n(M;\mathbb{Z}_2) & \xleftarrow{\mathrm{pr}^*} & H^n(S^n;\mathbb{Z}_2) \end{array}$$

We will

(1) $\pi_M^* : H^n(M/G;\mathbb{Z}_2) \to H^n(M;\mathbb{Z}_2)$ is an isomorphism
(2) $\pi_S^* : H^n(S^n/G;\mathbb{Z}_2) \to H^n(S^n;\mathbb{Z}_2)$ is 0.

Since $f^* = \pi^*\overline{\phi}^*(\pi_M^*)^{-1}\mathrm{pr}^*$, we see that $f^* = 0$, and the proof is complete.

It remains to prove (1) and (2). We first discuss (1).

Recall that the action of G on $M = S^n \times S^n - \Delta$ is given by $g\dot{(}x,y) = (gy, gx)$. One easily sees that the fixed point set

$$M^G = \{(x, gx) : x \in S^n\} \cong S^n$$

We claim that the inclusion $M^G \subset M$ is a homotopy equivalence. To see this it suffices to construct an explicit strong deformation retraction of M onto M^G. To this end define a homotopy

$$H : M \times [0,1] \to M$$

by

$$H((x,y),t) = (x, F(x,y,t))$$

where $F(x,y,t) \in S^n$ is the point where the line from x to $tgx + (1-t)y$ intersects S^n. It is straightforward now to verify that this defines the required strong deformation retraction.

It now follows that $H^*(M, M^G;\mathbb{Z}_2) = 0$. Then the Smith inequality mentioned above implies that $H^*(M/G, M^G;\mathbb{Z}_2) = 0$, hence that $\pi_M^* : H^*(M/G;\mathbb{Z}_2) \to H^*(M^G;\mathbb{Z}_2)$ is an isomorphism.

Now we discuss (2). The basic Smith sequence briefly reviewed above yields

$$H_\sigma^{n-1}(S^n;\mathbb{Z}_2) \to H_\sigma^n(S^n;\mathbb{Z}_2) \to H^n(S^n;\mathbb{Z}_2) \to H_\sigma^n(S^n;\mathbb{Z}_2) \to H_\sigma^{n+1}(S^n;\mathbb{Z}_2)$$

$$A \xrightarrow{\pi_S^*} H^n(S^n;\mathbb{Z}_2) \to A \to 0$$

where $A = H_\sigma^n(S^n;\mathbb{Z}_2) = H^n(S^n/G;\mathbb{Z}_2)$. From this it follows immediately that $A \approx \mathbb{Z}_2$ and that $\pi_S^* = 0$. □

Remark 1. Milnor shows that the theorem remains true for free involutions on an n-manifold Σ^n with the mod 2 homology of S^n. The main difficulty is to offer a different proof of the analogue of item (1) above.

13.4. **The linear case.** What finite groups act freely and linearly on a sphere? Equivalently, what groups G have an orthogonal representation so that no nontrivial element of the group has eigenvalue 1?

We say that a finite group satisfies the pq conditions if every subgroup of order pq is cyclic, for any two primes, the same or different.

Theorem 13.11. *A finite group G acts freely and orthogonally on a sphere if and only it satisfies the pq conditions for all primes p and q.*

13.5. **Theorem of Madsen-Thomas-Wall.** This is one striking culimination of a long series of investigations by numerous authors.

Theorem 13.12. *If G is a finite group with periodic cohomology (i.e., satisfying the p^2 for all primes p), and the $2p$ conditions for all odd primes p, then G acts freely on some sphere S^n.*

The action is topological, not smooth or PL, in general. The $n+1$ is a multiple of the period of $H^*(G)$, but not necessarily equal to that period. The proof requires deep results from surgery theory.

13.6. **Actions on the 3-sphere.** It follows from Perelman's work on Ricci flow and the Poincaré conjecture that a finite group acts freely on S^3 if and only if it acts freely and linearly on S^3 and, moreover, any free action is equivalent to a linear action. This breakthrough trumps many years of work on special cases going back to the late 1950's.

13.7. **Actions on $S^n \times \mathbb{R}^k$.** If follows easily from Swan's theorem that

Theorem 13.13. *A finite group acts freely on $S^n \times \mathbb{R}^k$ for some n and k if and only if it has periodic cohomology.*

Recent work has focused on free and properly discontinuous (deck transformations) groups acting on the non-compact manifolds $S^n \times \mathbb{R}^k$.

14. THE BOREL FIBERING

14.1. **Principal bundles.** (Free actions) A locally trivial fiber bundle $\varphi : X \to Y$ with fiber a topological group G is a principal G-bundle if there is a free action of G on X identifying φ with the orbit map for the action. For compact Lie groups acting freely on nice spaces, the orbit map is automatically a locally trivial fiber bundle by the existence of "slices," which is of course almost trivial in the case of finite groups.

14.2. **Classifying spaces.**

Proposition 14.1. *For any compact group G there is a contractible CW complex X on which G acts freely and cellularly.*

For a canonical construction, one may let $X = G * G * G * \cdots$, the infinite join, topologized as the direct limit of finite joins, which themselves are topologized with a suitable product/subspace topology. Alternatively, for a discrete group G, one chooses a set of generators and defining relations for the group, and starts by constructing the canonical 2-complex realizing this presentation. Then one simply attaches cells of ever higher dimensions to kill the homotopy groups π_i, $i \geq 2$

Proposition 14.2. *If X and Y are contractible G-CW complexes on which the compact group G acts freely, then $X/G \simeq Y/G$.*

This is a basic application of elementary obstruction theory.

Let E_G denote a contractible CW complex on which G acts freely. Then the orbit space $B_G = E_G/G$ is a *classifying space* for G. In particular it classifies equivalence classes of principal G-bundles over a space Y by $[Z; B_G]$. Up to equivalence any principal G-bundle over Y arises as the pullback f^*E_G of the bundle $E_G \to B_G$.

For technical reasons it is usually better to view E_G as being a free *right* G-space.

14.3. **Examples.**

(1) If $G = C_2$, then $B_G \simeq \mathbb{R}P^\infty$.
(2) If $G = C_p$, then $B_G \simeq L(p)^\infty$, an "infinite lens space".
(3) If $G = (C_p)^n$, then $B_G \simeq (L(p)^\infty)^n$.
(4) If $G = S^1$, then $B_G \simeq \mathbb{C}P^\infty$.

For nontrivial, finite or compact groups B_G is always infinite dimensional, but well-approximated by its finite skeleta.

14.4. **The Borel construction.** The Borel construction is a certain functor from G-spaces to spaces defined as follows. If X is a G-space, the Borel construction is the space $X_G = E_G \times_G X = (E_G \times X)/G$. A G-map $f : X \to Y$ induces a map $f_G : X_G \to Y_G$ by setting $f_G[e, x] = [e, f(x)]$.

The key point is that the map $X_G \to B_G$ given by $[e, x] \to [e]$ is a fiber bundle with fiber homotopy equivalent to X. This can be understood by reference to the following commutative diagram.

$$
\begin{array}{ccccc}
E_G & \xleftarrow{\mathrm{pr}_1} & E_G \times X & \xrightarrow{\mathrm{pr}_2} & X \\
{\scriptstyle \pi_G}\downarrow & & {\scriptstyle \pi_{E_G \times X}}\downarrow & & {\scriptstyle \pi_X}\downarrow \\
B_G & \xleftarrow{\overline{\mathrm{pr}}_1} & E_G \times_G X & \xrightarrow{\overline{\mathrm{pr}}_2} & X/G
\end{array}
$$

Note that $\mathrm{pr}_1^{-1}(e) = \{(e,x) : x \in X\} \cong X$, so that $\overline{\mathrm{pr}_1}^{-1}[e] = \{[e,x] : x \in X\} \cong X$, since G acts freely on E_G. With a little care one can check that this defines a locally trival fibering, known as the Borel fibering: the equivariant fibration $X \to X_G \to B_G$.

On the other hand, $\mathrm{pr}_2^{-1}(x) = \{(e,x) : e \in E_G\} \cong E_G$. The subgroup of G that leaves the latter subset of $E_G \times X$ invariant, rather than mapped off of itself, is exactly the isotropy group G_x. We find that $\overline{\mathrm{pr}_2}^{-1}[x] = \{(e,x) : e \in E_G\}/G_x \cong B_{G_x}$. Among other consequences, we see that if G acts freely on X so that all isotropy groups G_x are trivial, then $X_G \simeq X/G$.

If one has a G-map$f : X \to Y$, then one obtains a map of the Borel constructions $f_G : X_G \to Y_G$, given by $f_G([e,x]) = [e, f(x)]$.

14.5. **Equivariant cohomology.** The Borel construction provides a way of associating using cohmological invariants to a G-space.

Definition 2. If X is a G-space, the equivariant cohomology of X with coefficients in a commutative ring R is
$$H^*_G(X;R) := H^*(X_G;R)$$

There is a relative version for G pairs (X, A), and associated to the pair there is a long exact sequence of equivariant cohomology groups. The construction is natural in the sense that a G-map induces a homomorphism of equivariant cohomology groups. The construction is also natural in the group variable: if K is a subgroup of G, then there is a restriction homomorphism $\mathrm{res}^G_K : H^*_G(X;M) \to H^*_K(X;M)$, where the G-module M is also viewed as a K-module by restriction.

Now $H^*_G(\text{point}) = H^*(B_G;R)$ is a ring under cup product. Note that the G-map $X \to$ point induces a homomorphism
$$H^*(B_G;R) \to H^*_G(X;R)$$
making $H^*_G(X;R)$ into a module over $H^*(B_G;R)$

Suppose that $F = X^G$. Then $F_G \cong B_G \times F$. If we use field coefficients, then by the Künneth theorem $H^*_G(F) \approx H^*(B_G) \otimes H^*(F)$.

14.6. **The Borel Spectral Sequence.** Let G be a finite group acting on a finite-dimensional space X (which we assume is a G-CW complex). The as above we have the associated Borel construction, yielding a fibering
$$X \to X_G \to B_G$$
There is then Leray-Serre spectral sequence
$$E_2^{ij}(X) := H^i(B_G; H^j(X)) \Rightarrow H^{i+j}(X_G)$$
converging to E_∞^{ij}, the graded group associated with a filtration of $H^*(X_G)$.

The different is $d_r : E_r^{i,j} \to E_r^{i+r,j+1-r}$, such that $E_{r+1} = H(E_r, d_r)$.

The convergence statement means that there is an decreasing filtration
$$\cdots \supset \mathcal{F}^i \supset \mathcal{F}^{i+1} \supset \cdots$$
of $H^*(X_G)$ such that if we set $\mathcal{F}^{ij} = \mathcal{F}^i \cap H^{i+j}(X_G)$, then
$$E_\infty^{ij} = \mathcal{F}^{i,i+j}/\mathcal{F}^{i+1,i+j-1}$$
We sometimes refer to i above as the *filtration degree* and $i + j$ as the (co)homology degree.

The spectral sequence is natural for G-maps. In particular there is a restriction $E_2^{ij}(X) \to E_2^{ij}(X^G)$. For a G-invariant subspace $A \subset X$ there is also a relative spectral sequence $E_2^{ij}(X, A)$.

One can use the spectral sequence approach to reprove the basic results of Smith Theory , as well as to prove far-reaching extensions of the basic theory. To connect the abutment of the spectral sequence with the cohomology of the fixed point set we use the following result.

Lemma 14.3. *Let X be a finite-dimensional G-CW complex, and let S denote the invariant subcomplex of points with nontrivial isotropy group. Then $H^i(X_G, S_G) = 0$ for $i > \dim X$*

Proof. There is a natural map $f : (X_G, S_G) \to (X/G, S/G)$ induced by projection $E_G \times X$ to X such that the preimage of $x^* \in X/G$ is B_{G_x}. Therefore for $x \in X/G - S/G$ it follows that $f^{-1}(x)$ is contractible. Therefore $H^*(X/G, S/G) \to H^*(X_G, S_G)$ is an isomorphism. On the other hand $H^i(X/G, S/G) = 0$ for $i > \dim X$. □

Corollary 14.4. *Let X be a finite-dimensional G-CW complex, and let S denote the invariant subcomplex of points with nontrivial isotropy group. Then restriction $H^i(X_G) \to H^i(S_G)$ is an isomorphism for $i > \dim X$.*

We will ordinarily apply this when the action is semifree (e.g. $G = C_p$, p prime), in which case $S = X^G$.

14.7. Applications of the spectral sequence.

Theorem 14.5 (Smith Theorems)**.** *If $G = C_p$ acts on a finitistic space X that is mod p acyclic, then so is X^G. If $G = C_p$ acts on a finitistic space X that has the mod p cohomology of an n-sphere, then X^G has the mod p cohomology of a k-sphere for some $k \leq n$.*

Theorem 14.6 (Smith Theorems)**.** *If $G = C_p$ acts on a finitistic space X, then*

$$\dim H^*(X^G; \mathbb{Z}_p) \leq \dim H^*(X; \mathbb{Z}_p)$$

with equality if and only if

1. *The spectral sequence (defined using $\mathbb{Z}_p$ coefficients) collapses, and*
2. *The group G acts trivially on $H^*(X; \mathbb{Z}_p)$*

Theorem 14.7. *If $G = C_p$ acts on a compact simply connected 4-manifold X without boundary, and the action has a nonempty fixed point set, then the spectral sequence collapses.*

Proof. Because of the rows of zeroes in the spectral sequence one sees immediately that $E_2 = E_3$, $E_4 = E_5$, and $E_6 = E_\infty$. It remains to show that the differentials d_3 and d_5 must vanish if the fixed point set is nonempty. Using naturality of the spectral sequence and the relative versions $E(X, x)$ and $E(X - \{x\})$ (where $x \in F$), each of which only has two non-vanishing rows, one can factor the differentials d_3 and d_5 through 0 groups. □

Theorem 14.8. *If $G = C_p$ acts locally linearly on $\mathbb{C}P^2$, then the fixed point set is nonempty and satisfies $\dim H^*(F; \mathbb{Z}_p) = 3$. In particular, the fixed set consists of*

1. *Three isolated points (only when p is odd) or*
2. *An isolated point and a 2-sphere or*
3. *$\mathbb{R}P^2$ (only when $p = 2$)*

Proof. First note that the action is orientation-preserving, even when $p = 2$. (Indeed $\mathbb{C}P^2$ admits no orientation-reversing homeomorphism, since a cohomology top class $v \in H^4(X)$ can be expressed as $v = u \cup u$, where $u \in H^2(X)$ is a generator. Then $t^*(v) = T^*(u \cup u) = T^*(u) \cup T^*(u) = \pm u \cup \pm u = u \cup u = v$.)

By local linearity and the orientation preserving character of the action, the fixed point set F must consist of points and surfaces.

If p is odd or if $p = 2$ and G acts trivially on integral cohomology, then $\chi(F) = 3$ by the Lefschetz Fixed Point Theorem.

If $p = 2$ and G acts nontrivially on $H^2 = \widetilde{\mathbb{Z}}$, then $\chi(F) = 1$, again by LFPT.

In particular, $F \neq \emptyset$.

In all cases the action on mod p cohomology is trivial. Because the spectral sequence collapses and the action on the mod p cohomology must be trivial, we know that $\dim_{\mathbb{Z}_p} H^*(F; \mathbb{Z}_p) = 3$.

It follows that if the action on cohomology is trivial, then F consists of 3 points or of 1 point and a 2-sphere.

In the remaining case we have $p = 2$ and nontrivial action on cohomology. The only combination of points and surfaces with total mod 2 homology of rank 3 and Euler characteristic 1 is $F = \mathbb{R}P^2$. □

Addendum 1. When $p = 2$ the fixed set F *cannot* consist of exactly 3 points, as we show in the next theorem.

Question 10. *Is there such an action on a finite CW complex of the homotopy type of* $\mathbb{C}P^2$*?*

Theorem 14.9. *A locally linear C_2 action on a closed connected n-manifold with finite fixed point has an even number of fixed points.*

Proof. Let $D_1, D_2, \ldots, D_r$ be small invariant disk neighborhoods of the fixed points. Set $Y = \overline{X - \cup D_i}$ and $Z = Y/C_2$. Then Z is an n-manifold with boundary consisting of an odd number r of copies of $\mathbb{R}P^{n-1}$. And the orbit map $Y \to Z$ is a regular 2-fold covering map. Over the boundary one sees r copies of the standard covering $S^{n-1} \to \mathbb{R}P^{n-1}$. Each of these coverings is classified by a map $f : Z \to K(C_2, 1) = \mathbb{R}P^\infty$. This shows that $rf_*[\mathbb{R}P^{n-1}] = 0$ in $H_{n-1}(\mathbb{R}P^\infty; \mathbb{Z}_2)$. But f on each boundary can be viewed up to homotopy as the inclusion $\mathbb{R}P^{n-1} \subset \mathbb{R}P^\infty$. And we know that such a map is injective on mod 2 homology. If r is odd this gives a contradiction. □

Remark 2. A similar argument shows that a locally linear C_p action on a closed connected n-manifold cannot have exactly one fixed point.

Theorem 14.10. *If $G = C_p$, p odd, acts locally linearly on $S^2 \times S^2$, then the fixed point set is nonempty and satisfies* $\dim H^*(F; \mathbb{Z}_p) = 4$. *In particular, the fixed set consists of*

(1) *Four isolated points or*
(2) *Two* 2*-spheres or, possibly,*
(3) *Two isolated points and a single* 2*-sphere or*
(4) *A single* 2*-torus*

Proof. Since p is odd one can see that the action on homology is trivial. (There is a little bit more to say about the special case $p = 3$.) Assuming the fixed point set is nonempty, then the spectral sequence collapses, and we have $\dim H^*(F;\mathbb{Z}_p) = 4$. Since the action is locally liinear, and orientation preserving (since p is odd), the fixed point set, if nonempty, consists of isolated points and orientable surfaces. The list gives all possibilities. $\square$

Question 11. *The first two cases are standard. Can the third or fourth cases actually happen? Perhaps on a space of the homotopy type of $S^2 \times S^2$, but not on the actual manifold? What about the analogous question for $S^n \times S^n$, $n > 2$?*

Below we attempt to describe the situation for actions of C_2 on $S^2 \times S^2$. there are more possibilities for nontrivial actions on H^2, for some of which the action is orientation-reversing, with fixed set empty or of dimension one or three.

Theorem 14.11. *If $G = C_2$ acts on $S^2 \times S^2$, then the fixed point set is one of the following, where we organize the case according to the action on homology and the Lefschetz number of the generator*

Orientation preserving cases:

(1) *Four isolated points, where $H^2 = \mathbb{Z} \oplus \mathbb{Z}$, $\Lambda = 4$*
(2) *Two 2-spheres, where $H^2 = \mathbb{Z} \oplus \mathbb{Z}$, $\Lambda = 4$*
(3) *A single 2-spheres, where $H^2 = \mathbb{Z}[C_2]$, $\Lambda = 2$*
(4) *A 2-torus, where $H^2 = \widetilde{\mathbb{Z}} \oplus \widetilde{\mathbb{Z}}$, $\Lambda = 0$*
(5) *$\emptyset$, where $\widetilde{\mathbb{Z}} \oplus \widetilde{\mathbb{Z}}$, $\Lambda = 0$*

Orientation reversing cases:

(1) *$\emptyset$, where $H^2 = \mathbb{Z} \oplus \widetilde{\mathbb{Z}}$ or*
(2) *Two circles, where $H^2 = \mathbb{Z} \oplus \widetilde{\mathbb{Z}}$, $\Lambda = 0$*
(3) *$S^2 \times S^1$, where $H^2 = \mathbb{Z} \oplus \widetilde{\mathbb{Z}}$, $\Lambda = 0$ (perhaps also other 3-manifolds with the homology of $S^2 \times S^1$)*

Overall we know that $\dim H^*(F;\mathbb{Z}_2) \leq 4$. The Lefschetz number calculations show that the Euler characteristic, hence $\dim H^*(F;\mathbb{Z}_2)$ is even, hence $0, 2$, or 4.

Something that takes extra justification: The fixed point set can't have both points and surfaces, and can't have a nonorientable surface or 3-manifold.

15. Other topics that might have been discussed

15.1. The G-signature theorem.

15.1.1. *The g-signature.* We give a brief definition of this generalized Lefschets number. Suppose that the cyclic group C_m, with chosen generator g, acts by orientation preserving homeomorphisms on a closed oriented $2n$-manifold M. The middle homology $V = H_n(M)$ admits a $(-1)^n$-symmetric bilinear form

$$\cdot : V \times V \to \mathbb{Z}$$

induced by intersection of n-cycles. Then $H = H_n(M;\mathbb{C}) = V \otimes \mathbb{C}$ admits a corresponding Hermitian form

$$\varphi : H \times H \to \mathbb{C}$$

where

$$\varphi(x \otimes \alpha, y \otimes \beta) = \begin{cases} \alpha\bar{\beta}(x \cdot y) & n \text{ even} \\ i\alpha\bar{\beta}(x \cdot y) & n \text{ odd} \end{cases}$$

Now by averaging the g-module H can be split as $H^+ \oplus H^-$, where φ is positive definite on H^+ and negative definite on H^-, and each of H^+ and H^- is invariant under g_*. We define

$$\operatorname{sign}(g, M) = \operatorname{trace}\,(g_*|H^+) - \operatorname{trace}\,(g_*|H^-)$$

If we think of H^+ and H^- as representations of $G = C_m$, then we may define $\operatorname{Sign}(G, M)$ to be the virtual representation $H^+ - H^-$, whose character, then, is $\operatorname{sign}(g, M)$.

One can actually allow manifolds with boundary, with the only difference being that the decomposition of H into positive and negative parts must also allow for an orthogonal summand H_0 on which the the Hermitian form vanishes.

This G-signature has formal properties very like similar properties of the ordinary signature.

- $\operatorname{sign}(G, -M) = -\operatorname{sign}(G, M)$
- $\operatorname{sign}(M \times N) = \operatorname{sign}(M)\operatorname{sign}(G, N)$
- If $(G, M) = (G, M_1)\sup(G, M_2)$, possibly identified along some components of the boundaries of M_1 and M_2, then $\operatorname{sign}(G, M) = \operatorname{sign}(G, M_1) + \operatorname{sign}(G, M_2)$
- If $(G, M) = \partial(G, W)$ for some manifold W, then $\operatorname{sign}(G, M) = 0$.

15.1.2. *Dimension 2.* Suppose that the cyclic group C_m, with chosen generator g, acts by orientation preserving homeomorphisms on a closed oriented surface S with isolated fixed points $P_1, \ldots, P_k$. Suppose that in a neighborhood of P_i g acts by rotation through an angle of $2\pi a_i/m$.

Theorem 15.1.

$$sign(g, S) = \sum_{i=1}^{k} \frac{\zeta^{a_i} + 1}{\zeta^{a_i} - 1} = -i \sum_{i=1}^{k} \cot \frac{\pi a_i}{m}$$

An equivalent trigonometric form is

$$\operatorname{sign}(g, S) = -i \sum_{i=1}^{k} \cot \frac{\pi a_i}{m}$$

15.1.3. *Dimension 4.* Suppose that the cyclic group C_m, with chosen generator g, acts by orientation preserving homeomorphisms on a closed oriented 4-manifold M with isolated fixed points $P_1, \ldots, P_k$ and 2-dimensional fixed components $F_1, \ldots F_\ell$. Suppose that in a neighborhood of P_i g acts by rotation through angles of $2\pi a_i/m$ and $2\pi b_i/m$. Suppose that in the normal bundle to F_j g acts by rotation by $2\pi c_j/m$.

Theorem 15.2.

$$sign(g, M) = \sum_{i=1}^{k} \frac{\zeta^{a_i} + 1}{\zeta^{a_i} - 1} \frac{\zeta^{b_i} + 1}{\zeta^{b_i} - 1} - \sum_{j=1}^{\ell} e(F_j) \frac{4\zeta^{c_j}}{(\zeta^{c_j} - 1)^2}$$

An equivalent trigonometric form is

$$\operatorname{sign}(g, M) = \sum_{i=1}^{k} \cot \frac{\pi a_i}{m} \cot \frac{\pi b_i}{m} + \sum_{j=1}^{\ell} e(F_j) \operatorname{cosec}^2 \frac{\pi c_j}{m}$$

15.1.4. *Dimension n.* There are similar results in higher dimensions, but the result is yet more complicated. We omit further details.

15.2. **Localization.** A tool especially useful for understanding actions of non-cyclic p-groups.

Suppose that $S \subset H^*(B_G) = H^*_G(point)$ is a multiplicatively closed subset of homogeneous elements. Let $\mathcal{F}(S)$ denote the family of subgroups of G given by

$$\mathcal{F}(S) = \{H < G : S \cap \ker(H^*_G(point) \to H^*_G(G/H)) \neq \emptyset\}$$

Theorem 15.3. *If X is a finite-dimensional G-CW-complex and $A \subset X$ be a G-invariant subcomplex. Let $S \subset H^*(B_G)$ be a multiplicatively closed subset of homogeneous elements. Assume $X - A$ has only finitely many orbit types, all of the form G/H, $H \in \mathcal{F}(S)$. Then the inclusion $A \to X$ induces an isomorphism*

$$S^{-1}H^*_G(X) \to S^{-1}H^*_G(A)$$

15.3. **Converse of Smith theory.**

Theorem 15.4 (L.Jones)**.** *Let F be a compact polyhedron that is $\mathbb{Z}/p$-acyclic. Then there is an action of C_p on a compact contractible polyhedron X, with fixed point set $X^{C_p} = F$.*

Theorem 15.5 (Edmonds-Lee)**.** *Let p and q be distinct odd primes. A closed manifold M is the fixed point set of a C_{pq} action on some Euclidean space if and only if the tangent bundle of M stably admits a complex structure.*

15.4. **Groups acting without fixed points on disks.** Deep theorem of R. Oliver characterizing groups that act on some compact disk without a fixed point. Easier result:

Theorem 15.6 (Edmonds-Lee)**.** *A compact Lie group acts smoothly on some Euclidean space without fixed points if and only if it is not an extension of a torus group by a finite p-group.*

15.5. **Group actions on products of spheres.**

15.6. **Representations at fixed points.**

15.6.1. *Some elementary representation theory.*

15.6.2. *Isolated fixed points.* How closely related must the representations of a group action on the neighborhoods of isolated fixed points be, for example on a sphere?

15.7. **Group actions on 3-manifolds.**

15.7.1. *The Smith Conjecture.* The classical Smith Conjecture is that if the fixed point set K of a finite group G acting smoothly (or piecewise linearly) on the 3-sphere is a simple closed curve, then it is unknotted.

Ironically counterexamples to the analogous conjecture in higher dimensions were found earlier. Gordon found nontrivial knotted 2-spheres in S^4 fixed by a finite cyclic action.

15.7.2. *Groups acting freely on S^3.* Milnor compiled the list of possible finite groups that might act freely on the 3-sphere. The ones that do not correspond to linear actions have received a lot of scrutiny. It is a consequence of the Geometrization Conjecture, claimed by Perelman, that all free actions on S^3 are equivalent to linear actions.

15.7.3. *Periodic Knots.* A knot is periodic if it is invariant under (not fixed by) a cyclic group action. Many papers have been written on the subject, finding invariants of periodic knots, calculating the possible periods, etc.

15.7.4. *Circle actions.* Raymond. Classification of circle actions on 3-manifolds via their weighted orbit spaces (surfaces).

15.8. **Group actions on 4-manifolds.**

15.8.1. *Torus actions.* Raymond. Classification of torus actions on 4-manifolds via their weighted orbit spaces (surfaces).

15.8.2. *Circle actions.* Pao, Fintushel. Classification of circle actions on 4-manifolds via their weighted orbit spaces (3-manifolds).

15.8.3. *Applications of gauge theory and Seiberg-Witten invariants.*

15.8.4. *Finite group actions.* Edmonds, Ewing, Hambleton, Lee, Wilczynski, McCooey, Tanase.

15.9. **Group actions on aspherical manifolds.**

15.9.1. *Work of Conner and Raymond.*

15.9.2. *Totally Asymmetric Manifolds.* We survey the results of a search for manifolds that admit no effective action whatsoever. This first such examples were connected sums of aspherical manifolds; then isolated examples of asymmetric aspherical manifolds in various higher dimensions; then aspherical 3-manifolds; then geometric examples in higher dimensions.

15.10. **Simply connected asymmetric manifolds.**

15.10.1. *Work of Volker Puppe.* Simply connected 6-manifolds with no nontrivial orientation preserving periodic maps.

15.10.2. *Work of Matthias Kreck.* Simply connected 6-manifolds with no nontrivial periodic maps at all.

Appendix A. Topology of Manifolds

An n-manifold is a (Hausdorff, second countable, paracompact) topological space M such that each point $x \in M$ has a neighborhood $U \subset M$ such that U is homeomorphic to an open subset of $\mathbb{R}^n$.

An n-manifold with (possible) boundary is a (Hausdorff, second countable, paracompact) topological space M such that each point $x \in M$ has a neighborhood $U \subset M$ such that U is homeomorphic to an open subset of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$. Sometimes for emphasis we include an exponent, as in M^n, to indicate the dimension of the manifold.

The *boundary* of an n-manifold M with boundary is

$$\partial M = \{x \in M : x \text{ has no neighborhood homeomorphic to an open set in } \mathbb{R}^n\}$$

Then ∂M^n is an $(n-1)$-manifold. The *interior* of M is $M - \partial M$.

Examples of manifolds:

(1) Any open subset of $\mathbb{R}^n$
(2) The n-sphere $S^n = \{x \in \mathbb{R}^n : |x| = 1\}$
(3) The product of two manifolds, e.g., the product of two spheres, is a manifold. (Dimension adds.)
(4) Surfaces of higher genus.
(5) Projective spaces.
(6) Connected sum of two n-manifolds is an n-manifold: Remove the interior of a small disk from each and identify the boundary spheres. E.g. a genus two surface is a connected sum of two tori.

Examples of manifolds with boundary:

(1) Any open subset of $\mathbb{R}^n_+$
(2) The n-disk $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$
(3) The product of two manifolds, one with boundary and one without, e.g., the product of a spheres and a disk, is a manifold with boundary. (Dimension adds.) More generally the product of two manifolds with boundary.
(4) Annulus, Möbius band.
(5) Any n-manifold with the interior of a (nice) little n-ball removed.

Theorem A.1 (Classification of surfaces)**.** *A compact connected* 2*-manifold (without boundary) is a sphere, torus, projective plane, or connected sum of these. The only relations are generated by*

(1) $S^2 \# M^2 \cong M^2$
(2) $P^2 \# T^2 \cong P^2 \# P^2 \# P^2$

Compact surfaces without boundary are determined up to homeomorphism by Euler characteristic and orientability.

Fact: Every 2-manifold can be triangulated, that is, expressed as (the topological realization of) a simplical complex. Massey gives the classification of surfaces, assuming they are triangulated.

In dimension 3 all manifolds can also be triangulated. But in higher dimensions there are manifolds that cannot be triangulated.

All differentiable manifolds admit compatible triangulations. But not all triangulable manifolds have compatible (or any) differentiable structure.

Theorem A.2 (Boundary collars)**.** *If M is an n-manifold with boundary, then there is an embedding of $\partial M \times [0,1) \to M$ (identity on $\partial M \times \{0\} \to \partial M$) onto a neighborhood of ∂M in M.*

Two homeomorphisms of a manifold are said to be *isotopic* if they are homotopic through homeomorphisms. An isotopy of a submanifold $N^n \subset M^m$ is a homotopy $H : N^n \times I \to M^m$ such that it is an embedding at each level. I.e. each map $H_t : N^n \to M^m$ defined by $H_t(x) = H(x,t)$ is an embedding, and H_0 is the inclusion map. If N^n is a locally flat submanifold of M^m, then any such isotopy extends to an *ambient isotopy*, that is an isotopy of the identity defined an all of M^m.

One important use of boundary collars is to show that the result of gluing two manifolds together via a homeomorphism of (parts of) their boundaries depends only on the gluing map up to isotopy.

Theorem A.3 (Homogeneity)**.** *If M is a connected n-manifold and $x, y \in M$, then there exists a homeomorphism $f : M \to M$ such that $y = f(x)$. The homeomorphism can be chosen to be the identity outside any connected open set containing x and y. More generally, if $n > 1$, then any finite set of points can be moved to any other set of the same number of points, under any given correspondence of the points, by a homemorphism that is isotopic to the identity.*

Theorem A.4 (Alexander Trick)**.** *Any homeomorphism $f : D^n \to D^n$ that is the identity on the boundary S^{n-1} is isotopic to the identity, rel S^{n-1}.*

If M and N are n-manifolds with boundary and $f : \partial M \to \partial N$ is a homemorphism, then $M \cup_f N$ is an n-manifold without boundary. More generally one can identify suitable *parts* of the boundaries to produce a new manifold with boundary. We will content ourselves with the description of adding a handle. Let M be an n-manifold with boundary. Consider the n-ball written as a product $D^k \times D^{n-k}$. consider an embedding $f : S^{k-1} \times D^{n-k} \to \partial M$. (We require that the embedding be "tame" or "flat.") Then
$$M \cup_f D^k \times D^{n-k}$$
is also an n-manifold with boundary.

An embedded submanifold $N^n \subset M^m$ is *locally flat* if each point $x \in N^n$ has a neighborhood $U \subset M^m$ homeomorphic to $V \times \mathbb{R}^{m-n}$, where V is a neighborhood of x in N^n and the homeomorphism is the identity as a map $V \times \{0\} \to V \subset N^n$. smooth submanifolds are automatically locally flat.

If $D^n \to M^n$ is an embedding, we say it is locally flat, if the restriction $S^{n-1} \to M^n$ is locally flat. It is locally flat on the "inside," of course. But the example of the Alexander Horned Sphere shows that it need not be locally flat on the outside. The argument for the existence of boundary collars shows that locally flat implies flat in codimension one.

Theorem A.5 (Annulus Conjecture, Hard, due to Kirby, 1969)**.** *If $D_0^n \subset$ Int D_1^n is a locally flat embedding of one n-disk in the interior of another ($n \neq 4$), then $cl(D_1^n - D_0^n) \cong S^{n-1} \times [0,1]$.*

The Annulus Theorem is true and much easier in the differentiable or PL categories.

Corollary A.6. *Two locally flat embedded n-disks ($n \neq 4$) in the interior of a connected n-manifold are isotopic.*

Sketch. By homogeneity we may assume the two disks have an interior point in common. Using an exterior collar neighborhood we may shrink one disk by an isotopy so it lies wholly in the interior of the other. The Annulus Theorem says the difference between the two is an annulus. Using an exterior collar and the annulus coordinates we may isotope the interior disk until it coincides with the bigger one. □

APPENDIX B. GROUP ACTIONS ON SETS

We consider actions of a finite group G on a finite set X. Basic definitions are the same as we had earlier, viewing X as a space with the discrete topology. The emphasis here is on counting sets, orbits, and isotropy groups, etc. This technique arises in elementary group theory. The statements here are quite standard; we adapted from Rotman [7], Section 2.7.

An action of G on X is *transitive* if there is only one orbit.

Proposition B.1. *If G acts on a set X, then X is the disjoint union of the orbits and*

$$|X| = \sum_i |G(x_i)|$$

where the sum extends over a set of orbit representatives, one from each orbit.

Proposition B.2. *If G acts on a set X, and $x \in X$, then*

$$|G(x)| = [G : G_x]$$

the index of the isotropy subgroup of x in G.

Corollary B.3. *If G acts on a set X, then the cardinality of any orbit divides $|G|$, the order of G.*

A particular example important in finite group theory is the action of G on itself by conjugation. In this case, the isotropy group $G_x = C_G(x)$, the centralizer of x in G.

Corollary B.4 (Class Equation). *If G is a finite group, then*

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where the sum extends over a set of conjugacy class representatives, one from each class having more than just one element.

The class equation leads to a proof that p-groups have nontrivial center. And group action considerations also provide a proof that the alternating groups A_n, $n \geq 5$, are simple.

Theorem B.5 ("Burnside's Lemma"). *If a finite group G acts on a finite set X with exactly n orbits, then*

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where $X^g = \{x \in X : gx = x\}$.

Note that this fixed point notation clashes a bit with traditional notation in group theory, where if $x, g \in G$, then $x^g = gxg^{-1}$, and $x^G = \{gxg^{-1} : g \in G\}$.

Burnside's Lemma has applications in various combinatorial counting problems.

APPENDIX C. REPRESENTATION THEORY

We refer to Serre [9]

C.1. **Definitions.** If G is a finite group and V is a vector space over a field k, let $GL(V)$ denote the group of automorphisms of V. Then any group homomorphism $\rho : G \to GL(V)$ is a (linear) *representation* of G over k. For the most part we will restrict attention here to the case of $k = \mathbb{C}$ or $\mathbb{R}$.

A useful way of thinking about representations is as $k[G]$-modules, where $k[G]$ denotes the group algebra of G with coefficients in k. A representation $\rho : G \to GL(V)$ makes V into a $k[G]$-module.

If we identify V with k^n by choosing a basis, then each $\rho(g)$ can be viewed as a matrix.

Two representations $\rho_i : G \to GL(V_i)$, $i = 1, 2$, are *equivalent* if there is a vector space isomorphism $f : V_1 \to V_2$ such that $\rho_2(s)(f(v_1)) = f(\rho_1(s)(v_1))$ for all $v_1 \in V_1$ and $s \in G$.

The *direct sum* of two representations $\rho_i : G \to GL(V_i)$, $i = 1, 2$ is given by $\rho_1 \oplus \rho_2 : G \to V_1 \oplus V_2$ in the usual way.

A representation V is *irreducible* if it has no nontrivial submodule. One shows that any invariant submodule W of a $k[G]$-module V has a G-invariant complement, that is reducible representations are *decomposable*. For this the field k must have characteristic 0 or prime to the order of G. Then any representation is equivalent to a direct sum of irreducible representations. As we will see, it turns out that there are only finitely many irreducible representations for any finite group, and they are all contained in the so-called *regular representation* $k[G]$.

C.2. **New representations from old.** Let $\rho : G \to GL(V)$ is a representation.

C.2.1. *Restriction to a subspace.* Suppose W is a G-invariant subspace of V. Then there is a corresponding representation $\bar{\rho} : G \to GL(W)$ called the *restriction* of ρ to W. Sometimes we denote $\bar{\rho} = \rho_{|W}$.

C.2.2. *Restriction to a subgroup.* If H is a subgroup of G we also have a representation ρ_H of H given simply by restricting ρ to the subgroup. We might denote this as $\mathrm{Res}_H^G(\rho)$.

C.2.3. *Inflation.* Suppose Q is a quotient group of G and $\rho : Q \to GL(V)$ is a representation of Q. Then we can obtain a representation $\hat{\rho}$ of G simply by composing ρ with the quotient map $\pi : G \to Q$.

C.2.4. *Induction.* Suppose that H is a subgroup of G of index n, and that $\rho : H \to GL(V)$ is a given representation of H. Then we define the *induced representation* of G on the vector space $\mathrm{Maps}(G/H, V)$ by the following: If $\phi : G/H \to V$ and $s \in G$, define

$$(s \cdot \phi)(xH) = \phi(xs^{-1}H).$$

In terms of modules the induced representation can be written as $V \otimes_{k[H]} k[G/H]$. We will sometimes denote this representation by $\mathrm{Ind}_H^G(\rho)$.

C.3. **Examples.** Here we give direct, explicit descriptions of some standard representations, especially of groups that arise as groups of symmetries of regular polyhedra. One can use the theory of characters as described in the following two sections to see that we have described all the irreducible complex representations of these groups.

C.3.1. *Trivial representation.* For any group G this is the constant map $\rho_0 : G \to GL(V)$, where $\rho_0(s) = \mathrm{id}_V$ for all $s \in G$. This is reducible unless $\dim V = 1$.

C.3.2. *Regular representation.* For any group G this is the module $V = k[G]$. It has dimension, or *degree*, equal to $g = |G|$, the order of G. It turns out that all irreducible reresentations of G occur inside $k[G]$, with predictable multiplicity.

C.3.3. C_2. The cyclic group of order 2, with generator T, has two representations of degree 1, namely, the trivial representation, and the representation where T acts by multiplication by -1. It is an elementary exercise to see that any representation of C_2 is a sum of copies of these two representations. In particular $k[C_2]$ is equivalent to the sum of these two representations, for any field of characteristic different from 2. Over a field of characteristic 2, this is not true.

C.3.4. C_n. The cyclic group of order 2. Over the complex numbers we have exactly n 1-dimensional representations, where a fixed generator $T \in C_n$ goes to multiplication by any given nth root of unity. One can see that these nontrivial representations define degree 2 irreducible real representations (unless n is even and the generator goes to ± 1). Over the real numbers these representations correspond to sending the generator T to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta = 2\pi k/n$.

C.3.5. D_n. The dihedral group of order $2n$, with presentation

$$\left\langle S, T : S^n, T^2, TSTS \right\rangle$$

There will be two cases, depending whether n is even or odd.

- n even. In this case any combination of multiplication by ± 1 can be assigned to S and T independently, yielding 4 degree 1 representations. On the other hand if ω is any complex nth root of unity and h is any integer, then the assignments

  $$S^k \to \begin{pmatrix} \omega^{hk} & 0 \\ 0 & \omega^{-hk} \end{pmatrix} \quad \text{and} \quad TS^k \to \begin{pmatrix} 0 & \omega^{-hk} \\ \omega^{hk} & 0 \end{pmatrix}$$

  define a degree 2 representation ρ^h. These are already covered by the preceding for $h = 0$ or $h = n/2$. But for $0 < h < n/2$, we get $n/2 - 1$ representations of degree 2, an addition to the four of degree 1.
- n odd. In this case one only has two degree 1 representations defined by $S \to 1$ and $T \to \pm 1$. One still has the representations ρ^h, for $0 < h \leq (n-1)/2$.

C.3.6. S_3. The symmetric group of degree 3 and order 6. Of course this is the same as D_3. We describe the irreducible representations slightly differently as follows: We have the trivial representation; we have the inflation of the nontrivial degree 1 representation of the quotient C_2. and we have a nontrivial representaiton of the normal C_3 of degree 1, induced up to a degree 2 representation.

C.3.7. S_4. The symmetric group of degree 4 and order $4! = 24$. It can also be thought of as the full group of symmetries of a regular tetrahedron, or the group of orientation-preserving symmetries of the cube or octohedron. The latter description defines a degree 3 representation that can easily be seen to have no invariant subspace, hence be irreducible. Also the permutation representation defined on $\mathbb{C}^4$ by permuting coordinates, splits into a trivial representation (spanned by the sum of the permuted basis vectors) and an irreducible degree 3 representation.

C.3.8. A_4. The alternating group of degree 4 and order 12, also the group of orientation preserving symmetries of the tetrahedron. The action of A_4 on the tetrahedron defines a degree 3 representation that can be seen to have no invariant subspace, hence be irreducible. The group A_4 has a normal subgroup isomorphic to $C_2 \times C_2$, with quotient group C_3. As such there are degree 2 representation obtained by inflation from representations of C_3. Finally, there are degree 3 representations induced from degree 1 representations of $C_2 \times C_2$.

C.3.9. A_5. The alternating group of degree 5 and order 60, the group of orientation preserving symmetries of the regular icosahedron or dodecahedron, which describes an irreducible degree 3 complex representation (but defined over the reals). The permutations of 5 letters give a degree 5 representation containing a degree 1 trivial representation; its degree 4 complement is seen to have no invariant subspaces, hence be irreducible. Character theory, as described below, points to an irreducible degree 5 representation, which we need to describe explicitly.

C.4. **Characters and their properties.** If $\rho : G \to GL(V)$, then its character is defined to be the function $\chi_\rho : G \to k$ given by $\chi_\rho(s) = \mathrm{Tr}(M_{\rho(s)})$, where $M_{\rho(s)}$ is the matrix of the automorphism $\rho(s)$ with respect to some basis of V.

Here are some immediate properties that follow from well-known aspects of elementary matrix theory:

(1) χ_ρ is well-defined, independent of choice of basis.
(2) $\chi(1) = \dim(V) = \deg(\rho)$.
(3) $\chi(tst^{-1}) = \chi(s)$ for all $s, t \in G$. That is, χ is a "class function" on G.
(4) Equivalent representations have the same character.
(5) The character of a direct sum of representations is the sum of the characters of the individual representations.
(6) The character of a tensor product (not defined here) is the product of the characters.

Theorem C.1. *Representations with the same character are equivalent.*

Theorem C.2. ***Orthogonality Relations***

(1) *If χ_1 and χ_2 are two characters for diffferent irreducible representations, then*

$$\boxed{\sum_{s\in G} \chi_1(s)\chi_2(s)^* = 0}$$

(2) *If χ is the character of any irreducible representation, then*

$$\boxed{\sum_{s\in G} \chi(s)\chi(s)^* = |G|}$$

(3) *If $\chi_1, \ldots, \chi_k$ is the full list of all characters of irreducible represenations of G and $s \in G$ is fixed, then*

$$\boxed{\sum_{i=1}^{k} \chi_i(s)\chi_i(s)^* = |G|/|Conj(s)|}$$

where $Conj(s)$ denotes the conjugacy class of s.

A special case of the last statement above arises when one considers the characters evaluated on the identity element of the group.

Corollary C.3. *If $n_1, \ldots, n_k$ denote the degrees of the irreducible representations of G, then*

$$\boxed{\sum_{i=1}^{k} n_i^2 = |G|}$$

Remark 3. One can expand the idea of the orthogonality relations for characters to define an innner product on the space of all class functions on G by setting

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{s \in G} \phi(s) \psi(s)^*$$

It turns out that this defines a positive definite inner product and that the characters of the irreducible representations form an orthonormal basis.

The point of all of these formulas is that with some partial knowledge of representations of G one can often use this additional information to determine the rest of the characters of G.

C.5. **Complex character tables.** Here we record the character tables for some basic finite groups associated with symmetries of regular polyhedra in three dimensions, as described in an earlier section.

The top row of each table lists representatives of each conjugacy class. The second row lists the number of elements in each of the conjugacy classes, $\mathrm{conj}(s) = |G|/|C(s)|$, where $C(s)$ denotes the centralizer of s in G. Finally, each subsequent row lists the values on each conjugacy class of one of the characters of irreducible representations. Note that the orthogonality relations imply that the rows are orthonormal and the columns are also formally orthogonal with the column corresponding to $s \in G$ having norm $1/|C(s)|$.

C.5.1. C_n.

C_n	1	$T^k(1 \leq k \leq n-1$
$\mathrm{conj}(c)$	1	1
$\chi_j\ (0 \leq j \leq n-1)$	1	ω^{jk}

C.5.2. D_n. We must handle the cases when n is even and when n is odd separately.

$D_n(n \text{ even})$	1	$S^k(1 \leq k \leq n/2$	T	TS
$\mathrm{conj}(c)$	1	2	$n/2$	$n/2$
ψ_1	1	1	1	1
ψ_2	1	1	-1	-1
ψ_3	1	$(-1)^k$	1	-1
ψ_4	1	$(-1)^k$	-1	1
$\chi_j\ (0 < j < n/2)$	2	$\omega^{jk} + \omega^{-jk}$	0	0

$D_n(n \text{ odd})$	1	$S^k(1 \leq k \leq (n-1)/2$	T
$\mathrm{conj}(c)$	1	2	n
ψ_1	1	1	1
ψ_2	1	1	-1
$\chi_j\ (0 < j < n/2)$	2	$\omega^{jk} + \omega^{-jk}$	0

C.5.3. S_3. This is just D_3, which is given above.

C.5.4. A_4.

A_4	1	(12)(34)	(123)	(132)
conj(c)	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

C.5.5. S_4.

S_4	1	(12)	(12)(34)	(123)	(1234)
conj(c)	1	6	3	6	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

C.5.6. A_5.

A_5	1	(12)(34)	(123)	(12345)	(12354)
conj(c)	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	$-\alpha$	$-\beta$
χ_3	3	-1	0	$-\beta$	$-\alpha$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

C.6. **Orthogonal and unitary groups.** The canonical Gram-Schmidt orthogonalization process defines a strong deformation retraction of the general linear group $\mathrm{GL}(\mathbb{R}^n)$ into its compact subgroup $\mathrm{O}(n)$. Alternatively put, if V is an $\mathbb{R}[G]$-module, then a standard (but not equivariant) positive definite inner product on V can be averaged out over the group to yield a G-invariant positive definite inner product. But any positive definite inner product is equivalent to the standard one. This shows that the given G action on V is equivalent to an orthogonal action. Similarly a complex representation $G \to \mathrm{GL}(\mathbb{C}^n)$ is equivalent to one taking values in the unitary group $U(n)$ of matrices preserving a standard hermitian form.

Here are several well-known facts about the orthogonal groups, especially those in low dimensions:

(1) The group $\mathrm{SO}(n)$ acts transitively on the sphere S^{n-1} with isotropy group (conjugate to) $SO(n-1)$.
(2) The group $\mathrm{SO}(n)$ has dimension $n(n-1)/2$
(3) The group $\mathrm{SO}(2)$ is isomorphic to the multiplicative group S^1 of unit length complex numbers.
(4) The group $\mathrm{SO}(3)$ is homeomorphic to the real projective 3-space $\mathbb{R}P^3$. Its universal covering group $\mathrm{Spin}(3)$ is isomorphic to the group S^3 of unit quaternions. Define an action of S^3 on quaternions by conjugation. This preserve the purely imaginary 3-plane and defines the required 2 to 1 map $S^3 \to \mathrm{SO}(3)$.
(5) The quotient $\mathrm{SO}(4)/\{\pm I\}$ is isomorphic to $\mathrm{SO}(3) \times \mathrm{SO}(3)$. The action of $\mathrm{SO}(4)$ on $\mathbb{R}^4$ extends to one on $\Lambda^2\mathbb{R}^4$. The latter splits into eigenspaces for the Hodge star operator into two orthogonal 3-planes of self-dual and anti-self-dual 2-forms, preserved by the action of $\mathrm{SO}(4)$, and thus defining the required 2-to-1 map $\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$.

Here are a few observations relevant to finding the lowest dimensional sphere on which a particular group acts:

(1) The group C_2^n embeds in $\mathrm{SO}(n+1)$, but not in $\mathrm{SO}(n)$.
(2) If p is odd, then the group C_p^n embeds in $\mathrm{SO}(2n)$, but not in $\mathrm{SO}(2n-1)$.
(3) The group A_6 embeds in $\mathrm{SO}(5)$, but not in $\mathrm{SO}(4)$.

C.7. **Real representations from complex representations.** Here we will explain how to determine all irreducible real representations of a finite group, given knowledge of the complex representations.

APPENDIX D. COHOMOLOGY OF GROUPS

A nice reference is [5] We generally restrict attention to finite groups, however.

We take a topological point of view and proceed as follows. Given a group G there is a connected space $K(G,1)$ with $\pi_1 = G$ and $\pi_k = 0$ for $k \geq 2$. If M is any $\mathbb{Z}[G]$-module, then we define

$$H^k(G;M) := H^k(K(G,1);M)$$

where on the right singular cohomology and local coefficients are understood. Homology is similarly defined.

Alternatively one bypasses the topology and directly constructs a free or projective resolution P_* of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module, (that would correspond to the cellular chain complex of the universal covering of a $K(G,1)$-space):

$$\cdots \to P_3 \to P_2 \to P_1 \to \mathbb{Z} \to 0$$

and defines

$$H^k(G;M) := H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(P_*, M)) \text{ and } H_k(G;M) := H_k(P_* \otimes_{\mathbb{Z}[G]} M)$$

The space $K(G,1)$ is determined up to (based) homotopy equivalence. More generally, the resolution is well-defined up to chain homotopy equivalence. It follows that the homology and cohomology groups are well-defined.

In the case $G = C_n$, with generator $T \in G$, there is a simple and explicit resolution

$$\cdots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{D} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{D} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $N = \sum T^i$ (i.e., multiplication by this element of the group ring), $D = 1 - T$, and $\varepsilon(a_i T^i) = \sum a_i$ is the augmentation map. To compute $H_*(C_n;\mathbb{Z})$, just tensor with $\mathbb{Z}$, yielding the complex (after dropping the right hand $\mathbb{Z}$)

$$\cdots \xrightarrow{N} \mathbb{Z} \xrightarrow{D} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{D} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

It follows that

$$H_k(C_n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_n & \text{if } k > 0, \text{ odd} \\ 0 & \text{if } k > 0, \text{ even} \end{cases}$$

This provides the prototype for a finite group of periodic (co)homology. The groups of periodic cohomology of period 2 are precisely the cyclic groups.

APPENDIX E. TOPOLOGICAL GROUPS

A *topological group* is a group G that is endowed with a Hausdorff topology such that the group operations

$$\text{multiplication } \mu : G \times G \to G \quad \text{and} \quad \text{inversion } \iota : G \to G$$

are continuous. The Hausdorff assumption implies that a finite topological group is always discrete.

The main examples for us are $\mathrm{GL}_n(\mathbb{R})$, topologized as a subspace of $\mathbb{R}^{n^2}$ in the usual way, and its closed subgroups (with the induced topology).

Hilbert's Fifth Problem was a long-standing question in the area, asking for a characterization of Lie groups via transformation groups, where a group acting on a manifold must be a Lie group.

Gleason proved that a finite-dimensional, locally Euclidean group is a Lie group. Montgomery and Zippin showed that a locally compact, locally connected, finite-dimensional topological group is a Lie group.

An important step was to show that there is a neighborhood of the identity in G which contains no entire non-trivial subgroup of G.

There is a remaining open problem in this area, now known as the Hilbert-Smith Conjecture. There are several formulations. One is that no p-adic group can act effectively on a manifold. Another equivalent version is that no compact manifold M^m admits a homeomorphism $h : M^m \to M^m$ such that (1) Each orbit $\{h^n(z)\}$ is "small", and (2) $\{h^n : n \in \mathbb{Z}\}$ is relatively compact in the compact-open topology on $\mathrm{Homeo}(M^m)$. The question is can one of these compact, locally disconnected groups act effectively on a manifold.

It is a long standing conjecture in topology that if a compact group acts effectively on a manifold, then the group must be a Lie group–that is, the group itself must be locally euclidean. A version of this was proved by Montgomery-Zippin, but the most general statement still seems to be open. It comes down to a question whether certain weird groups like p-adic numbers or solenoids, or other inverse limit groups can act on a (finite-dimensional) manifold.

APPENDIX F. COMPACT LIE GROUPS

A nice reference is Adams, [1]. We satisfy ourselves with a few basic definitions and fundamental facts.

A Lie group is a smooth (C^∞) n-manifold G that is also a group, with multiplication map $\mu : G \times G \to G$ and inverse map $\iota : G \to G$ such that both μ and ι are smooth.

We are mainly interested in compact Lie groups. Examples:

(1) Tori $T^n = S^1 \times S^1 \times \cdots \times S^1$
(2) The orthogonal group $O(n)$ and its connected component of the identity the special orthogonal group $SO(n)$. The orthogonal group consists of all real $n \times n$ matrices A such that $AA^T = I$. These are the matrices that preserve the standard inner product on $\mathbb{R}^n$. Such matrices have determinant ± 1.The special orthogonal group is the subgroup consisting of orthogonal matrices of determinant $+1$.
(3) The unitary group $U(n)$ and the special unitary group $SU(n)$, the complex analogs of the orthogonal groups. The unitary group consists of complex $n \times n$ matrices A such that

$AA^* = I$, where A^* is the conjugate transpose of A. The special unitary group is the subgroup consisting of matrices of determinant 1, not just norm 1.
(4) The spinor group Spin(n), the 2-fold covering group of $SO(n)$, simply connected for $n \geq 3$.
(5) Some exceptional Lie groups, including G_2 and E_8.

Every compact Lie group G contains a maximal torus subgroup $T \subset G$, which plays a role somewhat analogous to that of the Sylow subgroups in finite group theory. Any two maximal tori are conjugate; and a compact connected Lie group is the union of its maximal tori.

If $T \subset G$ is a maximal torus in a compact connected Lie group G, then the group $W = N_G(T)/T$ is a finite group called the *Weyl group* of G. Note that W acts on T by conjugation in G, and hence acts on $H^*(T)$.

It turns out that $H^*(G) = H^*(T)^W$, at least with rational coefficients. One calculates $H^*(T)$ from the Künneth formula as an exterior algebra in k variables of degree 1, where $k = \dim T$. Then one needs to understand the action of W.

Similarly for complex representation rings: $R(G) = R(T)^W$. One understands $R(T)$ as being a polynomial algebra generated by basic 1-dimensional representations corresponding to the individual circle factors. Again one needs to understand the action of the Weyl group.

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