# INTRODUCTION TO COMPACT TRANSFORMATION GROUPS

Volume 46

Glen E. Bredon

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This is Volume 46 in
PURE AND APPLIED MATHEMATICS
A series of Monographs and Textbooks
Editors: PAUL A. SMITH AND SAMUEL EILENBERG
A complete list of titles in this series appears at the end of this volume

# INTRODUCTION TO COMPACT TRANSFORMATION GROUPS

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1972

ACADEMIC PRESS New York and London

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

*United Kingdom Edition published by*
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 75-180794

AMS(MOS) 1970 Subject Classification: 57E10

PRINTED IN THE UNITED STATES OF AMERICA

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# PREFACE

In topology, one studies such objects as topological spaces, topological manifolds, differentiable manifolds, polyhedra, and so on. In the theory of transformation groups, one studies the symmetries of such objects, or generally subgroups of the full group of symmetries. Usually, the group of symmetries comes equipped with a naturally defined topology (such as the compact-open topology) and it is important to consider this topology as part of the structure studied. In some cases of importance, such as the group of isometries of a compact riemannian manifold, the group of symmetries is a compact Lie group. This should be sufficient reason for studying *compact* groups of transformations of a space or of a manifold. An even more compelling reason for singling out the case of compact groups is the fact that one can obtain many strong results and tools in this case that are not available for the case of noncompact groups. Indeed, the theory of compact transformation groups has a completely different flavor from that of noncompact transformation groups.

There has been a good deal of research done on this subject in recent years, as a glance at the bibliography will show. This has convinced us of the need for a reasonably extensive introduction to the subject which would be comprehensible to a wide range of readers at the graduate level.

The main obstacle to the writing of a successful introduction to this subject is the fact that it draws on so many disparate parts of mathematics. This makes it difficult to write such an introducton which would be readable by most second-year graduate students, which would cover a large portion of the subject, and which would also touch on a good amount of interesting nontrivial mathematics of current interest. To overcome this obstacle, we have endeavored to keep the prerequisites to a minimum, especially in early parts of the book. (This does not apply to all of Chapter 0. For a reader with minimal background, we recommend the reading of the first three sections of that chapter, then skipping to Chapter I, with a return to parts of Chapter 0 when needed. Many readers would do well to skip Chapter 0 altogether.)

An indispensable prerequisite for reading this book is a first course in algebraic topology. The requirements in this direction are fairly minimal until the last half of Chapter III, where some Čech theory is needed. Poincaré duality is not used until Chapter IV, and spectral sequences appear only in Chapter VII. A considerable saving in the algebraic topological demands on the reader results from the fact that we do not consider the theory of generalized manifolds in this book. There is, of course, a resulting loss in the generality of some of the theorems, but we believe this is minimal. (Most current interest is in the case of smooth or locally smooth actions, and there the loss is practically nonexistent.)

Although we are almost entirely concerned with actions of compact *Lie* groups in this book, there is really very little about Lie groups which the reader needs to know, outside of a few simple facts about maximal tori which we develop in Chapter 0, Section 6. This results from the fact that we concentrate on those theorems for which the classification theory of compact Lie groups, detailed case by case calculations in representation theory, and similar considerations are not needed. (This restriction on the subject matter is also made for purely esthetic reasons.)

We have endeavored to minimize the prerequisite background in differential geometry by not treating smooth actions until Chapter VI. (Actually, much of the earlier parts of the book concern actions which are smooth, and we occasionally comment on this aspect for the benefit of readers, probably a majority, who understand it.) A major part of the book (Chapters IV and V) is devoted to the study of what we call locally smooth actions. This is a category of actions which we feel is the correct analogue of the category TOP of topological manifolds. Roughly speaking, it lies between the category of smooth actions on differentiable manifolds and that of continuous actions on topological manifolds, but avoids many of the pathological aspects of both of these categories. The consideration of locally smooth actions was partially prompted by our desire to delay the discussion of smooth actions and, on the other hand, to disregard the theory of generalized manifolds. However, locally smooth actions should be regarded as a basic subject for study (second only, in interest, to smooth actions), and should lead to interesting results in the future.

Since the theory of compact transformation groups can largely be regarded as a generalization of the theory of fiber bundles, it is not surprising that we make use of some elementary bundle theory in this book. The necessary background material in this direction is developed as needed.

The main activity in recent years on compact transformation groups has concerned smooth actions. Many of the recent results in this direction fall outside our scope because of the background restrictions we have placed on the book. It is also clear that many such results will be obsolete by publication time and we have ruled out their inclusion for this reason. A major area in the theory of smooth actions is occupied by the well-developed cobordism theory of group actions. We do not consider this theory here because of the prior existence of an excellent introduction to it in the pioneering monograph of Conner and Floyd [8].

The background demands on the reader are gradually increased throughout the book. This is especially true of the last four sections of Chapter VI and of parts of Chapter VII.

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## ACKNOWLEDGMENTS

I would like to acknowledge my indebtedness to the book of Montgomery and Zippin [4], which provided my own introduction to the subject. The major part of that book was devoted to the solution (by Gleason, Montgomery, and Zippin) of Hilbert's fifth problem on the structure of locally compact groups, a subject which we do not consider here. For this reason, and because of the numerous advances in transformation groups since its publication, there is very little overlap between it and this book. Many mathematicians owe a great deal to the friendship, encouragement, and help extended to them over the years by Professor Montgomery, and I am not the least of them.

I also wish to extend my thanks to Otto Morphy, who typed the entire manuscript and corrected the errors. All remaining mistakes, mathematical and otherwise, are his sole responsibility.

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# CHAPTER 0

## BACKGROUND ON TOPOLOGICAL GROUPS AND LIE GROUPS

In this chapter, we present some background material on topological groups, the classical groups, and compact Lie groups. Much of the material in this chapter is used only rarely in later chapters, so that it is not at all necessary for the reader to be familiar with all of the results presented here. In fact, readers with a scant background would probably be wise to read the first three sections of this chapter and then pass directly to Chapter I, returning to this chapter only when needed. Several of the proofs are omitted, when giving them would take us too far afield, but all the results are readily available in a number of standard sources on Lie groups, such as Chevalley [1], Hochschild [1], and Pontriagin [1].

## 1. ELEMENTARY PROPERTIES OF TOPOLOGICAL GROUPS

A **topological group** is a Hausdorff space $G$ together with a continuous multiplication $G \times G \to G$ [usually denoted by juxtaposition $(g, h) \mapsto gh$] which makes $G$ into a group, and such that the map $g \mapsto g^{-1}$ of $G \to G$ is continuous. The identity element of $G$ will usually be denoted by $e$.

For $g \in G$ there is the **left translation** map $L_g$: $G \to G$ defined by $L_g(h) = gh$. Clearly $L_{gh} = L_g \circ L_h$ and $L_{g^{-1}} = (L_g)^{-1}$ and it follows that each $L_g$ is a homeomorphism of $G$ onto itself. Similarly, right translation $R_g$: $G \to G$ defined by $R_g(h) = hg^{-1}$ satisfies $R_g \circ R_h = R_{gh}$ and $R_{g^{-1}} = (R_g)^{-1}$, and each $R_g$ is a homeomorphism.

For subsets $A$ and $B$ of $G$, we denote by $AB$, the subset $\{ab \mid a \in A,\ b \in B\}$ of $G$; that is, the image of $A \times B$ under multiplication $G \times G \to G$. Similarly, $A^{-1} = \{a^{-1} \mid a \in A\}$, and $A^n = \{a_1 a_2 \cdots a_n \mid a_i \in A\}$.

Since $L_g$ is a homeomorphism we see that $gU$ is a neighborhood of $g$ iff $U$ is a neighborhood of $e$. Moreover, if $U$ is a neighborhood of $e$, then so are $U^{-1}$ and $U \cap U^{-1}$. A neighborhood $U$ of $e$ is called **symmetric** if $U = U^{-1}$, and it follows that the symmetric neighborhoods of $e$ form a neighborhood basis at $e$ and this basis completely describes the topology of $G$.

From continuity of the multiplication map $G \times G \times G \to G$ at the point $(e, g, e)$, we obtain the following lemma.

**1.1. Lemma** *Given $g \in G$ and a neighborhood $U$ of $g$, there exists a neighborhood $V$ of $e$ with $VgV \subset U$.* ▮

Similarly, from multiplication $G \times \cdots \times G \to G$ we have the following lemma.

**1.2. Lemma** *Given a neighborhood $U$ of $e$ and an integer $n$ there exists a neighborhood $V$ of $e$ with $V^n \subset U$.* ▮

**1.3. Proposition** *If $H$ is a subgroup of a topological group $G$, then $\bar{H}$ is also a subgroup. If $H$ is normal, then so is $\bar{H}$.*

*Proof* Let $\mu\colon G \times G \to G$ be $\mu(g, h) = gh^{-1}$. Then

$$\mu(\bar{H} \times \bar{H}) = \mu(\overline{H \times H}) \subset \overline{\mu(H \times H)} = \bar{H}$$

so that $\bar{H}$ is a subgroup. If $H$ is normal, then, since $L_g R_g\colon G \to G$ taking $h \mapsto ghg^{-1}$ is a homeomorphism, we have

$$g\bar{H}g^{-1} = L_g R_g(\bar{H}) = \overline{L_g R_g(H)} = \bar{H}$$

so that $\bar{H}$ is normal. ▮

**1.4. Proposition** *Let $H$ be a closed subgroup of $G$. Then the space $G/H$ of left cosets $gH$ of $H$ in $G$, with the quotient topology induced by the canonical map $\varphi\colon G \to G/H$, is a Hausdorff space and $\varphi$ is continuous and open.*

*Proof* The map $\varphi$ is continuous by definition and it is open since $\varphi^{-1}\varphi(U) = UH = \bigcup_{h\in H} Uh$ is open for all open $U \subset G$. To show that $G/H$ is Hausdorff, suppose that $g_1H \neq g_2H$; that is $g_1^{-1}g_2 \notin H$. Since $H$ is closed, 1.1 implies that there is a *symmetric* neighborhood $U$ of $e$ with $(Ug_1^{-1}g_2U) \cap H = \varnothing$. Thus $g_1^{-1}g_2U \cap UH = \varnothing$ which implies that $g_2U \cap g_1UH = \varnothing$ and hence that $g_2UH \cap g_1UH = \varnothing$. However, $\varphi(g_iU) = g_iUH$ are then disjoint open sets in $G/H$ containing $g_1H$ and $g_2H$. ▮

**1.5. Proposition** *If $H$ is a closed normal subgroup of a topological group $G$, then $G/H$ is a topological group.*

*Proof* Consider the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\eta} & G \\ \downarrow \psi & & \downarrow \varphi \\ (G/H) \times (G/H) & \xrightarrow{\mu} & G/H \end{array}$$

where $\psi = \varphi \times \varphi$, $\mu(g_1H, g_2H) = g_1^{-1}g_2H$, and $\eta(g_1, g_2) = g_1^{-1}g_2$. It suffices to show that $\mu$ is continuous. However, for $W \subset G/H$ open, we have

$$\mu^{-1}(W) = \psi\psi^{-1}\mu^{-1}(W) = \psi\eta^{-1}\varphi^{-1}(W)$$

which is open since $\varphi$ and $\eta$ are continuous and $\psi = \varphi \times \varphi$ is open. ∎

**1.6. Proposition** *If $G$ is a topological group and $G_0$ is the component of $G$ containing the identity $e$, then $G_0$ is a closed normal subgroup of $G$.*

*Proof* Since components are closed it suffices to show that $G_0$ is a subgroup and is normal. However, $G_0 \times G_0$ is connected and hence $G_0G_0^{-1}$ (its continuous image) is connected and contains $e$. Thus $G_0G_0^{-1} \subset G_0$ which means that $G_0$ is a subgroup of $G$. If $\omega$: $G \to G$ is any automorphism of $G$ which is also a homeomorphism, then $\omega(G_0)$ is a component of $G$ and contains $e$, whence $\omega(G_0) = G_0$. In particular, this holds for the inner automorphisms $L_gR_g$, and hence $G_0$ is normal. ∎

If $G$ and $G'$ are topological groups, then by a **homomorphism** $\varphi$: $G \to G'$ we mean a homomorphism of groups which is also continuous. The reader may supply the easy details of the following result.

**1.7. Proposition** *If $\varphi$: $G \to H$ is a surjective homomorphism of topological groups, then $K = \ker \varphi$ is a closed normal subgroup of $G$ and the induced map $\varphi'$: $G/K \to H$ is continuous and bijective. If $G$ is compact, then $\varphi'$ is open, and hence is an isomorphism of topological groups.* ∎

We now turn to some special facts for compact groups.

**1.8. Proposition** *Let $G$ be a compact group and let $g \in G$. Let $A = \{g^n \mid n = 0, 1, 2, \ldots\}$. Then $\bar{A}$ is a subgroup of $G$.*

*Proof* Put $B = \{g^n \mid n \in \mathbf{Z}\}$ which is a subgroup of $G$. Then $\bar{B}$ is also a subgroup of $G$ by 1.3. If $e$ is isolated in $\bar{B}$, then $\bar{B}$ is compact and discrete and hence finite and we must then have $g^n = e$ for some $n > 0$. Thus suppose that $e$ is not isolated in $\bar{B}$. Then for any symmetric neighborhood $U$ of $e$ in $G$ there is an integer $n \neq 0$ with $g^n \in U$. Since $U$ is symmetric we may suppose that $n > 0$. Then $g^{n-1} \in (g^{-1}U) \cap A$. Since the $g^{-1}U$ form a neighborhood basis at $g^{-1}$ it follows that $g^{-1} \in \bar{A}$ and this clearly implies that $\bar{A} = \bar{B}$. ∎

If $H$ is a subgroup of $G$, then the normalizer $N(H)$ of $H$ in $G$ is the subset of $G$ consisting of those elements $g \in G$ such that $gHg^{-1} = H$. This is clearly a subgroup of $G$ and is closed when $H$ is closed.

**1.9. Proposition** *If $G$ is a compact group and $H \subset G$ is a closed subgroup, then $gHg^{-1} = H$ [i.e., $g \in N(H)$] iff $gHg^{-1} \subset H$.*

*Proof* Let $\varphi$: $G \times G \to G$ be $\varphi(g, k) = gkg^{-1}$. Suppose that $g \in G$ is such that $gHg^{-1} \subset H$ and let $A = \{g^n \mid n = 0, 1, 2, \ldots\}$ as in 1.8. Then $\varphi(A \times H) \subset H$ and continuity of $\varphi$ implies that $\varphi(\bar{A} \times H) \subset H$. By 1.8 we have that $g^{-1} \in \bar{A}$ and hence the latter inclusion implies that $g^{-1}Hg \subset H$. Thus $H \subset gHg^{-1}$. ∎

**1.10. Proposition** *If $G$ is a compact group, then every neighborhood $U$ of $e$ in $G$ contains a neighborhood $V$ of $e$ which is invariant under conjugation.*

*Proof* Let $\varphi$: $G \times G \to G$ be $\varphi(g, k) = gkg^{-1}$. If $U$ is an open neighborhood of $e$, then $G - U$ is compact and $\varphi(G \times (G - U))$ is a compact set whose complement $V \subset U$ is open, invariant, and contains $e$. ∎

We remark that 1.9 and 1.10 would be false in general without the compactness assumption (or some other assumption).

**1.11. Proposition** *If $N$ is a totally disconnected normal subgroup of a connected topological group $G$, then $N$ is central in $G$.*

*Proof* For $k \in N$ the map $\psi_k$: $G \to N$, defined by $\psi_k(g) = gkg^{-1}$, is continuous. Since $G$ is connected, $\psi_k(G)$ is connected. Since $N$ is totally disconnected, $\psi_k(G)$ must consist of exactly one point $\{k\}$. Thus $gkg^{-1} = k$ for all $g \in G$ and $k \in N$. ∎

The following fact is an elementary exercise that we leave to the reader.

**1.12. Proposition** *Let $G$ be a topological group and $H \subset G$ a closed subgroup. If $H$ and $G/H$ are connected, then so is $G$.* ▌

## 2. THE CLASSICAL GROUPS

Let $K$ stand for either the field **R** of real numbers or the field **C** of complex numbers. For an $n \times n$ matrix $A$ over $K$ we denote by $A'$ its transpose, by $\bar{A}$ its complex conjugate, and let $A^* = \bar{A}'$. Let $\mathbf{M}_n(K)$ denote the algebra of all $n \times n$ matrices over $K$ and note that this is a $K$-vector space of dimension $n^2$. The group, under multiplication, of nonsingular $n \times n$ matrices over $K$ is called the **general linear group** $\mathbf{Gl}(n, K)$ and the subgroup of matrices of determinant 1 is called the **special linear group** $\mathbf{Sl}(n, K)$. The **orthogonal group** $\mathbf{O}(n, K)$ consists of those $A \in \mathbf{Gl}(n, K)$ with $AA' = I$ and $\mathbf{SO}(n, K) = \mathbf{O}(n, K) \cap \mathbf{Sl}(n, K)$. We put $\mathbf{O}(n) = \mathbf{O}(n, \mathbf{R})$ and $\mathbf{SO}(n) = \mathbf{SO}(n, \mathbf{R})$. The **unitary group** $\mathbf{U}(n)$ consists of those $A \in \mathbf{Gl}(n, \mathbf{C})$ with $AA^* = I$ and $\mathbf{SU}(n) = \mathbf{U}(n) \cap \mathbf{Sl}(n, \mathbf{C})$. Since $A'$, $A^*$, and $\det A$ are continuous functions of $A$ and multiplication $\mathbf{M}_n(K) \times \mathbf{M}_n(K) \to \mathbf{M}_n(K)$ is continuous, it is clear that these are all closed subgroups of $\mathbf{Gl}(n, K)$. Since $\mathbf{Gl}(n, K)$ is defined by the inequality $\det A \neq 0$, it is an open subset of $\mathbf{M}_n(K)$. The formula for $A^{-1}$ in terms of the coefficients of $A$ shows that $A \mapsto A^{-1}$ is continuous on $\mathbf{Gl}(n, K)$; and in fact that it is analytic. (This also follows from the Implicit Function Theorem since multiplication $\mathbf{M}_n(K) \times \mathbf{M}_n(K) \to \mathbf{M}_n(K)$ is analytic.) It follows that all of the above groups are topological groups with the relative topology from $\mathbf{M}_n(K)$.

Since $A^*$ is a continuous function of $A$ the equation $AA^* = I$ defining $\mathbf{U}(n)$ shows that $\mathbf{U}(n)$ is closed in $\mathbf{M}_n(\mathbf{C}) \approx \mathbf{C}^{n^2}$. It also shows that the coefficients of $A$ are bounded by one in absolute value, so that $\mathbf{U}(n)$ is bounded in $\mathbf{M}_n(\mathbf{C})$. Thus $\mathbf{U}(n)$ is *compact*. Since $\mathbf{SU}(n)$, $\mathbf{O}(n)$, and $\mathbf{SO}(n)$ are closed subgroups of $\mathbf{U}(n)$, they are also compact groups.

Let $A$ be any $n \times n$ matrix over $K$. If the absolute values of the coefficients of $A$ are bounded by $c$, then those of $A^k$ are bounded by $(nc)^k$ as is easily seen by induction on $k$. Thus the series of coefficients of the matrix series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

are all absolutely dominated by the convergent series

$$1 + nc + \frac{(nc)^2}{2!} + \frac{(nc)^3}{3!} + \cdots.$$

Thus this series of matrices converges to a matrix

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

and the convergence is *uniform* in any compact subset of $\mathbf{M}_n(K)$. Thus the coefficients of $e^A$ are *analytic* functions of the coefficients of $A$. That is, the map exp: $\mathbf{M}_n(K) \to \mathbf{M}_n(K)$, taking $A$ to $\exp(A) = e^A$, is analytic. Note that $e^0 = I$. To compute the Jacobian of exp at 0 we use the usual matrix coordinates $x_{ij}$ and note that, for $X = [x_{ij}]$,

$$\begin{aligned} e^X &= I + [x_{ij}] + \frac{1}{2!}[x_{ij}]^2 + \cdots \\ &= [\delta_{ij} + x_{ij} + \text{higher degree terms}]. \end{aligned}$$

Thus the Jacobian is the $n^2 \times n^2$ matrix with entries

$$\left.\frac{\partial(\delta_{ij} + x_{ij} + \text{higher degree terms})}{\partial x_{kl}}\right|_{X=0} = \begin{cases} 1 & \text{if } \langle i, j\rangle = \langle k, l\rangle, \\ 0 & \text{otherwise,} \end{cases}$$

and hence is the *identity* $n^2 \times n^2$ matrix. From the Implicit Function Theorem it follows that exp is one–one on some neighborhood of $O \in \mathbf{M}_n(K)$ to some neighborhood of $I \in \mathbf{Gl}(n, K)$ and has an analytic inverse there.

Note that

$$\begin{aligned} e^{BAB^{-1}} &= I + BAB^{-1} + \frac{BA^2B^{-1}}{2!} + \cdots \\ &= B(I + A + \frac{A^2}{2!} + \cdots)B^{-1} = Be^AB^{-1}. \end{aligned}$$

Also exp over $\mathbf{R}$ is just the restriction of exp over $\mathbf{C}$ to $\mathbf{M}_n(\mathbf{R}) \subset \mathbf{M}_n(\mathbf{C})$. For any $A \in \mathbf{M}_n(\mathbf{C})$ there is a $B \in \mathbf{Gl}(n, \mathbf{C})$ such that $BAB^{-1}$ has super diagonal form

$$BAB^{-1} = \begin{bmatrix} \lambda_1 & & \star \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

where the $\lambda_i$ are the eigenvalues of $A$. From this it is trivial to calculate that

$$Be^A B^{-1} = e^{BAB^{-1}} = \begin{bmatrix} e^{\lambda_1} & & & \star \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix}$$

and thus the $e^{\lambda_i}$ are the eigenvalues of $e^A$ with the correct multiplicities. In particular, their product shows that

$$\det e^A = e^{\mathrm{trace}(A)}$$

[where $\mathrm{trace}(A) = \lambda_1 + \cdots + \lambda_n$]. In particular, $\det e^A \neq 0$ for all $A$, and hence the image of exp is in $\mathbf{Gl}(n, K)$.

**2.1. Lemma** *If $A$ and $B$ commute, then $e^{A+B} = e^A e^B$.*

*Proof* For $A$ and $B$ fixed, let $s$ and $t$ be real variables and consider the analytic functions $e^{sA+tB}$ and $e^{sA}e^{tB}$ of $s$ and $t$. We have

$$\begin{aligned} e^{sA+tB} &= \sum_n \frac{(sA + tB)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(sA)^j (tB)^{n-j}}{j!(n-j)!} \\ &= \left(\sum_{i=0}^{\infty} \frac{(sA)^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{(tB)^j}{j!}\right), \end{aligned}$$

where the last product is to be regarded as a formal product of matrices of power series in $s$ and $t$ and the last equality means an equality of matrices of formal power series in $s$ and $t$. However, the last term is just the power series expansion of $e^{sA}e^{tB}$, as is seen by computation of the partial derivatives in $s$ and $t$. Since $e^{sA+tB}$ and $e^{sA}e^{tB}$ are analytic functions of $s$ and $t$ it follows that they coincide for all $s$ and $t$. ▌

It follows that for any given $A \in \mathbf{M}_n(K)$ the map

$$K \to \mathbf{Gl}(n, K)$$

given by

$$t \mapsto e^{tA}$$

is a homomorphism of topological groups (with the additive group structure on $K$). In particular, $e^{-A} = (e^A)^{-1}$.

Since $\mathbf{Gl}(n, K)$ is open in the $K$-vector space $\mathbf{M}_n(K)$, the tangent vector to a curve in $\mathbf{Gl}(n, K)$, or in $\mathbf{M}_n(K)$, can be regarded as an element of $\mathbf{M}_n(K)$. In particular, for the curve $t \mapsto e^{tA}$ of $\mathbf{R} \to \mathbf{Gl}(n, K)$ [for fixed $A \in \mathbf{M}_n(K)$] we compute that its tangent vector at $t = 0$ is

$$\lim_{t\to 0} \frac{e^{tA} - I}{t} = \lim_{t\to 0} \left(A + t\frac{A^2}{2!} + \cdots\right) = A.$$

Thus the real tangent space at $I$ in $\mathbf{Gl}(n, K)$ coincides with the space of tangent vectors at $t = 0$ to the real one-parameter groups $t \mapsto e^{tA}$.

In the same way, for the operation of $\mathbf{Gl}(n, K)$ on $K^n$, the tangent vector to the curve $t \mapsto e^{tA} \cdot v$ in $K^n$, where $A \in \mathbf{M}_n(K)$ and $v \in K^n$ are fixed, is

$$\lim_{t\to 0} \frac{e^{tA} \cdot v - v}{t} = \lim_{t\to 0}\left(Av + t\frac{A^2 v}{2!} + \cdots\right) = Av.$$

If $e^{tA} \cdot v = v$ for all $t$, then it follows that $Av = 0$. Conversely, if $Av = 0$, then $e^{tA} \cdot v = (I + tA + \cdots)v = v$. Thus we have the following proposition.

**2.2. Proposition** *For $v \in K^n$ and $A \in \mathbf{M}_n(K)$, the tangent vector to the curve $t \mapsto e^{tA} \cdot v$ at $t = 0$ is 0 iff $e^{tA} \cdot v = v$ for all $t$.* ∎

**2.3. Theorem** *For $A \in \mathbf{M}_n(\mathbf{C})$ we have the following facts:*

(1) *If $A$ is real, then $e^A$ is real.*
(2) *If $A$ is skew symmetric ($A' = -A$), then $e^A \in \mathbf{O}(n, \mathbf{C})$.*
(3) *If $A$ is skew Hermitian ($A^* = -A$), then $e^A \in \mathbf{U}(n)$.*
(4) *If $A$ has trace 0, then $e^A \in \mathbf{Sl}(n, \mathbf{C})$.*

*Moreover, for $A$ in a suitable neighborhood of $0 \in \mathbf{M}_n(\mathbf{C})$, the reverse implications all hold.*

*Proof* For example, suppose that $A' = -A$. Then $(e^A)' = e^{A'} = e^{-A} = (e^A)^{-1}$ so that $e^A \in \mathbf{O}(n, \mathbf{C})$. Conversely, if $(e^A)' = (e^A)^{-1}$, then $e^{A'} = e^{-A}$ which implies that $A' = -A$ provided that $A$ is sufficiently near 0, since exp is one–one near 0. The proof of the other implications are all similar to this and will be omitted. ∎

Note the various combinations of the conditions in 2.3. It is also important to note that the conditions of 2.3 are all real linear and define real vector subspaces of $\mathbf{M}_n(\mathbf{C})$ whose dimensions are easy to calculate. Thus we have the following easy consequence of 2.3.

**2.4. Corollary** *Let* $G \subset \mathbf{Gl}(n, \mathbf{C})$ *be one of the groups* $\mathbf{Gl}(n, \mathbf{C})$, $\mathbf{Gl}(n, \mathbf{R})$, $\mathbf{Sl}(n, \mathbf{C})$, $\mathbf{Sl}(n, \mathbf{R})$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{O}(n, \mathbf{C})$, $\mathbf{SO}(n, \mathbf{C})$, $\mathbf{O}(n, \mathbf{R})$, *or* $\mathbf{SO}(n, \mathbf{R})$. *Then there is a real vector subspace* $T_G$ *of* $\mathbf{M}_n(\mathbf{C})$ *such that* exp: $\mathbf{M}_n(\mathbf{C}) \to \mathbf{Gl}(n, \mathbf{C})$ *maps a neighborhood of* 0 *in* $T_G$ *homeomorphically onto a neighborhood of* $I$ *in* $G$. *The dimensions of these subspaces are, respectively,* $2n^2$, $n^2$, $2n^2 - 2$, $n^2 - 1$, $n^2$, $n^2 - 1$, $n(n-1)$, $n(n-1)$, $\frac{1}{2}n(n-1)$, *and* $\frac{1}{2}n(n-1)$. ∎

Let $U \subset \mathbf{Gl}(n, \mathbf{C})$ be a symmetric open neighborhood of $I$ on which the inverse, log: $U \to \mathbf{M}_n(\mathbf{C})$, of exp is defined and analytic. Let $V$ be a symmetric open neighborhood of $I$ with $V^2 \subset U$ and put $W = \log(V)$. Then the composition

$$W \times W \xrightarrow{\exp \times \exp} V \times V \xrightarrow{\varphi} U \xrightarrow{\log} \mathbf{M}_n(\mathbf{C})$$

is real analytic, where $\varphi(gh) = gh^{-1}$, and takes

$$(W \cap T_G) \times (W \cap T_G) \to T_G.$$

This is just the expression of the multiplication map $\varphi$ in terms of the coordinates about $I$ in $G$ given by log. Thus each of the above groups has coordinates about $I$ in which multiplication and inversion are expressed as real analytic functions of the coordinates.

We now turn to the discussion of one family of classical groups that we have thus far omitted, the **symplectic groups** $\mathbf{Sp}(n)$. The group $\mathbf{Sp}(n)$ is defined to be the group of all $n \times n$ quaternionic matrices $A$ with $AA^* = I$. Here $A^* = \bar{A}'$, where $\bar{A}$ is the quaternionic conjugate of $A$; that is $\bar{A} = [\bar{a}_{ij}]$ when $A = [a_{ij}]$ and where $q = a + bi + cj + dk$ has conjugate $\bar{q} = a - bi - cj - dk$. For a quaternionic $n \times n$ matrix $P$ we can write

$$P = A + Bj,$$

where $A, B \in \mathbf{M}_n(\mathbf{C})$. Note that $jB = \bar{B}j$ and thus multiplication is given by

$$(A + Bj)(C + Dj) = (AC - B\bar{D}) + (AD + B\bar{C})j.$$

For $P = A + Bj$ we define

$$\varphi(P) = \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}.$$

A short calculation shows that $\varphi$ is an isomorphism of the algebra of all $n \times n$ quaternionic matrices *into* that of $2n \times 2n$ complex matrices, and that $\varphi(P^*) = \varphi(P)^*$. Thus $P$ is *symplectic* ($PP^* = I$) iff $I = \varphi(I) = \varphi(PP^*) = \varphi(P)\varphi(P^*) = \varphi(P)\varphi(P)^*$ which holds iff $\varphi(P)$ is *unitary*. It is easy to calculate that a $2n \times 2n$ complex matrix $Q$ has the form

$$Q = \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}$$

iff

$$JQJ^{-1} = \bar{Q},$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

For $Q$ *unitary*, this equation becomes

$$Q'JQ = J.$$

Thus the symplectic group $\mathbf{Sp}(n)$ is isomorphic via $\varphi$ to the subgroup of $\mathbf{U}(2n)$ consisting of those unitary matrices preserving the bilinear form $J$ (i.e., $Q'JQ = J$).

As in 2.3 one sees that if $A \in \mathbf{M}_{2n}(\mathbf{C})$ and if $A^* = -A$ and $JA + A'J = 0$, then $e^A \in \varphi(\mathbf{Sp}(n))$, and the converse holds for $A$ sufficiently close to 0. Thus 2.4 also holds for $G = \varphi(\mathbf{Sp}(n)) \approx \mathbf{Sp}(n)$ and the dimension of $T_G$ can be computed to be $2n^2 + n$.

We conclude this section by defining the notion of a **representation.** If $G$ is a topological group, then a real representation of $G$ is just a (continuous) homomorphism of $G$ into $\mathbf{Gl}(n, \mathbf{R})$. This can be thought of as an action of $G$ on $\mathbf{R}^n$ by linear transformations. Similarly, a complex representation of $G$ is a homomorphism $G \to \mathbf{Gl}(n, \mathbf{C})$. Since $\mathbf{Gl}(n, \mathbf{R}) \subset \mathbf{Gl}(n, \mathbf{C})$ each real representation gives rise to a complex one called its **complexification.** By a **unitary** or an **orthogonal** representation of $G$ we mean a homomorphism of $G$ into $\mathbf{U}(n)$ or $\mathbf{O}(n)$, respectively.

Two representations $\varphi, \psi$: $G \to \mathbf{Gl}(n)$ are said to be **equivalent** if there is an $A \in \mathbf{Gl}(n)$ with $\psi(g) = A^{-1}\varphi(g)A$ for all $g \in G$. They are orthogonally, or unitarily, equivalent if $A \in \mathbf{O}(n)$ or $A \in \mathbf{U}(n)$, respectively.

Since $A\colon \mathbf{R}^n \xrightarrow{\approx} \mathbf{R}^n$ can be thought of as a change of basis, the equivalence of two representations $\varphi, \psi\colon G \to \mathbf{Gl}(n)$ simply means that they have identical matrix form with respect to suitable nonstandard bases (different for $\varphi$ and $\psi$) of $\mathbf{R}^n$. In particular, a representation $\varphi\colon G \to \mathbf{Gl}(n, \mathbf{R})$ is equivalent to an orthogonal representation iff there exists a (positive definite) inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^n$ preserved by $\varphi$ (that is, $\langle \varphi(g)v, \varphi(g)w \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in \mathbf{R}^n$), for then each $\varphi(g)$ would have an orthogonal matrix form with respect to an orthonormal basis on $\mathbf{R}^n$ for this inner product.

## 3. INTEGRATION ON COMPACT GROUPS

Let $G$ be a topological group and $f\colon G \to \mathbf{R}$ a real-valued function. For $h \in G$, define functions $R_h f$ and $L_h f$ by $(R_h f)(g) = f(gh)$ and $(L_h f)(g) = f(h^{-1}g)$.

**3.1. Theorem** *Let $G$ be a compact group. Then there is a unique real-valued function $I$ (called the Haar integral) defined for continuous real-valued functions $f$ on $G$, such that*

(a) $I(f_1 + f_2) = I(f_1) + I(f_2)$.
(b) $I(cf) = cI(f)$, *where* $c \in \mathbf{R}$.
(c) *If* $f(g) \geq 0$ *for all* $g \in G$, *then* $I(f) \geq 0$.
(d) $I(1) = 1$.
(e) $I(R_h f) = I(f) = I(L_h f)$ *for all* $h \in G$. ∎

We shall not give the proof of this theorem here. A simple elementary proof for compact groups was given by von Neumann and this proof can be found in Pontriagin [1]. There is a generalization of this integral for locally compact groups, proved by Haar, with weaker properties. This can be found in many places, including Loomis [1] and Montgomery and Zippin [4]. (These references give versions of Haar's original proof. A totally different proof can be found in Bredon [9].)

We shall often use the notation $\int f(g)\, dg$ for $I(f)$, so that (e) can be written

$$\int f(gh)\, dg = \int f(g)\, dg = \int f(h^{-1}g)\, dg.$$

Note that if $f$ is nonnegative and not identically 0 then, by compactness of

$G$, we can find elements $h_1, \ldots, h_n$ in $G$ with $\sum R_{h_i} f$ everywhere positive. Then $\sum R_{h_i} f \geq c$ for some constant $c > 0$ and we have

$$nI(f) - c = I((\sum R_{hi} f) - c) \geq 0.$$

Thus $I(f) > 0$.

Similarly, if $f_1(g) \geq f_2(g)$ for all $g$, then $I(f_1) \geq I(f_2)$. Also $| I(f) | \leq \sup\{| f(g) | \mid g \in G\}$.

**3.2. Proposition** *Let $f$: $G \times A \to \mathbf{R}$ be continuous, where $A$ is any topological space and $G$ is a compact group. Then the function $F$: $A \to \mathbf{R}$ defined by*

$$F(a) = \int f(g, a)\, dg$$

*is continuous.*

*Proof* Let $\varepsilon > 0$ and $a \in A$ be given. By continuity of $f$ and compactness of $G$ there is a neighborhood $U$ of $a$ such that

$$| f(g, b) - f(g, a) | < \varepsilon$$

for all $b \in U$. Then

$$| F(b) - F(a) | = \left| \int f(g, b)\, dg - \int f(g, a)\, dg \right|$$

$$= \left| \int (f(g, b) - f(g, a))\, dg \right| < \varepsilon$$

for all $b \in U$. ∎

**3.3. Theorem** *Let $f$: $G \times \mathbf{R} \to \mathbf{R}$ be continuous, where $G$ is a compact group, and suppose that $f(g, t)$ is differentiable in $t$ with a derivative $df(g, t)/dt$ which is continuous on $G \times \mathbf{R}$. Then $F(t) = \int f(g, t)\, dg$ is also differentiable and*

$$\frac{d}{dt} F(t) = \int \frac{d}{dt} f(g, t)\, dg.$$

*Proof* For real number $s \neq 0$ we have

$$\frac{f(g, t + s) - f(g, t)}{s} = \frac{d}{dt} f(g, t + \varphi s)$$

for some $0 \leq \varphi \leq 1$. Since $df(g, t)/dt$ is uniformly continuous on compact subsets of $G \times \mathbf{R}$, it follows that, for fixed $t \in \mathbf{R}$, this difference quotient converges to $df(g, t)/dt$ *uniformly in* $g \in G$ as $s \to 0$. Therefore, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|s| < \delta$, then

$$\left|\frac{f(g, t+s) - f(g, t)}{s} - \frac{d}{dt} f(g, t)\right| < \varepsilon$$

for all $g \in G$. Then also

$$\left| \int \frac{f(g, t+s) - f(g, t)}{s} \, dg - \int \frac{d}{dt} f(g, t) \, dg \right| < \varepsilon,$$

that is,

$$\left| \frac{F(t+s) - F(t)}{s} - \int \frac{d}{dt} f(g, t) \, dg \right| < \varepsilon$$

which implies the desired conclusion. ∎

Note that an inductive use of 3.3 shows that if $f\colon G \times \mathbf{R}^n \to \mathbf{R}$ is a function such that $f(g, t_1, \ldots, t_n)$ is $C^\infty$ in the $t_i$ for fixed $g \in G$ and such that the partial derivatives of all orders are continuous on $G \times \mathbf{R}^n$, then $F(t_1, \ldots, t_n) = \int f(g, t_1, \ldots, t_n) \, dg$ is also $C^\infty$.

**3.4. Proposition** *If* $f\colon G \times G \to \mathbf{R}$ *is continuous, then* $\int \int f(g, h) \, dg \, dh = \int\int f(g, h) \, dh \, dg$.

***Proof*** Each of these satisfies the conditions for the integral on $G \times G$ so that uniqueness of the integral implies this fact. ∎

By simply integrating componentwise, the integral can be extended to an integral on vector-valued continuous functions $f\colon G \to \mathbf{R}^n$ giving $\int f(g) \, dg \in \mathbf{R}^n$. The linearity conditions (a) and (b) of 3.1 imply immediately that if $T\colon \mathbf{R}^n \to \mathbf{R}^k$ is any linear transformation, then

$$\int T(f(g)) \, dg = T\left(\int f(g) \, dg\right).$$

If $V$ is a vector space, then $\mathbf{Gl}(V)$ denotes the group of linear automorphisms of $V$ and a representation of $G$ on $V$ is a homomorphism $G \to \mathbf{Gl}(V)$. For a real vector space $V$ of finite dimension $n$, each such representation is clearly equivalent, in the obvious sense, to a representation $G \to \mathbf{Gl}(n, \mathbf{R})$ on euclidean space, and we require continuity of this.

The major use for the Haar integral is in constructing invariant things by integrating noninvariant things. An illustration of this is the proof of the following theorem:

**3.5. Theorem** *Each representation of a compact group $G$ on an n-dimensional real vector space $V$ is equivalent to an orthogonal representation of $G$ on $\mathbf{R}^n$.*

*Proof* We can assume that $V = \mathbf{R}^n$. Let us denote the action of $G$ on $\mathbf{R}^n$ via the given representation by $(g, v) \mapsto gv$. As pointed out in Section 2, it suffices to find a positive definite inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^n$ which is invariant under $G$, i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$ and $v$, $w \in \mathbf{R}^n$. Let us denote the usual euclidean inner product by $v \cdot w$; that is, $v \cdot w = \sum v_i w_i$, where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. Define

$$\langle v, w \rangle = \int (gv) \cdot (gw)\, dg.$$

This is clearly bilinear and symmetric. It is positive definite since, for $v \neq 0$,

$$\langle v, v \rangle = \int (gv) \cdot (gv)\, dg > 0$$

since the integrand is positive. This inner product is also invariant since

$$\langle hv, hw \rangle = \int (ghv) \cdot (ghw)\, dg = \int (gv) \cdot (gw)\, dg = \langle v, w \rangle$$

by property (e) of the integral. ∎

The same proof shows that every complex representation of a compact group is equivalent to a unitary representation.

If $\varphi\colon G \to \mathbf{Gl}(V)$ and $\psi\colon G \to \mathbf{Gl}(W)$ are representations, then $\varphi \oplus \psi$ denotes the representation $G \to \mathbf{Gl}(V \oplus W)$ given by $(\varphi \oplus \psi)(g) \cdot (v, w) = (\varphi(g)v, \psi(g)w)$.

A (finite-dimensional) representation $\varphi\colon G \to \mathbf{Gl}(V)$ is said to be **reducible** if there is a subspace $0 \neq W \neq V$ of $V$ such that $\varphi(g)(W) = W$ for all $g \in G$. Otherwise it is called **irreducible**. It is called **completely reducible** if it is equivalent to the direct sum $\varphi \approx \psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_k$ of irreducible representations $\psi_i$. It is not hard to see, in this case, that the $\psi_i$ are unique up to order and equivalence.

**3.6. Proposition** *Every finite-dimensional real representation $G \to \mathbf{Gl}(V)$ of a compact group is completely reducible.*

*Proof* By 3.5 we can assume that $V = \mathbf{R}^n$ and the representation is orthogonal $\varphi$: $G \to \mathbf{O}(n)$. If $W \subset \mathbf{R}^n$ is invariant, then we claim that $W^\perp = \{v \in V = \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \in W\}$ is also invariant. This follows from the equation

$$(\varphi(g)v) \cdot w = v \cdot \varphi(g)'w = v \cdot \varphi(g^{-1})w.$$

Thus $\varphi \approx (\varphi \mid W) \oplus (\varphi \mid W^\perp)$ and the result follows by an easy induction. ∎

## 4. CHARACTERISTIC FUNCTIONS ON COMPACT GROUPS

Let $G$ be a compact group. The letters $x$ and $y$ will be used for elements of $G$ in this section. For a given positive integer $n$ let $V$ denote the (infinite-dimensional) real vector space of all continuous functions $f$: $G \to \mathbf{R}^n$. (For most purposes $n$ can be taken to be 1, but it will be convenient to allow the general case.) For $f, g$: $G \to \mathbf{R}^n$ define

$$\langle f, g \rangle = \int f(x) \cdot g(x)\, dx,$$

where the integrand is the usual euclidean inner product.

Suppose we are given a continuous function

$$k\colon G \times G \to \mathbf{R}$$

with $k(x, y) = k(y, x)$, called a "kernel function." Then we define an operator $K$: $V \to V$ by

$$(Kf)(x) = \int k(x, y) f(y)\, dy.$$

It is easily seen that $K$ maps the unit sphere $\{f \mid \langle f, f \rangle = 1\}$ to a uniformly bounded and equicontinuous set of functions. Also $K$ is symmetric, $\langle Kf, g \rangle = \langle f, Kg \rangle$, by interchange of order of integration.

A function $0 \neq \varphi \in V$ is said to be a **characteristic function** (of $K$) belonging to the **characteristic value** $\lambda \in \mathbf{R}$ if $K\varphi = \lambda\varphi$.

Let us list, without proof, three well-known facts which are consequences of the above properties of $K$. The proofs can be found in many places, such as Hochschild [1, pp. 14–18] or Chevalley [1, pp. 204–209].

(a) For any $c > 0$ there are finitely many characteristic values $\lambda$ with $|\lambda| \geq c$.

(b) For $\lambda \neq 0$ the characteristic functions belonging to $\lambda$ form a finite-dimensional vector space.

Note that from these facts we can find a sequence $\varphi_1, \varphi_2, \ldots$ of characteristic functions belonging to the nonzero characteristic values $c_1, c_2, \ldots$ such that the $|c_i|$ converge monotonically to zero, such that $\langle \varphi_i, \varphi_j \rangle = \delta_{i,j}$, and such that each characteristic function belonging to $\lambda \neq 0$ is a linear combination of those $\varphi_i$ with $c_i = \lambda$. Then another basic property is:

(c) If $g = Kf$ for some $f \in V$, then the "Fourier series" $\sum \langle g, \varphi_i \rangle \varphi_i$ converges uniformly to $g$.

Recall that for $f \in V$ the right translate $R_y f \in V$ is defined by $(R_y f)(x) = f(xy)$ and the left translate $L_y f \in V$ by $(L_y f)(x) = f(y^{-1}x)$. Since $R_x R_y = R_{xy}$ and $L_x L_y = L_{xy}$ these both represent $G$ by linear transformations on the infinite-dimensional vector space $V$.

We shall assume from now on that the kernel function $k$ has the form

$$k(x, y) = h(xy^{-1})$$

where $h\colon G \to \mathbf{R}$ satisfies $h(x) = h(x^{-1})$.

For any linear map $A\colon \mathbf{R}^n \to \mathbf{R}^n$, composition with $f\colon G \to \mathbf{R}^n$ defines $Af\colon G \to \mathbf{R}^n$ and $f \mapsto Af$ defines a linear transformation $A\colon V \to V$ denoted by the same symbol.

**4.1. Lemma** *In the above situation we have $R_y \circ K = K \circ R_y$ for all $y$, and $A \circ K = K \circ A$. If $h$ also satisfies $h(yxy^{-1}) = h(x)$ for all $x$, then $L_y \circ K = K \circ L_y$.*

***Proof*** We compute

$$\begin{aligned} K(R_y f)(x) &= \int h(xz^{-1}) f(zy)\, dz \\ &= \int h\big(xy(zy)^{-1}\big) f(zy)\, dz \\ &= \int h\big((xy)z^{-1}\big) f(z)\, dz \\ &= (Kf)(xy) = \big(R_y(Kf)\big)(x). \end{aligned}$$

Also

$$K(Af)(x) = \int h(xy^{-1})A\big(f(y)\big)\,dy$$

$$= \int A\big(h(xy^{-1})f(y)\big)\,dy$$

$$= A\int h(xy^{-1})f(y)\,dy = A(Kf)(x).$$

For the last statement we compute

$$K(L_y f)(x) = \int h(xz^{-1})f(y^{-1}z)\,dz$$

$$= \int h\big(yy^{-1}x(y^{-1}z)^{-1}y^{-1}\big)\,f(y^{-1}z)\,dz$$

$$= \int h(yy^{-1}xz^{-1}y^{-1})f(z)\,dz$$

$$= \int h(y^{-1}xz^{-1})f(z)\,dz$$

$$= (Kf)(y^{-1}x) = \big(L_y(Kf)\big)(x). \quad \blacksquare$$

It follows from 4.1 that each $R_y$ and each $A$ (as above) transforms the space of characteristic functions belonging to $\lambda$ into itself. For $c > 0$ we let $V_c$ denote the subspace of $V$ spanned by those characteristic functions belonging to characteristic values $\lambda$ with $|\lambda| \geq c$. Then $V_c$ is finite-dimensional by (a) and (b), and is preserved by each $R_y$ and each $A$. Put $V_c^{\perp} = \{g \in V \mid \langle g, f\rangle = 0 \text{ for all } f \in V_c\}$. Now $V_c$ is spanned by some of the $\varphi_i$, say $\varphi_1, \ldots, \varphi_r$. Then $f \in V$ can be written

$$f = \sum_{i=1}^{r} \langle f, \varphi_i\rangle\varphi_i + \Big(f - \sum_{i=1}^{r} \langle f, \varphi_i\rangle\varphi_i\Big)$$

which shows that $V = V_c \oplus V_c^{\perp}$.

Since $\langle Ag, f\rangle = \langle g, A'f\rangle$, for $A\colon \mathbf{R}^n \to \mathbf{R}^n$ linear and $A'$ its transpose, we see that $A$ preserves $V_c^{\perp}$. Also computation gives

$$\langle R_y g, f\rangle = \langle g, R_{y^{-1}} f\rangle$$

which shows that $R_y$ preserves $V_c^{\perp}$.

Let $P_c: V \to V_c$ be the orthogonal projection $P_c f = \sum_{i=1}^{r} \langle f, \varphi_i \rangle \varphi_i$ taking $V_c^{\perp}$ to 0. Then we see that

$$P_c \circ R_y = R_y \circ P_c \qquad \text{and} \qquad P_c \circ A = A \circ P_c$$

in the above situation. In the same way, if $h(yxy^{-1}) = h(x)$ for all $x \in G$, then $L_y$ commutes with $K$ by 4.1 and hence also commutes with $P_c$.

**4.2. Theorem** *Let $G$ be a compact group and $H$ a closed subgroup. Let $\varrho\colon H \to \mathbf{Gl}(n, \mathbf{R})$ be a representation of $H$. Then there exists a representation $\eta\colon G \to \mathbf{Gl}(m, \mathbf{R})$ for some $m$, such that $\varrho$ is contained in the restriction of $\eta$ to $H$; that is, $\eta \mid H \approx \varrho \oplus \mu$ for some representation $\mu$ of $H$.*

*Proof* It clearly suffices to prove this for the case in which $\varrho$ is *irreducible.* As above, let $V$ be the space of continuous functions $G \to \mathbf{R}^n$. Let $h\colon G \to \mathbf{R}$ be a given map satisfying $h(x) = h(x^{-1})$ and $h(yxy^{-1}) = h(x)$ for all $x$, $y \in G$. Define $(Kf)(x) = \int h(xy^{-1}) f(y)\, dy$ as above. We shall make an appropriate choice of $h$ later.

Let $W \subset V$ be the subspace consisting of those functions $f\colon G \to \mathbf{R}^n$ such that

$$f(yx) = \varrho(y) f(x) \qquad \text{for all } y \in H \text{ and } x \in G.$$

That is,

$$L_{y^{-1}} f = \varrho(y) \circ f \qquad \text{for all } y \in H.$$

With $A = \varrho(y)$ in the above remarks, we see that for any $c > 0$ (to be chosen later) the projection $P_c$ preserves $W$ and $K$ also preserves $W$, since both of these commute with each $L_{y^{-1}}$ and each $A = \varrho(y)$. Also note that each $R_x$ preserves $W$ for $x \in G$ (but the $L_y$ do not).

Put $W_c = P_c W \subset W$. Then the $R_x$, for $x \in G$, define a finite-dimensional representation $\eta$ of $G$ on $W_c$. Consider the linear map

$$J\colon W_c \to \mathbf{R}^n$$

given by $J(f) = f(e)$. For $y \in H$ we have

$$J(R_y f) = (R_y f)(e) = f(y) = \varrho(y) f(e) = \varrho(y) J(f)$$

which means that $J$ carries the representation $\eta \mid H$ to $\varrho$. The kernel of $J$ is a representation $\mu$ of $H$ which has a complement $\tau$ by 3.6, since $H$ is compact. Thus $\eta \mid H = \mu \oplus \tau$ and $J$ takes $\tau$ monomorphically to $\varrho$. If

$J \neq 0$, then $J\colon \tau \to \varrho$ would be an equivalence of representations of $H$, since $\varrho$ is irreducible, and we would be done. Thus it suffices to show that we can pick the function $h$ (and hence the operator $K$) and $c > 0$ so that there is a function $f \in W_c$ with $f(e) = J(f) \neq 0$.

To this end let $v = (1, 0, \ldots, 0) \in \mathbf{R}^n$ and define $g\colon H \to \mathbf{R}^n$ by $g(y) = \varrho(y) \cdot v$. This extends to a map $\bar{g}\colon G \to \mathbf{R}^n$ by the Tietze Theorem since $G$ is compact and hence normal. Put

$$g^*(x) = \int_H \varrho(y^{-1})\bar{g}(yx)\, dy.$$

For $x, y \in H$ we have $\bar{g}(yx) = \varrho(yx)v = \varrho(y)\varrho(x)v$ and thus $g^* \mid H = g$. For all $x \in G$ and $z \in H$ we have

$$\begin{aligned} g^*(zx) &= \int_H \varrho(y^{-1})\bar{g}(yzx)\, dy \\ &= \int_H \varrho(z(yz)^{-1})\bar{g}(yzx)\, dy \\ &= \int_H \varrho(zy^{-1})\bar{g}(yx)\, dy \\ &= \varrho(z) \int_H \varrho(y^{-1})\bar{g}(yx)\, dy \\ &= \varrho(z)g^*(x) \end{aligned}$$

and thus $g^* \in W$. Since $g^*$ extends $g$ we shall now drop the star, giving $g \in W$ with $g(e) = v = (1, 0, \ldots, 0)$.

Let $g_1(x)$ be the first component of $g(x)$, so that $g_1(e) = 1$. By continuity there is a neighborhood $U$ of $e$ in $G$ on which $g_1$ is positive. We may assume that $U$ is symmetric and invariant under conjugation by 1.10. Let $q\colon G \to \mathbf{R}$ be a nonnegative map vanishing outside $U$ and with $q(e) = 1$. Put

$$h(x) = \tfrac{1}{2} \int_G \left(q(yxy^{-1}) + q(yx^{-1}y^{-1})\right) dy.$$

Then $h$ is nonnegative, vanishes outside $U$, has $h(e) = 1$, and satisfies $h(x) = h(x^{-1})$ and $h(yxy^{-1}) = h(x)$. Then, for this choice of $h$, we have

$$(Kg)(e) = \int_G h(ey^{-1})g(y)\, dy \neq 0,$$

since its first component $\int_G h(y^{-1})g_1(y)\, dy$ is positive. Recall that by property (c) above, $P_cKg$ converges uniformly to $Kg$ as $c \to 0$. Thus we can choose $c > 0$ sufficiently small so that $(P_c Kg)(e) \neq 0$. Then $f = P_c Kg \in W_c$ is our desired function. ▮

If $\varrho\colon G \to \mathrm{Gl}(V)$ is a representation and if $v \in V$, we put

$$G_v = \{g \in G \mid \varrho(g)v = v\}$$

which is called the **isotropy subgroup** of $G$ at $v$. (Also see Chapter I, Section 2.) It is clearly a closed subgroup of $G$.

**4.3. Theorem** *Let $G$ be a compact group and $H$ a closed subgroup. Let $U$ be a neighborhood of $e$ in $G$. Then there exists a representation $\varrho\colon G \to \mathbf{Gl}(n, \mathbf{R})$ for some $n$, and a point $v \in \mathbf{R}^n$ with $H \subset G_v \subset UH$.*

*Proof* Let $W$ be a symmetric neighborhood of $e$ in $G$ with $W^2 \subset U$ and let $h\colon G \to \mathbf{R}$ be a nonnegative continuous function satisfying $h(e) = 1$, $h(x) = h(x^{-1})$, and $h(x) = 0$ for $x \notin W$. Let the operator $K$ be defined by

$$(Kf)(x) = \int_G h(xy^{-1}) f(y)\, dy.$$

Since $G/H$ is compact Hausdorff, and hence normal, there is a continuous nonnegative function $f'\colon G/H \to \mathbf{R}$ with $f'(eH) = 1$ and $f'(xH) = 0$ for $x \notin WH$. Let $f\colon G \to \mathbf{R}$ be $f(x) = f'(xH)$. Then $f(e) = 1$, $f(x) = 0$ for $x \notin WH$ and $R_y f = f$ for all $y \in H$. Put $g = Kf$. Since $R_y$ commutes with $K$ we have $R_y g = g$ for $y \in H$.

Now if

$$0 \neq g(x) = \int h(xy^{-1}) f(y)\, dy,$$

then for some $y \in G$ we must have $xy^{-1} \in W$ and $y \in WH$, so that $x \in W^2H \subset UH$. Thus

$$g(x) = 0 \qquad \text{for} \quad x \notin UH.$$

On the other hand,

$$g(e) = \int h(y^{-1}) f(y)\, dy > 0.$$

Since $P_c g$ converges uniformly to $g$ as $c \to 0$, by property (c) above, we can take $c$ so small that $(R_x P_c g)(e) = (P_c g)(x) < (P_c g)(e)$ for all $x \in G - UH$. For this value of $c$, let $\varphi = P_c g$. Then $G$ acts on the finite-dimensional vector space $V_c$ via the $R_x$, and we have the vector $\varphi \in V_c$ with

$$R_x \varphi \neq \varphi \qquad \text{for} \quad x \notin UH$$

(since they differ at $e$) and

$$R_y\varphi = \varphi \qquad \text{for all } y \in H,$$

since $R_y$ commutes with $P_c$.

Thus the isotropy group $G_\varphi$ contains $H$ and is contained in $UH$. ∎

By 3.5 and 4.3 applied to $H = \{e\}$, we have the following corollary.

**4.4. Corollary** *Let $G$ be a compact group and $U$ a neighborhood of $e$ in $G$. Then there is a homomorphism $\psi$: $G \to \mathbf{O}(n)$ with* $\ker \psi \subset U$. ∎

We say that a group $G$ has **no small subgroups** if there is a neighborhood $U$ of $e$ in $G$ containing no subgroup of $G$ other than $\{e\}$.

**4.5. Corollary** *A compact group $G$ has no small subgroups iff it is isomorphic to a closed subgroup of* $\mathbf{O}(n)$.

*Proof* By 4.4 there is a monomorphism $\psi$: $G \to \mathbf{O}(n)$. Since $\psi$ is one–one, $\mathbf{O}(n)$ is Hausdorff, and $G$ is compact, $\psi$ is a homeomorphism (and an isomorphism of groups) to its image.

For the converse it suffices to show that $\mathbf{Gl}(n, \mathbf{R})$ has no small subgroups. For this, consider a convex neighborhood $U$ of 0 in $\mathbf{M}_n(\mathbf{R}) \approx \mathbf{R}^{n^2}$ for which $\exp$: $2U \to \mathbf{Gl}(n, \mathbf{R})$ is a homeomorphism to some neighborhood of $I$ in $\mathbf{Gl}(n, \mathbf{R})$; see Section 2. Suppose that $H \subset \exp(U)$ is a subgroup of $\mathbf{Gl}(n, \mathbf{R})$ and that $I \neq B \in H$. Let $0 \neq A \in U$ be such that $e^A = B$. Then $2^rA \in 2U - U$ for some integer $r \geq 1$ and $B^{2^r} = e^{2^rA} \in \exp(2U) - \exp(U)$ by 2.1. This contradicts the assumption that $B^{2^r} \in H \subset \exp(U)$. ∎

*Remarks* Theorem 4.2 is well known, as is the case with all the results of this chapter. The author is indebted to Wallach for the idea of the present proof. Theorem 4.3 (and 5.2 of the next section) is due independently to Mostow [1] and Palais [2], while the special case 4.4 (and 4.5) was previously proved by von Neumann.

## 5. LIE GROUPS

In this section we shall outline the elementary theory of Lie groups, omitting most of the standard proofs and discussing only those things of importance for later chapters. The discussion is intended for readers who are not already familiar with Lie groups (and, possibly, not even with differentiable manifolds). It is designed simply to indicate what a Lie group is and to indicate a few elementary facts which we shall need. Those few things which we need are easily understood and believable, and we feel that the reader need not know how to prove these results. For the proofs, and indeed for much better expositions, of these facts, the reader may consult any number of readily available sources, such as Chevalley [1] or Hochschild [1].

Let $G$ be a topological group. Then $G$ is called a **Lie group** if there is an open neighborhood $U$ of $e$ in $G$ and a homeomorphism $x\colon U \to W$ onto an open set $W \subset \mathbf{R}^n$, for some $n$ and with $x(e) = 0$, such that the group operations are real analytic near $e$ in these local coordinates. More precisely, let $x_i(g)$ denote the $i$th coordinate of $x(g) \in \mathbf{R}^n$ for $g \in U$. Then there are real analytic functions $\varphi_i$ defined on some neighborhood of $0 \in \mathbf{R}^{2n}$ such that

$$x_i(gh) = \varphi_i(x_1(g), \ldots, x_n(g),\ x_1(h), \ldots, x_n(h))$$

for all $g$ and $h$ in some open neighborhood $V \subset U$ of $e$. Similarly

$$x_i(g^{-1}) = \psi_i(x_1(g), \ldots, x_n(g))$$

for $g$ near $e$, where the $\psi_i$ are real analytic and defined near $0 \in \mathbf{R}^n$.

A one-parameter group $\gamma\colon \mathbf{R} \to G$ [i.e., a continuous homomorphism, $\gamma(s + t) = \gamma(s)\gamma(t)$] is called **analytic** if each $x_i \circ \gamma$ is analytic near $0 \in \mathbf{R}$. (We can give $G$ the structure of an analytic manifold for which the group operations are analytic, and it follows that $\gamma$ is everywhere analytic.) The derivatives $a_i = dx_i(\gamma(t))/dt\,|_{t=0}$ are the coordinates of the "tangent vector" to $\gamma$ at $t = 0$ with respect to the given coordinate system $x$. From the elementary theory of differential equations it can be shown that $\gamma$ is completely determined by this vector $X = (a_1, \ldots, a_n) \in \mathbf{R}^n$ and, moreover, that every $X \in \mathbf{R}^n$ arises in this way from some analytic one-parameter group $\gamma_X$. It also follows from this theory that the map

$$\exp\colon \mathbf{R}^n \to G$$

defined by $\exp X = \gamma_X(1)$ is analytic for $X$ near 0. Since computation shows that $t \mapsto \gamma_{sX}(t)$ and $t \mapsto \gamma_X(st)$ are analytic one-parameter groups having the same tangent vector at $t = 0$ we have

$$\gamma_{sX}(1) = \gamma_X(s)$$

for all $s \in \mathbf{R}$. Then an easy calculation shows that the Jacobian matrix of $x \circ \exp$ at 0 is just the identity. The Inverse Function Theorem then shows that, near 0, exp has an inverse function, called "log," which is also analytic (i.e., $\log \circ x^{-1}$ is analytic). Note that log then defines a coordinate system near $e$ (i.e., we could take $x = \log$). These coordinates are called "canonical coordinates of the first kind." In these coordinates the analytic one-parameter groups are just the straight lines through the origin. It is easy to see from this that every continuous one-parameter group $\gamma$: $\mathbf{R} \to G$ is automatically analytic, and hence has the form $\gamma(t) = \exp(tX)$ for some $X \in \mathbf{R}^n$. This significant fact implies, in turn, that every continuous homomorphism $G \to H$ between two Lie groups is analytic near $e$. Consequently, a coordinate system $x$ near $e$ on $G$, making $G$ into a Lie group, is *unique* near $e$ up to *analytic* change of coordinates. (This is the reason that a *given* such structure is not taken as part of the definition of a Lie group.)

Just as in the proof of 4.5 it can be shown that a Lie group $G$ has no small subgroups.

Suppose that $H$ is a closed subgroup of the Lie group $G$. By a careful investigation of the exponential map it can be shown that $H$ coincides near $e$ with the image under exp: $\mathbf{R}^n \to G$ of a linear subspace $V \subset \mathbf{R}^n$ (and, in fact, of any neighborhood of 0 in $V$). If $V^\perp$ is the orthogonal complement (or any complement) of $V$ in $\mathbf{R}^n$, then the map $\varphi$: $\mathbf{R}^n = V^\perp \oplus V \to G$ given by

$$\varphi(w, v) = (\exp w)(\exp v)$$

has (with respect to local coordinates) the identity as its Jacobian matrix. Thus the local inverse of $\varphi$ gives a system $x = (x_1, \ldots, x_n)$ of local coordinates at $e$ in $G$, defined, for example, on $U = \{g \in G \mid |x_i(g)| < 1\}$, and having the property that each $U \cap gH$, for $g \in G$, is given by the equations

$$x_i = c_i, \qquad i = 1, \ldots, k$$

(where the $c_i$ are constants depending on the choice of the coset $gH$, and $n - k$ is the dimension of $H$). In particular, it follows that $H$ is itself a Lie group.

Applying this to $G = \mathbf{O}(n)$, the next result follows from 4.5.

**5.1. Theorem** *A compact group is a Lie group iff it is isomorphic to a closed subgroup of* $\mathbf{O}(n)$ *for some n.* ▌

Given $H \subset G$ closed, and local coordinates $x = (x_1, \ldots, x_n)$ as above, let $C$ be the set of points in the coordinate neighborhood with $x_{k+1} = \cdots = x_n = 0$ (i.e., $C = \exp W$, where $W$ is a small neighborhood of 0 in

$V^\perp$). Then $CH$ is open in $G$ and the multiplication

$$C \times H \to CH$$

is a bijection. Put $C' = \exp \frac{1}{2}W$. Suppose that $K \subset G$ is a subgroup with $H \subset K \subset C'H$. If $K \neq H$, then it follows that $K$ contains an element $e \neq k \in C'$. However, lines through the origin of $C$ (in the given coordinates) are one-parameter groups. Thus it follows (as in the proof of 4.5) that some power of $k$ is in $C - C'$ and this contradicts the assumption that $K \subset C'H$. Since $C'H$ is open in $G$ this, together with 4.3 and 3.5, implies the following result.

**5.2. Theorem** *Let $G$ be a compact Lie group and $H \subset G$ a closed subgroup. Then there exists a representation $\varrho$: $G \to \mathbf{O}(m)$, for some $m$, and a point $v \in \mathbf{R}^m$ with $G_v = H$.* ∎

With the above notation, note that the local cross section $C$ of the left cosets of $H$ in $G$ at $e$ maps homeomorphically to a neighborhood of $eH$ in $G/H$. The inverse of this can be used to define local coordinates on $G/H$ near $eH$. Suppose that $H = G_v$ for a given representation $\varrho$: $G \to \mathbf{O}(m)$ as in 5.2 (with $G$ compact Lie). Then $gH \mapsto g(v)$ defines a map $\theta$: $G/H \to \mathbf{R}^m$ which is easily seen to be a homeomorphism into. In terms of the indicated local coordinates, the differential of $\theta$ at $eH$ can be thought of as the map taking $w \in W$ to the tangent vector $\theta_*(w)$ at $t = 0$ to the curve $t \mapsto \varrho(\exp tw)(v)$ in $\mathbf{R}^m$. Since $\varrho(\exp tw)$ is a one-parameter group in $\mathbf{O}(m)$, it has the form $e^{tA}$ for some $m \times m$ matrix $A$. By 2.2 it follows that $\theta_*(w) = 0$ iff $v = e^{tA}v = \varrho(\exp tw)(v)$ for all $t$. However, this holds iff $\exp(tw) \in G_v = H$ for all $t$, which holds only for $w = 0$ by the definition of $W \subset V^\perp$.

Thus the differential of $\theta$: $G/H \to \mathbf{R}^m$ at $eH$ is a monomorphism. In fact, coordinate systems can be defined near the other points of $G/H$ by left translation of the given coordinates, and it follows that, with this analytic structure on $G/H$, $\theta$ embeds $G/H$ as an analytic submanifold of $\mathbf{R}^m$.

We conclude this section with a discussion of the simply connected covering group of a Lie group $G$. Since the underlying topological space of a connected Lie group $G$ has nice local properties, there is a simply connected covering space $\tilde{G}$ of $G$. We select a point $\tilde{e} \in \tilde{G}$ over $e \in G$, once and for all. Since $\tilde{G} \times \tilde{G}$ is a simply connected covering space of $G \times G$, there is a unique map $\tilde{G} \times \tilde{G} \to \tilde{G}$ which takes $(\tilde{e}, \tilde{e}) \mapsto \tilde{e}$ and covers the multiplication map $G \times G \to G$. We denote this map by $(\tilde{g}, \tilde{h}) \mapsto \tilde{g}\tilde{h}$. The two maps $(\tilde{g}, \tilde{h}, \tilde{k}) \mapsto (\tilde{g}\tilde{h})\tilde{k}$ and $(\tilde{g}, \tilde{h}, \tilde{k}) \mapsto \tilde{g}(\tilde{h}\tilde{k})$ of $\tilde{G} \times \tilde{G} \times \tilde{G} \to \tilde{G}$ both take $(\tilde{e}, \tilde{e}, \tilde{e})$ to $\tilde{e}$ and cover the same map $G \times G \times G \to G$. Thus they coincide. The map $\tilde{G} \to \tilde{G}$ taking $\tilde{g}$ to $\tilde{g}\tilde{e}$ covers the identity on $G$ and takes $\tilde{e}$ to $\tilde{e}$. Thus $\tilde{g}\tilde{e} = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$. Similarly, $\tilde{e}\tilde{g} = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$. There is also a unique map $\tilde{G} \to \tilde{G}$ taking $\tilde{e}$ to $\tilde{e}$ and covering $g \mapsto g^{-1}$ on $G$. If we denote by $\tilde{g}^{-1}$ the image of $\tilde{g} \in \tilde{G}$ under this map, then the map $\tilde{g} \mapsto \tilde{g}\tilde{g}^{-1}$ of $\tilde{G} \to \tilde{G}$ covers the constant map $g \mapsto e$ and takes $\tilde{e}$ to $\tilde{e}$. Thus $\tilde{g}\tilde{g}^{-1} = \tilde{e}$ for all $\tilde{g} \in \tilde{G}$.

These remarks show that, with the given choice of $\tilde{e} \in \tilde{G}$, there is a unique topological group structure on $\tilde{G}$ such that the projection $\tilde{G} \to G$ is a homomorphism. Since $\tilde{e}$ has a neighborhood $\tilde{U}$ in $\tilde{G}$ mapping homeomorphically onto a neighborhood $U$ of $e$ in $G$, it follows that $\tilde{G}$ is a Lie group. It is called the **universal covering group** of $G$. The kernel $K$ of the projection $\tilde{G} \to G$ is a discrete normal subgroup of $\tilde{G}$ and hence is central in $\tilde{G}$ by 1.11. Thus every connected Lie group $G$ can be obtained from some simply connected Lie group $\tilde{G}$ by factoring out by a discrete subgroup $K$ in the center of $\tilde{G}$. Two connected Lie groups are said to be **locally isomorphic** if they have isomorphic universal covering groups.

Let us now consider the case in which $G$ is connected and *abelian.*

**5.3. Lemma** *Let $G$ be a connected abelian $n$-dimensional Lie group. Then* exp: $\mathbf{R}^n \to G$ *is a homomorphism (where* $\mathbf{R}^n$ *has its additive group structure) and identifies* $\mathbf{R}^n$ *with the universal covering group of $G$.*

*Proof* For $u, v \in \mathbf{R}^n$ and $t \in \mathbf{R}$ the map $\varphi\colon \mathbf{R} \to G$ given by $\varphi(t) = \exp(tu)\exp(tv)$ clearly defines a one-parameter group, since $G$ is abelian. The tangent vector to this at $t = 0$ is just $\mu_*(u, v)$, where $\mu_*\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is the differential at $(e, e)$ of the multiplication map $\mu\colon G \times G \to G$. However, $\mu_*$ is linear and $\mu_*(u, 0) = u$ and $\mu_*(0, v) = v$, since $\mu(g, e) = g$ and $\mu(e, g) = g$. Thus $\mu_*(u, v) = u + v$ and we conclude that $\varphi$ is just the one-parameter group $\varphi(t) = \exp(t(u + v))$ with tangent vector $u + v$. Putting $t = 1$, we have

$$\exp(u + v) = \exp(u)\exp(v),$$

so that exp is a homomorphism. The image of exp is then clearly an open subgroup of $G$. It is also closed since its complement is a union of cosets, which must also be open. Thus exp: $\mathbf{R}^n \to G$ is a covering. ∎

It follows, for a connected abelian Lie group $G$, that $G \approx \mathbf{R}^n/N$, where $N$ is a discrete subgroup of $\mathbf{R}^n$. It is not very difficult to show that there is a basis $v_1, \ldots, v_n$ of $\mathbf{R}^n$ such that $N$ is just the subgroup generated by $v_1, \ldots, v_k$ for some $k$. Thus $G \approx \mathbf{T}^k \times \mathbf{R}^{n-k}$, where $\mathbf{T}^k$ is the $k$-dimensional torus (i.e., the product of $k$ copies of the circle group $\mathbf{S}^1 = \mathbf{U}(1) \approx \mathbf{R}/\mathbf{Z}$). Thus we have the following characterization of connected abelian Lie groups:

**5.4. Theorem** *A connected abelian Lie group $G$ is isomorphic to $\mathbf{T}^k \times \mathbf{R}^{n-k}$ for some $n$, $k$.* ▌

## 6. THE STRUCTURE OF COMPACT LIE GROUPS

In this section we shall first develop the properties of maximal tori in compact Lie groups. We shall do this is some detail since it will frequently be important to us in later chapters. Then we shall outline the classification theory of compact connected Lie groups. Since we will not make use of the latter material in this book, we shall make no attempt at indicating the proofs of most of the results on classification.

A **toral group** is a compact, connected, abelian, Lie group, and hence a product of circle groups. A **maximal torus** $T$ in a compact Lie group $G$ is a toral subgroup of $G$ which is not properly contained in any larger toral subgroup of $G$.

**6.1. Lemma** *The automorphism group* Aut $T$ *of a torus $T$ is discrete in the compact-open topology.*

*Proof* Regard $\mathbf{R}^n$ as the universal covering group of $T = \mathbf{R}^n/\mathbf{Z}^n$. Each automorphism of $T$ clearly lifts to an automorphism of $\mathbf{R}^n$ preserving $\mathbf{Z}^n$, and conversely. Thus Aut $T$ may be regarded as the subgroup of Aut $\mathbf{R}^n$ $= \mathbf{Gl}(n, \mathbf{R})$ preserving the integer lattice $\mathbf{Z}^n$. That is, these are matrices with integer coefficients and determinant $\pm 1$, and these form a discrete subgroup of $\mathbf{Gl}(n, \mathbf{R})$. ▌

**6.2. Corollary** *If $T$ is a maximal torus of $G$, then $N(T)/T$ is finite, where $N(T)$ is the normalizer of $T$ in $G$.*

*Proof* The normalizer $N(T)$ is closed and hence is a Lie group. Its connected component $N_0(T)$ of the identity operates trivially on $T$ (by conjugation) by 6.1. If $\dim N_0(T) > \dim T$, let $S$ be the image of a one-parameter group in $N_0(T)$ which is not in $T$. Then $S$ and $T$ generate a connected abelian group whose closure $T'$ is also abelian, connected, and compact. This contradicts maximality of $T$, and hence $N_0(T) = T$. Since $N(T)$ is a Lie group, this implies that $T$ is open in $N(T)$ and hence that $N(T)/T$ is finite. ▮

We remark that $N(T)/T$ is called the **Weyl group** of $G$.

We use

$$\chi(M) = \sum (-1)^i \operatorname{rank} H_i(M)$$

to denote the euler characteristic of a space $M$.

**6.3. Proposition** *Let $T$ be a maximal torus in the compact Lie group $G$ and put $N = N(T)$. Then $\chi(G/N) = 1$ and $\chi(G/T) = \operatorname{ord}(N/T)$.*

*Proof* Consider the action of $T$ on $G/N$ by left translation (see Chapter I). The point $eN$ of $G/N$ is clearly fixed by this action, since $T \subset N$. The differential of this action at the fixed point $eN$ (with respect to local coordinates about this point) is a representation of $T$. By 3.5, this representation is equivalent to an orthogonal representation, and this means that local coordinates may be taken in $G/N$ about the point $eN$ in terms of which $T$ acts orthogonally in the coordinate patch. In particular, there is a disk $D^m \subset G/N$ about $eN$ as origin which is preserved by the action of $T$. Thus $D^m$, $\partial D^m$, $\operatorname{int} D^m$, and $M = (G/N) - \operatorname{int} D^m$ are preserved by this action of $T$.

It is not hard to see that there is an element $t \in T$ whose powers are dense in $T$. (Such an element is called a "generator" of $T$.) Now $gN$ is fixed by $t$ iff $tgN = gN$. This holds iff $g^{-1}tg \in N$ and hence iff $g^{-1}Tg \subset N$ since the powers of $t$ are dense in $T$. However, this holds iff $g^{-1}Tg = T$ (i.e., $g \in N$) since $T$ is the identity component of $N$. Thus $eN$ is the only fixed point of $t$ on $G/N$. Since $t$ has no fixed points on $\partial D^m = S^{m-1}$ and is homotopic to the identity (since it is contained in the action of the connected group $T$), the Lefschetz Fixed Point Theorem implies that $m$ is even [i.e., $\chi(S^{m-1}) = 0$]. Similarly, $t$ has no fixed points on $M = (G/N) - \operatorname{int} D^m$ and is homotopic to the identity there, so that $\chi(M) = 0$ by the Lefschetz Fixed Point Theorem. From the exact homology sequence

$$\cdots \to H_i(M) \to H_i(G/N) \to H_i(G/N, M) \to H_{i-1}(M) \to \cdots$$

and the fact that

$$H_i(G/N, M) \approx H_i(D^m, \partial D^m)$$

we see that

$$\chi(G/N) = \chi(M) + (-1)^m = 1$$

since $m$ is even. Now $G/T$ is a covering space of $G/N$ with $\operatorname{ord}(N/T)$ sheets. By counting simplices in a triangulation of $G/N$ and those in the induced triangulation of $G/T$ over it, we see that

$$\chi(G/T) = \operatorname{ord}(N/T)\chi(G/N) = \operatorname{ord}(N/T). \quad \blacksquare$$

**6.4. Theorem** *Let $T$ be a maximal torus of the compact connected Lie group $G$ and let $g \in G$. Then there exists an element $k \in G$ such that $k^{-1}gk \in T$.*

***Proof*** Consider the transformation $G/T \to G/T$ taking $hT$ to $ghT$. This is homotopic to the identity since $G$ is connected. Thus the Lefschetz number of this map is $\chi(G/T) \neq 0$, and it follows that there is a fixed point $kT$. Then $gkT = kT$, so that $k^{-1}gk \in T$. $\blacksquare$

**6.5. Corollary** *Any two maximal tori of a compact Lie group $G$ are conjugate in $G$.*

***Proof*** If $T$ and $T'$ are maximal tori, let $t \in T'$ generate $T'$ (i.e., its powers are dense in $T'$). Then $g^{-1}tg \in T$ for some $g \in G$ and it follows that $g^{-1}T'g \subset T$. Thus $gTg^{-1} \supset T'$ and they must be equal by maximality. $\blacksquare$

**6.6. Corollary** *Let $T$ be a maximal torus of a compact Lie group $G$ and let $A \subset T$ be any subset. Suppose that $g \in G$ is such that $gAg^{-1} \subset T$. Then there exists an element $k \in N(T)$ such that $kak^{-1} = gag^{-1}$ for all $a \in A$.*

***Proof*** Let $H$ be the identity component of the subgroup $\{h \in G \mid hah^{-1} = a \text{ for all } a \in A\}$. (This is closed in $G$ and hence a Lie group.) Then $T \subset H$ and, since $A \subset g^{-1}Tg$, we also have $g^{-1}Tg \subset H$. Thus these are both maximal tori of $H$ and 6.5 implies that there is an element $h \in H$ with $hTh^{-1} = g^{-1}Tg$. Put $k = gh \in N(T)$. Then $kak^{-1} = ghah^{-1}g^{-1} = gag^{-1}$ for all $a \in A$. $\blacksquare$

**6.7. Corollary** *Let $T$ be an arbitrary torus in the compact connected Lie group $G$ and let $g \in G$ be an element that commutes with each element of $T$. Then there is a maximal torus of $G$ containing both $g$ and $T$.*

***Proof*** Let $g \in T_0$, a maximal torus of $G$. Let $H$ be the identity component of the subgroup $\{h \in G \mid hg = gh\}$. Then $T \subset H$ and $T_0 \subset H$, so that

$hT_0h^{-1} \supset T$ for some $h \in H$ by 6.5 applied to $H$. However, $hgh^{-1} = g$, so that $hT_0h^{-1}$ is the required maximal torus. ▮

**6.8. Corollary** *Let $T$ be an arbitrary torus in the compact connected Lie group $G$. Then the centralizer $Z(T)$ of $T$ in $G$ is connected and is precisely the identity component of $N(T)$.*

*Proof* From 6.7 we see that $Z(T)$ is the union of the maximal tori containing $T$, and hence it is connected. Thus $Z(T) \subset N_0(T)$. Conversely, $N_0(T) \subset Z(T)$ by 6.1. ▮

As promised, we now turn to the classification theory of compact connected Lie groups $G$, omitting most of the proofs.

Consider the action of $G$ on itself by conjugation. If $n = \dim G$, then the differential of this action at $e \in G$ gives a representation $\mathrm{Ad}\colon G \to \mathbf{Gl}(n, \mathbf{R})$ called the **adjoint representation**. Since $G$ is compact, this is an orthogonal representation with respect to a suitable basis. By 3.6 this representation is the direct sum of irreducible representations. Let $V_0$ denote the subspace of fixed vectors and let $V_1, \ldots, V_k$ denote the nontrivial irreducible components of the adjoint representation. Thus $\mathbf{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ as a representation space of $G$. If $v \in V_0$ then $\exp(tv)$ is fixed under conjugation, so that this is a one-parameter group in the center of $G$. From this it is clear that $\exp(V_0)$ is precisely the identity component $T_0$ (a torus) of the center of $G$. If $i \geq 1$, then it can be shown that $\exp(V_i)$ is a compact connected *subgroup* $G_i \subset G$ and has $\exp(U_i)$ as a neighborhood of $e$, where $U_i$ is a small neighborhood of 0 in $V_i$. (This is difficult.) It is clear that $G_i$ is a normal subgroup of $G$.

Moreover, it is clear that the $G_i$ are **simple**, by which we mean that they are *nonabelian* and have no nontrivial, proper, *connected* normal subgroups (since the $V_i$ are irreducible). Conversely, it can be seen that each simple normal subgroup of $G$ is one of the $G_i$. Thus this decomposition is unique. If $g \in G_i$ and $h \in G_j$ for $i \neq j$, then $ghg^{-1}h^{-1} \in G_i \cap G_j$. For $g$ and $h$ both near $e$, this implies that $gh = hg$. Since the $G_i$ are connected, we conclude that $G_i$ commutes with $G_j$ for $i \neq j$. The inclusions then induce a homomorphism of Lie groups

$$T_0 \times G_1 \times \cdots \times G_k \to G$$

which is a homeomorphism in the neighborhood of $e$. Thus the kernel of this is a finite normal (hence central) subgroup of the left-hand side. This shows the following basic fact.

**6.9. Theorem** *Every compact connected Lie group G has the form*

$$G \approx (T_0 \times G_1 \times \cdots \times G_k)/K,$$

*where K is a finite subgroup of the center of the product, $T_0$ is the identity component of the center of G, and the $G_i$ are the simple normal subgroups of G.* ∎

We remark that if $T_0$ is trivial, then $G$ is called **semisimple**; that is, it is locally isomorphic to a product of simple compact Lie groups. The following fact shows that the $G_i$ may be replaced by their simply connected covering groups in 6.9 without losing compactness.

**6.10. Theorem** *The universal covering group of a semisimple compact Lie group G is also compact.*

*Proof* We shall only indicate the argument for this. It suffices to show that $\pi_1(G)$ is finite. Since it is abelian, we need only show that $H_1(G)$ is finite. By the Universal Coefficient Theorem it suffices to show that $H^1(G; \mathbf{R}) = 0$. By the de Rham Theorem and an integration argument, it suffices to show that there are no nonzero differential one-forms $\omega$ on $G$ which are invariant under both right and left translation (and hence under conjugation). However such a one-form $\omega \neq 0$ annihilates an $(n-1)$-dimensional subspace $V$ of $\mathbf{R}^n$ (the tangent space at $e$ in $G$) and this subspace $V$ is invariant under the adjoint representation. The orthogonal complement $V^\perp$ is then fixed under the adjoint representation (since it has dimension 1 and $G$ is connected and compact). Thus exp: $V^\perp \to G$ defines a one-parameter group in the center of $G$, contrary to $G$ being semisimple. ∎

By 6.9 we know all compact connected Lie groups once we know the simply connected, simple, compact Lie groups and their centers. We shall simply state the well-known results on this. Let $\mathbf{A}_k$ denote the local isomorphism class of $\mathbf{SU}(k+1)$; $\mathbf{B}_k$, that of $\mathbf{SO}(2k+1)$; $\mathbf{C}_k$, that of $\mathbf{Sp}(k)$; and $\mathbf{D}_k$, that of $\mathbf{SO}(2k)$. (The subscripts $k$ represent the rank of the group, which is defined to be the dimension of its maximal torus.)

There are five so-called "exceptional" simple Lie groups, whose local isomorphism classes are denoted by $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$. (The dimensions of these groups are, respectively, 14, 52, 78, 133, and 248.)

It is known that the $\mathbf{A}_k$, $k \geq 1$; $\mathbf{B}_k$, $k \geq 2$; $\mathbf{C}_k$, $k \geq 3$; $\mathbf{D}_k$, $k \geq 4$; $\mathbf{G}_2$; $\mathbf{F}_4$; $\mathbf{E}_6$; $\mathbf{E}_7$; and $\mathbf{E}_8$ form a complete list, without repetition, of the local isomorphism classes of the compact simple Lie groups. For low values of the rank $k$, there are the following local isomorphisms: $\mathbf{D}_1 \approx \mathbf{S}^1$ (not semisimple); $\mathbf{C}_1 \approx \mathbf{B}_1 \approx \mathbf{A}_1$; $\mathbf{C}_2 \approx \mathbf{B}_2$; $\mathbf{D}_2 \approx \mathbf{A}_1 \times \mathbf{A}_1$; and $\mathbf{D}_3 \approx \mathbf{A}_3$.

By an inductive argument using fibrations over spheres [$\mathbf{SU}(k+1)/\mathbf{SU}(k) \approx \mathbf{S}^{2k+1}$, etc.] one sees that $\mathbf{SU}(k+1)$ and $\mathbf{Sp}(k)$ are already simply connected, while $\mathbf{SO}(n)$ has fundamental group $\mathbf{Z}_2$ for $n \geq 3$. The universal (double) covering group of $\mathbf{SO}(n)$ is called the "spinor group" and is denoted by $\mathbf{Spin}(n)$. The centers of the *simply connected* representatives of the simple groups are given by the following table.

| $G$: | $\mathbf{A}_k$ | $\mathbf{B}_k$ | $\mathbf{C}_k$ | $\mathbf{D}_{2k}$ | $\mathbf{D}_{2k+1}$ | $\mathbf{G}_2$ | $\mathbf{F}_4$ | $\mathbf{E}_6$ | $\mathbf{E}_7$ | $\mathbf{E}_8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| center $\tilde{G}$: | $\mathbf{Z}_{k+1}$ | $\mathbf{Z}_2$ | $\mathbf{Z}_2$ | $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ | $\mathbf{Z}_4$ | 0 | 0 | $\mathbf{Z}_3$ | $\mathbf{Z}_2$ | 0 |

# CHAPTER I

## TRANSFORMATION GROUPS

In the first five sections of this chapter we shall give the definitions and a few simple properties of the basic notions which we shall deal with in this book. Thus a topological transformation group is defined in Section 1. Equivariant maps, equivalence of transformation groups, and the notion of an isotropy group are discussed in Section 2 and an equivariant version, due to Gleason, of the Tietze Extension Theorem is proved. Orbits and orbit spaces are discussed in Section 3 and the notion of orbit type is studied in Section 4. Section 5 contains a discussion of fixed point sets.

In Section 6 we discuss three general constructions of new transformation groups out of old ones, twisted products, fibered products or pullbacks, and equivariant attachings.

In Sections 7 and 8 we construct several strange examples of transformation groups, due to Conner, Floyd, Richardson, and the author. These will be of fundamental interest throughout the book.

In Section 9 we study the problem of lifting a transformation group on a space $X$ to one on a covering space of $X$.

## 1. GROUP ACTIONS

By a **topological transformation group** we mean a triple $(G, X, \Theta)$, where $G$ is a topological group, $X$ is a Hausdorff topological space, and $\Theta$: $G \times X \to X$ is a map such that:

(1) $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$ for all $g$, $h \in G$ and $x \in X$;

(2) $\Theta(e, x) = x$ for all $x \in X$, where $e$ is the identity of $G$.

The map $\Theta$ is called an **action** of $G$ on $X$. The space $X$, together with a given action $\Theta$ of $G$, is called a $G$-**space** (or, more precisely, a *left* $G$-space; the obvious analogous notion of a *right* $G$-space will also be used on a few occasions).

We shall often abuse precise terminology and use the same notation for a $G$-space as for the underlying Hausdorff topological space, regarding

$\Theta$ as understood. When $\Theta$ is understood from the context we shall often use the notation $g(x)$ or $gx$ for $\Theta(g, x)$ (or $xg$ in the case of a *right* $G$-space) so that (1) and (2) become $g(h(x)) = (gh)(x)$ and $e(x) = x$. Similarly, for $C \subset G$ and $A \subset X$ we put $C(A) = \{g(x) \mid g \in C, \ x \in A\}$. A set $A$ is said to be **invariant** under $G$ if $G(A) = A$.

For $g \in G$ let $\theta_g$: $X \to X$ be the map defined by $\theta_g(x) = g(x) = \Theta(g, x)$. Then $\theta_g\theta_h = \theta_{gh}$ and $\theta_e = 1_X$, the identity map of $X$, by (1) and (2). Thus

$$\theta_g\theta_{g^{-1}} = \theta_e = 1_X = \theta_{g^{-1}}\theta_g$$

which shows that each $\theta_g$ is a homeomorphism of $X$.

If **Homeo**($X$) denotes the group (under composition) of all homeomorphisms of $X$ onto itself, then $g \mapsto \theta_g$ defines a homomorphism

$$\theta\colon \ G \to \mathbf{Homeo}(X).$$

The kernel of this homomorphism $\theta$ will be called the "kernel of the action $\Theta$." Thus

$$\ker \Theta = \{g \in G \mid g(x) = x \text{ for all } x \in X\}$$

and is a normal subgroup of $G$. It is clearly *closed* in $G$.

The action $\Theta$ is called **effective** if $\ker \Theta$ is trivial (that is, if $\theta$ is an injection) and it is called **almost effective** if $\ker \Theta$ is a discrete subgroup of $G$.

For most purposes it suffices to consider only effective actions because of the following elementary proposition. It is, however, convenient for some purposes to allow some actions to be noneffective.

**1.1. Proposition** *Let $\Theta$ be an action of $G$ on $X$ and let $N = \ker \Theta$. Then there is a canonically induced effective action $\Theta/\ker \Theta$ of $G/N$ on $X$.*

*Proof* Define $\Theta/\ker \Theta$: $(G/N) \times X \to X$ by $(gN)(x) = g(x)$. We must check continuity of this map. The projection $\pi$: $G \to G/N$ is open and thus from the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\ \Theta\ } & X \\ {\scriptstyle \pi\times 1}\downarrow & \nearrow{\scriptstyle \Theta/\ker\Theta} & \\ G/N \times X & & \end{array}$$

the continuity of $\Theta/\ker \Theta$ follows from the equation

$$(\Theta/\ker \Theta)^{-1}(U) = (\pi \times 1)(\Theta^{-1}(U))$$

for $U$ open in $X$. Clearly, $\Theta/\ker \Theta$ is effective. ∎

Now we shall prove some basic facts which hold only for actions of *compact* groups.

**1.2. Theorem** *If $\Theta$: $G \times X \to X$ is an action of a compact group $G$ on $X$, then $\Theta$ is a closed map.*

***Proof*** Recall our standing assumption that $X$ is Hausdorff. Let $C \subset G \times X$ be closed and let $y$ be in the closure of $\Theta(C)$. Then there is a net $(g_\alpha, x_\alpha)$ in $C$ such that $\Theta(g_\alpha, x_\alpha) = g_\alpha(x_\alpha)$ converges to $y$. Passing to a subnet we may assume that $g_\alpha$ converges to $g$, since $G$ is compact. Then $x_\alpha = \Theta(g_\alpha^{-1}, g_\alpha(x_\alpha))$ converges to $\Theta(g^{-1}, y) = g^{-1}(y)$. Thus $(g_\alpha, x_\alpha)$ converges to $(g, g^{-1}(y)) \in C$ since $C$ is closed. Thus $y = \Theta(g, g^{-1}(y)) \in \Theta(C)$. ▮

**1.3. Corollary** *If $G$ is compact and $X$ is a $G$-space, then $G(A)$ is closed in $X$ for each closed $A \subset X$, and $G(A)$ is compact if $A$ is compact.* ▮

We shall now give some examples of actions. First there are some actions associated with any topological group $G$. Let $H \subset G$ be a closed subgroup. Then $G$ acts on $G/H$ through left translation $L_g(g'H) = gg'H$. The kernel of this action is clearly $\bigcap_{g \in G} gHg^{-1}$. If $N(H)$ is the normalizer of $H$ in $G$, then $N(H)$ acts on $G/H$ by right translation $R_n(gH) = gn^{-1}H$ and $H$ is clearly the kernel of this action, so that there is an induced effective action of $N(H)/H$ on $G/H$. Another example is the action of $G$ on itself by conjugation $C_g(h) = ghg^{-1}$. The kernel of this is the center of $G$. The continuity of all of these actions is an immediate consequence of the continuity of multiplication $G \times G \to G$ and of the inverse $G \to G$ in $G$.

The general linear group $\mathbf{Gl}(n, \mathbf{R})$ acts on $\mathbf{R}^n$ (similarly over the complex numbers or the quaternions). Thus any representation $G \to \mathbf{Gl}(n, \mathbf{R})$ defines an action of $G$ on $\mathbf{R}^n$. Orthogonal representations $G \to \mathbf{O}(n, \mathbf{R})$ also give actions of $G$ on the unit disk $\mathbf{D}^n$ and on the unit sphere $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$.

If $\mathbf{S}^1$ denotes the group of complex numbers of norm 1, and we let $\mathbf{T}^2$ be the torus $\mathbf{S}^1 \times \mathbf{S}^1$, then there is an interesting action of the additive group $\mathbf{R}$ of real numbers on $\mathbf{T}^2$, called an "irrational flow," defined by

$$\theta_r(e^{2\pi ix}, e^{2\pi iy}) = (e^{2\pi i(x+r)}, e^{2\pi i(y+\alpha r)}),$$

where $\alpha$ is irrational. Of course this is just the homomorphism $r \mapsto (e^{2\pi ir}, e^{2\pi i\alpha r})$ of $\mathbf{R} \to \mathbf{T}^2$ followed by left translation of $\mathbf{T}^2$ on itself.

The examples given above are all important ones, but, of course, they are rather elementary. Later we shall give many examples of a deeper and more interesting nature.

## 2. EQUIVARIANT MAPS AND ISOTROPY GROUPS

For a fixed topological group $G$, the $G$-spaces form a category whose morphisms are called "equivariant maps." An **equivariant map** (or a $G$-map) $\varphi: X \to Y$ between $G$-spaces is a map which commutes with the group actions, that is,

$$\varphi(g(x)) = g(\varphi(x)) \qquad \text{for all } g \in G \text{ and } x \in X.$$

An equivariant map $\varphi: X \to Y$ which is also a homeomorphism is called an **equivalence** of $G$-spaces. In this case we note that the inverse $\varphi^{-1}$ of $\varphi$ is also equivariant, for if $y = \varphi(x)$, then

$$\varphi^{-1}(g(y)) = \varphi^{-1}(g(\varphi(x))) = \varphi^{-1}\varphi(g(x)) = g(x) = g(\varphi^{-1}(y)).$$

Two actions which are equivalent cannot be topologically distinguished from one another. Thus they are regarded as essentially the same. It is also reasonable to regard two actions as essentially the same if they differ only by an automorphism of $G$. Thus we say that two $G$-spaces $X$ and $Y$ are **weakly equivalent** if there is an automorphism $\alpha$ (continuous) of $G$ and a homeomorphism $\varphi: X \to Y$ with

$$\varphi(g(x)) = \alpha(g)(\varphi(x)) \qquad \text{for all } g \in G \text{ and } x \in X.$$

*Example* Let $G$ be the cyclic group of order 5, whose elements are the fifth roots of unity. Let $X$ be the unit circle in the complex plane. Then the action given by $(\gamma, z) \mapsto \gamma z$ is inequivalent to that given by $(\gamma, z) \mapsto \gamma^2 z$. (Why?) However, these two actions are weakly equivalent since $\gamma \mapsto \gamma^2$ is an automorphism of $G$.

Now suppose that $X$ is a $G$-space and let $x \in X$. The set

$$G_x = \{g \in G \mid g(x) = x\}$$

of elements of $G$ leaving $x$ fixed is clearly a closed subgroup of $G$. Thus $G_x$ is called the **isotropy subgroup** (or **stability subgroup**) of $G$ at $x$. Since $gG_xg^{-1}(g(x)) = gG_x(x) = g(x)$ we have that $gG_xg^{-1} \subset G_{g(x)}$. Conversely $g^{-1}G_{g(x)}g \subset G_{g^{-1}g(x)} = G_x$ so that

**(2.1)** $$G_{g(x)} = gG_xg^{-1}.$$

Thus the translates by $G$ of a given point all have conjugate isotropy subgroups, and all conjugates appear.

**2.2. Proposition** *If $\varphi: X \to Y$ is an equivariant map between G-spaces, then $G_x \subset G_{\varphi(x)}$ for all $x \in X$.* ∎

Note that the kernel of an action is just $\bigcap_{x \in X} G_x$. An action of $G$ on $X$ is said to be **free** if $G_x$ is trivial for *each* $x \in X$. We shall see later that free actions of compact Lie groups are particularly simple. An action is called **semifree** if, for each $x \in X$, $G_x$ is either trivial or is all of $G$.

We rephrase these definitions: An action of $G$ on $X$ is effective if each $g \neq e$ in $G$ moves at least one point. It is semifree if each point of $X$ is either left fixed by all elements of $G$ or is moved by all nontrivial elements of $G$. It is free if each nontrivial element of $G$ moves every point of $X$.

To conclude this section we shall prove the following important result concerning extension of equivariant maps.

**2.3. Tietze–Gleason Theorem**[†] *Let $G$ be a compact group acting on a normal space $X$ and let $A \subset X$ be a closed invariant subspace. Let $\varrho: G \to \mathbf{Gl}(n, \mathbf{R})$ be a representation of $G$ and let $\varphi: A \to \mathbf{R}^n$ be equivariant; that is $\varphi(g(a)) = \varrho(g) \cdot \varphi(a)$. Then there exists an equivariant extension $\psi: X \to \mathbf{R}^n$ of $\varphi$.*

*Proof* By the Tietze Extension Theorem $\varphi$ extends to a map $\varphi': X \to \mathbf{R}^n$. To produce an *equivariant* extension we simply "average" $\varphi'$ as follows. Define

$$\psi(x) = \int \varrho(g^{-1})\varphi'(g(x))\, dg$$

where the integral is the normalized Haar integral on $G$ (applied to the indicated vector-valued function); see Chapter 0, Section 3. Then for $h \in G$ we compute

$$\begin{aligned} \psi(h(x)) &= \int \varrho(g^{-1})\varphi'(g(h(x)))\, dg \\ &= \int \varrho(h(gh)^{-1})\varphi'(gh(x))\, dg \\ &= \int \varrho(h)\varrho((gh)^{-1})\varphi'(gh(x))\, dg \\ &= \varrho(h) \int \varrho(k^{-1})\varphi'(k(x))\, dk \\ &= \varrho(h)\psi(x) \end{aligned}$$

[†] See Gleason [1], Palais [3, 4].

by the linearity of integration and the invariance of the integral under right translation (by $h$). Thus $\psi$ is equivariant. Moreover, for $a \in A$ we have

$$\begin{aligned}\psi(a) &= \int \varrho(g^{-1})\varphi'(g(a))\,dg \\ &= \int \varrho(g^{-1})\varphi(g(a))\,dg \\ &= \int \varphi(a)\,dg = \varphi(a)\int dg = \varphi(a)\end{aligned}$$

by the equivariance of $\varphi$ and normalization of the integral. Thus $\psi$ does extend $\varphi$. Finally, $\psi$ is continuous by 0.3.2. ∎

## 3. ORBITS AND ORBIT SPACES

If $X$ is a $G$-space and $x \in X$, then the subspace

$$G(x) = \{g(x) \in X \mid g \in G\}$$

is called the **orbit** of $x$ (under $G$). Note that if $g(x) = h(y)$ for some $g,\ h \in G$ and $x,\ y \in X$, then for any $g' \in G$, $g'(x) = g'g^{-1}g(x) = g'g^{-1}h(y) \in G(y)$ so that $G(x) \subset G(y)$; conversely $G(y) \subset G(x)$. Thus the orbits $G(x)$ and $G(y)$ of any two points $x$, $y$ in $X$ are either equal or disjoint.

We let $X/G$ denote the set whose elements are the orbits $x^* = G(x)$ of $G$ on $X$. (Thus $x^* = y^*$ iff $x$ and $y$ are in the same orbit.) Let $\pi = \pi_X$: $X \to X/G$ denote the natural map taking $x$ into its orbit $x^* = G(x)$. Then $X/G$ endowed with the quotient topology ($U \subset X/G$ is open iff $\pi^{-1}(U)$ is open in $X$) is called the **orbit space** of $X$ (with respect to $G$).

If $A \subset X$, then $\pi^{-1}\pi(A) = \{g(a) \mid g \in G \text{ and } a \in A\} = G(A)$ is the union of the orbits of elements of $A$ and is called the **saturation** of $A$.

If $U$ is open, then $G(U) = \bigcup_{g\in G} g(U)$ is open since each $g(U) = \theta_g(U)$ is open ($\theta_g$ being a homeomorphism of $X$). Thus $\pi^{-1}\pi(U) = G(U)$ is open for $U$ open in $X$ and, by definition, this means that $\pi(U)$ is open in $X/G$. Thus $\pi$: $X \to X/G$ is a continuous *open* map.

The reader may note that for the irrational flow on the torus, defined in Section 1, the orbit space has the trivial (indiscrete) topology and hence is not a very interesting space. However, for actions of *compact* groups $G$, the orbit space has reasonably pleasant properties, as we show below.

**3.1. Theorem** *If $X$ is a $G$-space with $G$ compact, then*

(1) *$X/G$ is Hausdorff.*
(2) *$\pi: X \to X/G$ is closed.*
(3) *$\pi: X \to X/G$ is proper [$\pi^{-1}$(compact) is compact].*
(4) *$X$ is compact iff $X/G$ is compact.*
(5) *$X$ is locally compact iff $X/G$ is locally compact.*

*Proof* To prove (2) let $A \subset X$ be closed. Then $G(A)$ is closed by 1.3. But $G(A) = \pi^{-1}\pi(A)$ so that $\pi(A)$ is closed by definition of the topology of $X/G$ (look at complements).

To prove (1) suppose $G(x) \neq G(y)$. Since $G \to G \times \{y\} \to G(y)$ is continuous, each orbit $G(y)$ is compact. It is a standard fact that in a Hausdorff space any two disjoint compact subsets can be separated by open sets. In particular, there is an open neighborhood $U$ of $x$ with $\bar{U} \cap G(y) = \emptyset$. Since $\pi(y) \notin \pi(\bar{U})$ it follows that $\pi(U)$ and $X/G - \pi(\bar{U})$ are disjoint open sets separating $x^*$ and $y^*$.

Since the orbits are compact, (3) follows from the general fact that a closed map $\pi: X \to Y$ is proper if $\pi^{-1}(y)$ is compact for each $y \in Y$. For completeness we shall prove this. Let $C \subset Y$ be compact and let $\{U_\alpha \mid \alpha \in A\}$ be an open covering of $\pi^{-1}(C)$. For each $y \in C$ there is a finite subset $A_y \subset A$ of indices such that the $U_\alpha$, $\alpha \in A_y$, cover $\pi^{-1}(y)$. Put $U_y = \bigcup\{U_\alpha \mid \alpha \in A_y\} \supset \pi^{-1}(y)$ and let $V_y = Y - \pi(X - U_y)$ which is open. Note that $\pi^{-1}V_y \subset U_y$, and $y \in V_y$. Let $V_{y_1}, \ldots, V_{y_n}$ cover $C$. Then

$$\pi^{-1}(C) \subset \pi^{-1}V_{y_1} \cup \cdots \cup \pi^{-1}V_{y_n} \subset U_{y_1} \cup \cdots \cup U_{y_n} = \bigcup\{U_\alpha \mid \alpha \in A_{y_i};\ i = 1, \ldots, n\},$$

a finite union.

Part (4) is clear from (1) and (3). To prove (5) suppose that $U \subset X$ is open, $x \in U$, and $\bar{U}$ is compact. Then $\pi(x) \in \pi(U) \subset \pi(\bar{U})$ so that $\pi(\bar{U})$ is a compact neighborhood of $\pi(x)$. Conversely, if $C$ is a compact neighborhood of $\pi(x)$, then $\pi^{-1}(C)$ is a compact neighborhood of $x$ by (3). ∎

We can now state a long-standing conjecture.

**Conjecture** (Conner [9]) The orbit space of any action of a compact Lie group on euclidean space $\mathbf{R}^n$, or on a disk $\mathbf{D}^n$, is contractible.

More general conjectures of the same type and some interesting partial information can be found in Conner [9], Conner and Floyd [1, 2], and Floyd [1].

A **cross section** for $\pi\colon X \to X/G$ is a continuous map $\sigma\colon X/G \to X$ such that $\pi\sigma$ is the identity on $X/G$. A **local cross section** defined on $U \subset X/G$ is a cross section of $\pi|\pi^{-1}(U)$. Local cross sections do not generally exist even for actions of compact Lie groups. For example, consider the action of the subgroup $G = \{I, -I\}$ of $\mathbf{O}(3)$ on $\mathbf{R}^3$. Then there is no local cross section defined in the neighborhood of $\pi(0)$ in $\mathbf{R}^3/G$. However, there are many cases for which local cross sections do exist and they are useful. A closely related notion, called a "slice," does exist under very general conditions for actions of compact Lie groups, and will be studied later.

**3.2. Proposition** *Let $X$ be a $G$-space with $G$ compact. Let $C$ be a closed subset of $X$ touching each orbit in exactly one point. Then the map $\sigma\colon X/G \to X$ defined by $\{\sigma(x^*)\} = G(x) \cap C$ is a cross section. Conversely, the image of a cross section is closed in $X$.*

*Proof* We need to show that $\sigma$ is continuous. For this let $A \subset C$ be closed. By 3.1, $\sigma^{-1}(A) = \pi(A)$ is closed, as desired. For the converse, let $C = \sigma(X/G)$ and let $\{x_\alpha\}$ be a net in $C$ converging to $x \in X$. We have $\lim \pi(x_\alpha) = \pi(x)$ so that

$$x = \lim x_\alpha = \lim \sigma\pi(x_\alpha) = \sigma\pi(x) \in C$$

and hence $C$ is closed. ∎

Because of this result, we shall often use the term "cross section" for the closed set which is the image of a cross section.

**3.3. Theorem** *Let $G$ be a compact group acting on the spaces $X$ and $Y$. Let $C \subset X$ be any closed set and let $\varphi\colon C \to Y$ be a map such that whenever $c$ and $g(c)$ are both in $C$ (for some $g \in G$), then $\varphi(g(c)) = g(\varphi(c))$. Then $\varphi$ can be extended uniquely to an equivariant map $\varphi'$ of $G(C)$ into $Y$.*

*Proof* For $g \in G$ and $c \in C$ we put $\varphi'(g(c)) = g(\varphi(c))$ which is clearly the only possibility for an equivariant extension. To see that $\varphi'$ is well defined let $g(c) = g'(c')$. Then $c = g^{-1}g'(c')$ so that $\varphi(c) = \varphi(g^{-1}g'(c'))$ $= g^{-1}g'\varphi(c')$ by assumption. Thus $g\varphi(c) = g'\varphi(c')$ as desired. To see that $\varphi'$ is continuous, let $\{x_\alpha\}$ be a net in $G(C)$ converging to $x \in G(C)$. Put $x_\alpha = g_\alpha(c_\alpha)$. By passing to a subnet we may assume that $\{g_\alpha\}$ converges to $g$.

Then $\lim c_\alpha = \lim g_\alpha^{-1}(x_\alpha) = g^{-1}(x) \in C$ since $C$ is closed. Let $c = g^{-1}(x)$. Then

$$\begin{aligned}\lim \varphi'(x_\alpha) &= \lim \varphi'(g_\alpha(c_\alpha)) \\ &= \lim g_\alpha \varphi(c_\alpha) \\ &= g\varphi(c) = \varphi'(g(c)) = \varphi'(x). \quad \blacksquare\end{aligned}$$

**3.4. Corollary** *Let $G$ be a compact group acting on the spaces $X$ and $Y$ and let $C \subset X$ be a cross section of $\pi$: $X \to X/G$. Let $\varphi$: $C \to Y$ be any map such that $G_c \subset G_{\varphi(c)}$ for all $c \in C$. Then there is a unique extension of $\varphi$ to an equivariant map $\varphi'$: $X \to Y$.*

***Proof*** If $c$, $g(c)$ are both in $C$, then $g \in G_c \subset G_{\varphi(c)}$ so that $\varphi(g(c)) = \varphi(c) = g\varphi(c)$. $\blacksquare$

## 4. HOMOGENEOUS SPACES AND ORBIT TYPES

An action of $G$ on $X$ is said to be **transitive** if there is precisely one orbit, $X$ itself. An example is a left coset space $G/H$ of a topological group $G$ by a closed subgroup $H$ together with the action of $G$ by left translation: $\Theta(g, g'H) = gg'H$.

If $x$ is a point in any $G$-space $X$, then there is the natural map

$$\alpha_x\colon G/G_x \to G(x)$$

defined by $\alpha_x(gG_x) = g(x)$. By the definition of the topology on $G/G_x$ and by the continuity of $g \mapsto g(x)$, $\alpha_x$ is continuous. It is also clear that $\alpha_x$ is one–one and onto. However, $\alpha_x$ may not be a homeomorphism. For example, it is not a homeomorphism in the case of the irrational flow on the torus. Since, however, a one–one map of a compact space onto a Hausdorff space is a homeomorphism, we have the following proposition.

**4.1. Proposition** *If $G$ is compact, then $\alpha_x$: $G/G_x \to G(x)$ is a homeomorphism.* $\blacksquare$

Note also that $\alpha_x$ is equivariant with respect to the left translation of $G$ on $G/G_x$ and the restricted action of $G$ on $G(x) \subset X$. Thus, for $G$ compact, $\alpha_x$ is an equivalence of transitive $G$-spaces.

We shall now restrict our attention in this section exclusively to actions of a given *compact* group $G$. The class of such $G$-spaces forms a category whose morphisms are the equivariant maps. The full subcategory of transitive $G$-spaces will be called the **category of $G$-orbits.** Note that by 4.1 any object in this category is isomorphic to some coset space $G/H$. Thus we shall characterize the morphisms on the coset spaces of $G$.

**4.2. Theorem** *Let $G$ be compact and $H$ and $K$ closed subgroups. Then:*

(1) *There exists an equivariant map $G/H \to G/K$ iff $H$ is conjugate to a subgroup of $K$.*

(2) *If $a \in G$ and $aHa^{-1} \subset K$ let $R_a^{K,H}\colon G/H \to G/K$ be given by $R_a^{K,H}(gH) = ga^{-1}K$. Then $R_a^{K,H}$ is equivariant and well defined.*

(3) *Every equivariant map $G/H \to G/K$ has the form $R_a^{K,H}$ for some $a \in G$ with $aHa^{-1} \subset K$.*

(4) $R_a^{K,H} = R_b^{K,H}$ *iff $ab^{-1} \in K$.*

*Proof* Suppose $f\colon G/H \to G/K$ is any map and put $f(H) = a^{-1}K$ for some $a \in G$. Then $f$ is equivariant iff $f(gH) = ga^{-1}K$ for all $g \in G$. Conversely, the formula $f(gH) = ga^{-1}K$ defines a map (i.e., is well defined), necessarily equivariant, provided only that $f(ghH) = f(gH)$ for all $h \in H$. That is, we must have $gha^{-1}K = ga^{-1}K$ for all $h \in H$. This holds iff $aha^{-1} \in K$ for all $h \in H$, that is, $aHa^{-1} \subset K$. This establishes (1)–(3). Clearly $R_a^{K,H} = R_b^{K,H}$ iff $a^{-1}K = b^{-1}K$, that is, $ab^{-1}K = K$. ∎

Note that if $K = aHa^{-1}$, then $R_a^{K,H}$ is right translation $gH \mapsto gHa^{-1} = (ga^{-1})aHa^{-1} = ga^{-1}K$. More generally for $K \supset aHa^{-1}$, then $R_a^{K,H} = R_e^{K,aHa^{-1}} \circ R_a^{aHa^{-1},H}$ which is the composition of right translation $gH \mapsto gHa^{-1} = (ga^{-1})aHa^{-1}$ with the natural map $G/aHa^{-1} \to G/K$ induced by the inclusion $aHa^{-1} \subset K$.

Also note that if $aHa^{-1} \subset H$, then $aHa^{-1} = H$ by 0.1.9 since $G$ is compact.

**4.3. Corollary** *Every equivariant map $G/H \to G/H$ is right translation by an element of $N(H)$ and is an equivalence of $G$-spaces. The map $a \mapsto R_a^{H,H}$ induces an isomorphism of $N(H)/H$ onto the group* $\mathbf{Homeo}^G(G/H)$ *(under composition) of self-equivalences of the $G$-space $G/H$ with the compact-open topology.*

*Proof* The right translation action $G/H \times N(H) \to G/H$ is clearly continuous, and this implies the continuity of the map $N(H) \to \mathbf{Homeo}^G(G/H)$ taking $a \mapsto R_a^{H,H}$ (see Dugundji [1]). Thus $N(H)/H \to \mathbf{Homeo}^G(G/H)$ is continuous, one–one, and onto, and hence it is a homeomorphism since $N(H)/H$ is compact. ▌

**4.4. Corollary** *If there exist equivariant maps $G/H \to G/K$ and $G/K \to G/H$, then each is an equivalence and $H$ is conjugate to $K$.* ▌

If we divide the category of $G$-orbits by equivalences, then we obtain the **category of $G$-orbit types.** This is the category of equivalence classes of transitive $G$-spaces (orbits). If $X$ is a $G$-orbit, then we let type($X$) denote its **type**, that is, its equivalence class under equivariant homeomorphisms. From the discussion above, type($X$) contains a coset space $G/H$. Moreover, type($G/H$) = type($G/K$) iff $H$ and $K$ are conjugate in $G$. A morphism type($G/H$) $\to$ type($G/K$) exists iff there is an equivariant map $G/H \to G/K$ and this holds iff $H$ is conjugate to a subgroup of $K$. If $X$ and $Y$ are $G$-orbits and a morphism type($X$) $\to$ type($Y$) exists (i.e., if there is an equivariant map $X \to Y$), then we write

$$\text{type}(X) \geq \text{type}(Y).$$

This gives a partial ordering of the orbit types with type($\star$) = type($G/G$) a minimum and type($G$) a maximum. Note that this partially ordered set is not the same as the category of $G$-orbit types, but results from it by the identification of certain morphisms; see V.4.3.

By (2.1) the isotropy groups which occur at points in an orbit form a complete conjugacy class of subgroups of $G$. Thus for an orbit $X$ we let its **$G$-isotropy type**($H$) be the conjugacy class of $H$ in $G$, where $X$ is equivalent to $G/H$. Then for $H$, $K$ subgroups of $G$ we have type($G/H$) $\geq$ type ($G/K$) iff $H$ is conjugate to a subgroup of $K$; written $(H) \leq (K)$. Thus the partially ordered set of $G$-orbit types is canonically antiisomorphic to that of $G$-isotropy types.

[In the literature, an isotropy type is called an "orbit type" and the notion we have called "orbit type" has no name. We prefer the terminology introduced here as being more descriptive.]

**Example** Let $G$ be the rotation group $\mathbf{SO}(3)$. Let $X$ be the space of all symmetric $3 \times 3$ real matrices of trace 0. (Note that this is a real vector space of dimension 5, so that $X \approx \mathbf{R}^5$.) For $g \in G$ and $x \in X$ define

$$\theta_g(x) = gxg^{-1}$$

(using matrix multiplication). It is a well-known fact from linear algebra that $x$ and $y$ are in the same orbit of this action iff the eigenvalues of the matrices $x$ and $y$ are equal, counting multiplicities. Thus, let $Y$ be the (topological) subspace of $\mathbf{R}^3$ consisting of triples $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Then the map $\pi$: $X \to Y$, taking the matrix $x$ into its eigenvalues in decreasing order, satisfies $\pi(x) = \pi(x') \Leftrightarrow G(x) = G(x')$.

This map has a right inverse $\sigma$: $Y \to X$, where

$$\sigma(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

It follows that $Y \approx X/G$ with $\pi$ corresponding to the orbit map and $\sigma$ corresponding to a cross section.

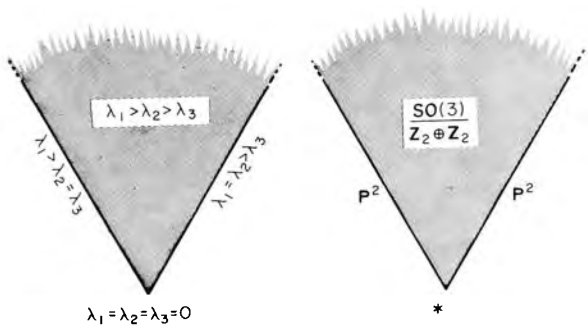



FIGURE I-1

Now $\lambda_1 + \lambda_2 + \lambda_3 = 0$ defines a plane in $\mathbf{R}^3$ and the inequalities $\lambda_1 \geq \lambda_2$ and $\lambda_2 \geq \lambda_3$ show that $Y$ is the region in the plane formed by two rays from the origin at an angle of 60°, together with the interior. It is illustrated in Figure I-1.

The isotropy group of the point $\sigma(\lambda_1, \lambda_2, \lambda_3)$ is

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

for $\lambda_1 > \lambda_2 > \lambda_3$. This group is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ [since $G = \mathbf{SO}(3)$]. For $\lambda_1 = \lambda_2 > \lambda_3$ the isotropy group consists of matrices of the form

$$\left[\begin{array}{cc|c} & & 0 \\ \multicolumn{2}{c|}{\star} & \\ & & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right]$$

which is the normalizer of $\mathbf{SO}(2)$. Similarly for $\lambda_1 > \lambda_2 = \lambda_3$. Such orbits are projective planes. Of course, for $\lambda_1 = \lambda_2 = \lambda_3 = 0$ the isotropy group is $\mathbf{SO}(3)$ itself and the orbit is a point. The orbit space together with representatives for the orbit types are illustrated on the right-hand side of Figure I-1.

## 5. FIXED POINTS

A point $x$ in a $G$-space $X$ is said to be **stationary** if $G(x) = \{x\}$, that is, $G_x = G$. A stationary point is also called a **fixed point** of $G$ on $X$. [In some contexts a point which is left fixed by *some* nontrivial element of $G$ (i.e., $G_x \neq \{e\}$) is called a "fixed point." However, it is a general practice (which we shall adhere to), in the theory of compact transformation groups, to regard the terms "stationary" and "fixed" as synonymous.] We denote the subspace of fixed points of $G$ on $X$ by

$$X^G = \{x \in X \mid g(x) = x \text{ for all } g \in G\}$$

and *also* by

$$F(G, X) = X^G.$$

[Of course when $H$ is a subgroup of $G$, the set of points fixed under $H$ is denoted by $F(H, X)$ or $X^H$.]

For a linear representation of $G$ on $\mathbf{R}^{n+1}$, $F(G, \mathbf{R}^{n+1})$ is clearly a linear subspace of $\mathbf{R}^{n+1}$. Thus for an orthogonal action of $G$ on $\mathbf{S}^n$ the fixed point set is a sphere $\mathbf{S}^r$. Consequently, if a $G$-space $X$ is equivalent to an orthogonal action on $\mathbf{S}^n$, then $X^G$ is homeomorphic to a sphere. It is natural to ask whether any action of a compact group on a sphere is equivalent to an orthogonal action. A weaker question is whether the fixed point set of an action on a sphere must be a sphere. The answer to this question was unknown for a long period of time. The first results concerning it were the positive results of Smith. We shall consider these important classical results of Smith in Chapter III. The first negative results in this direction were the examples of Floyd [2, 7, 9]. Other counterexamples of this type were given in Bredon [12, 19], Conner and Floyd [3], Conner and Montgomery [2], Floyd and Richardson [1], and Kister [1]. In Sections 7 and 8 we shall construct some of these counterexamples.

In the study of transformation groups it is often useful to consider the fixed point sets of subgroups $H$ of $G$. In this regard we note that there is a natural action of the normalizer $N(H)$ on $X^H$, where $X$ is a $G$-space; that is, the restriction of the $G$-action on $X$ to $N(X)$ on $X^H$. One need only note that $N(H)$ leaves $X^H$ invariant since, for $x \in X^H$ and $n \in N(H)$, we have $Hn(x) = nH(x) = n(x)$ and thus $n(x) \in X^H$.

As an example consider $F(H, G/H)$. A coset $gH$ is fixed under $H$ iff $HgH \subset gH$, which holds iff $g^{-1}Hg \subset H$. If $H$ is a compact Lie group, then this implies that $g^{-1}Hg = H$ since $g^{-1}Hg$ has the same dimension and number of components as does $H$. Thus, in this case, we have that $g \in N(H)$ so that

$$F(H, G/H) = N(H)/H.$$

Similarly we see that, for $K \subset G$, $F(K, G/H) \neq \varnothing$ iff $K$ is conjugate to a subgroup of $H$.

For another example let $H$ be a closed subgroup of a compact Lie group $G$ and let $T$ be a maximal torus of $H$. Consider $F(T, G/H)$. As above we see that $gH \in F(T, G/H)$ iff $g^{-1}Tg \subset H$. Since $g^{-1}Tg$ is also a maximal torus of $H$ and all maximal tori are conjugate in $H$ there exists an $h \in H$ with $(gh)^{-1}Tgh = h^{-1}g^{-1}Tgh = T$; that is, $gh \in N(T)$. Thus $g \in N(T)H$ and

$$gH \in \frac{N(T)H}{H} \approx \frac{N(T)}{H \cap N(T)}.$$

The converse clearly holds and thus we have

$$F(T, G/H) = \frac{N(T)H}{H} \approx \frac{N(T)}{H \cap N(T)}.$$

From the above two examples, one might expect that for any $K \subset H$, $F(K, G/H)$ is a coset space of $N(K)$ (as an $N(K)$-space). We shall disabuse the reader of this expectation by the following example.

Let $G = \mathbf{U}(3)$ act on complex projective space $\mathbf{CP}^2$ by the usual matrix action on homogeneous coordinates $(z_0\colon z_1\colon z_2)$. Let $x$ be the point with coordinates $(0\colon 0\colon 1)$. Then $G_x = \mathbf{U}(2) \times \mathbf{U}(1) \subset \mathbf{U}(3)$ and by Section 4 this gives an equivalence of the $G$-spaces $\mathbf{CP}^2$ and $\mathbf{U}(3)/(\mathbf{U}(2) \times \mathbf{U}(1))$. Letting $H = \mathbf{U}(2) \times \mathbf{U}(1)$ and $K$ the subgroup of order 2 generated by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the action of $A$ on $\mathbf{CP}^2$ is just $(z_0\colon z_1\colon z_2) \mapsto (-z_0\colon z_1\colon z_2)$ and we see easily that its fixed point set is the disjoint union of the point $(1:0:0)$ and the two-sphere $\{(0\colon z_1\colon z_2)\}$. Since these components of $F(K, \mathbf{CP}^2)$ have different dimensions, $F(K, G/H)$ is not a coset space.

## 6. ELEMENTARY CONSTRUCTIONS

In this section we shall discuss three elementary methods of constructing new transformation groups out of old ones.

**(A) Twisted products** Let $H$ be a *compact* subgroup of $G$ and let $H$ act on a space $A$. Then $H$ acts on $G \times A$ by $(h, (g, a)) \mapsto (gh^{-1}, ha)$. Let $G \times_H A$ denote the orbit space of this $H$-action. The $H$-orbit of $(g, a)$ will be denoted by $[g, a]$, so that $[g, a] = [g', a']$ iff there is an $h \in H$ with $g' = gh^{-1}$ and $a' = ha$. Define a $G$-action on $G \times_H A$ by putting

$$g'[g, a] = [g'g, a].$$

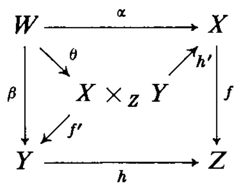
Let $i_e\colon A \to G \times_H A$ be $i_e(a) = [e, a]$. Then $i_e$ is $H$-equivariant since $[e, ha] = [h, a] = h[e, a]$. It is clearly continuous and one–one. Also $i_e$ is closed since it is the composition $A \to G \times A \to G \times_H A$ of closed maps. Thus $i_e$ is an embedding (homeomorphism onto its image).

The projection $G \times A \to G$ induces an equivariant map $p\colon G \times_H A \to G/H$ (given by $[g, a] \mapsto gH$). Suppose that $\pi\colon G \to G/H$ has local cross sections (this is the case when $G$ is Lie). Let $C \subset G/H$ be a closed neighborhood of a point and let $\sigma\colon C \to G$ be a cross section, so that $c \times h \mapsto \sigma(c)h$ is a product representation $C \times H \xrightarrow{\approx} \pi^{-1}(C)$. Consider the map $C \times A \to G \times_H A$ defined by $c \times a \mapsto [\sigma(c), a]$. This is clearly one–one onto $p^{-1}(C)$. It is also continuous and is closed since it is the composition $C \times A \to G \times A \to G \times_H A$ of closed maps. Thus, this map is a homeomorphism, and shows that $p\colon G \times_H A \to G/H$ is a locally trivial fiber map with fiber $A$. (It is, in fact, just the $A$-bundle which is associated with the principal $H$-bundle $G \to G/H$. We shall study this in more detail and generality in Chapter II.)

As we shall see in Chapter II, twisted products are basic to the study of the structure of transformation groups.

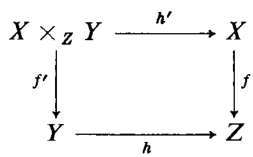
**(B) Fibered products** Let $X$, $Y$, and $Z$ be $G$-spaces and let $f\colon X \to Z$ and $h\colon Y \to Z$ be equivariant maps. The **fibered product** (or **pull-back**) $X \times_Z Y$ is the subspace of $X \times Y$ consisting of pairs $(x, y)$ such that $f(x) = h(y)$. The diagonal $G$-action $(g, (x, y)) \mapsto (gx, gy)$ clearly preserves $X \times_Z Y$ and thus makes it into a $G$-space. The projections $f'\colon X \times_Z Y \to Y$ and $h'\colon X \times_Z Y \to X$ are clearly equivariant.

The fibered product $X \times_Z Y$ satisfies the universal property of pull-backs. That is, if $W$ is a $G$-space and $\alpha\colon W \to X$ and $\beta\colon W \to Y$ are equivariant and such that $f\alpha = h\beta$, then there is a unique map $\theta\colon W \to X \times_Z Y$ such that the diagram




commutes. In fact $\theta$ is given by $\theta(w) = (\alpha(w), \beta(w))$.

In the pull-back diagram




we note that $f'$ is onto if $f$ is. Also note that $f'$ is open if $f$ is open (and similarly with $h'$ and $h$). To see this, let $(x, y)$ be in $X \times_Z Y$ [that is, $f(x) = h(y)$] and let $U$ be an open neighborhood of $x$. Then $f(U)$ is open and $y \in h^{-1}(fU)$. Let $V \subset h^{-1}(fU)$ be any open neighborhood of $y$. Now $(X \times_Z Y) \cap (U \times V)$ clearly projects *onto* $V$ via $f'$, which implies that $f'$ is open.

An important special case is that for which $Z = X/G$, $f = \pi_X\colon X \to X/G$ is the orbit map, and $Y$ has trivial $G$-action. In this case, the fibered product will be denoted by $h^*X$ and called the **pull-back of** $X$ **via** $h$. We have the diagram

$$\begin{array}{ccc} h^*X & \xrightarrow{h'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[h]{} & X/G \end{array}$$

and $\pi'$ is equivariant from the $G$-space $h^*X$ to the trivial $G$-space $Y$. Thus $\pi'$ induces a map $\sigma\colon (h^*X)/G \to Y$. Now $\pi'$ is open and onto, since $\pi$ is, and thus $\sigma$ is also open and onto. If $(x, y)$ and $(x', y)$ are both in $h^*X$, then $\pi(x) = h(y) = \pi(x')$, so that $x$ and $x'$ are in the same orbit, whence $(x, y)$ and $(x', y)$ are in the same orbit.

This shows that $\sigma$ is one–one, and hence that

$$\sigma\colon (h^*X)/G \xrightarrow{\approx} Y$$

is a homeomorphism. Since $\sigma$ is canonical, we may regard $Y$ as the orbit space $(h^*X)/G$.

**(C) Equivariant attachings** Let $X$ and $Y$ be $G$-spaces and let $A \subset X$ be a closed invariant subspace. If $\varphi\colon A \to Y$ is equivariant, then the adjunction space $X \cup_\varphi Y$ (when Hausdorff) inherits a natural $G$-action. In particular, mapping cones and cylinders of equivariant maps between $G$-spaces are again $G$-spaces.

Of particular interest is the case in which $\varphi$ is an equivariant homeomorphism of $A$ to its image. In this case, it is no loss of generality to suppose that $A = X \cap Y$ is a subspace of both $X$ and $Y$ with the same $G$-action on $A$ in both cases. (Thus $X \cup Y$ is the adjunction by the identity on $A$.) We are interested in changing $X \cup Y$ by cutting it apart and gluing it together again by means of some other equivariant homeomorphism $\varphi\colon A \to A$ (precisely, we adjoin disjoint copies of $X$ and $Y$ by the map induced by $\varphi$). If $\varphi\colon A \to A$ extends to an equivariant homeomorphism of $X \to X$ (or of $Y$), then $X \cup_\varphi Y \approx X \cup Y$ as a $G$-space, the homeomorphism being given by $\varphi$ on $X$ and the identity on $Y$. However, if $\varphi$ does not extend to $X$ or to $Y$, then $X \cup_\varphi Y$ may be a new $G$-space. To construct such examples we must find such suitable equivariant self-homeomorphisms of a $G$-space $A$. We shall now discuss one method of finding such maps.

Notice that the equivariant self-homeomorphisms of a $G$-space $A$ form a group $\mathbf{Homeo}^G(A)$ under composition. Now suppose that we are given an equivariant map $\theta\colon A \to G$ taking $a \mapsto \theta_a$ (note this notation) where the action on $G$ is by conjugation $C_g(h) = ghg^{-1}$. Such maps $\theta$ form a group by $(\theta\theta')_a = \theta_a\theta'_a$ since $(\theta\theta')_{ga} = \theta_{ga}\theta'_{ga} = (g\theta_a g^{-1})(g\theta'_a g^{-1}) = g\theta_a\theta'_a g^{-1} = g(\theta\theta')_a g^{-1}$. We shall denote this group by $\mathbf{Map}^G(A, G)$.

We define a function $\mathbf{Map}^G(A, G) \to \mathbf{Homeo}^G(A)$ by $\theta \mapsto \theta^*$, where $\theta^*(a) = \theta_a(a)$. To prove the claimed properties we first note that $\theta^*$ is equivariant since $\theta^*(ga) = \theta_{ga}(ga) = (g\theta_a g^{-1})(ga) = g(\theta_a(a)) = g(\theta^*(a))$. Next we note that $\theta \mapsto \theta^*$ is an *antihomomorphism* since $(\theta^* \circ \lambda^*)(a) = \theta^*(\lambda^*(a)) = \theta^*(\lambda_a(a)) = \lambda_a(\theta^*(a)) = \lambda_a(\theta_a(a)) = (\lambda_a\theta_a)(a) = (\lambda\theta)_a(a) = (\lambda\theta)^*(a)$. In particular, this implies that $(\theta^{-1})^* = (\theta^*)^{-1}$ and hence that $\theta^*$ is indeed a homeomorphism. (It is worth noting that if $A$ is a differentiable manifold and $G$ is a Lie group acting differentiably on $A$, then differentiable maps $\theta$ go into diffeomorphisms $\theta^*$.)

Of course these remarks do not solve our problem of finding equivariant homeomorphisms, but it is generally easier to find equivariant *maps* than *homeomorphisms*, and, in fact, the construction given above will prove to be useful in the next section. We will also make use of the following remarks. Suppose that we are given an equivariant map $\theta\colon A \to G$ as above, and let $A \times A$ have the diagonal $G$-action $g(a, a') = (g(a), g(a'))$. Then the map $A \times A \to G$ given by $(a, a') \mapsto \theta_a$ is obviously equivariant. By the remarks above, we deduce that $(a, a') \mapsto (\theta_a(a), \theta_a(a'))$ is an equivariant homeomorphism of $A \times A$. Now $(a, a') \mapsto (a', a)$ is also equivariant so that $(a, a') \mapsto (\theta_a(a'), \theta_a(a))$ is an equivariant homeomorphism.

In the next section we shall be interested in the case for which $\theta_a(a) = a$. Thus we note, in particular, that if $a \mapsto \theta_a$ is an equivariant map $A \to G$ such that $\theta_a(a) = a$ for all $a$, then the map $A \times A \to A \times A$ defined by

$$(x, y) \mapsto (\theta_x(y), x)$$

is an equivariant homeomorphism. (Of course, it is not hard to check this directly, and the reader is invited to do so.)

## 7. SOME EXAMPLES OF O($n$)-SPACES

Using the remarks at the end of the last section, we shall construct some $\mathbf{O}(n)$-spaces by attaching $\mathbf{S}^{n-1} \times \mathbf{D}^n$ to $\mathbf{S}^{n-1} \times \mathbf{D}^n$ via an equivariant self-homeomorphism $\varphi$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$; the $\mathbf{O}(n)$-action on these spaces being the standard action on $\mathbf{S}^{n-1}$ and on $\mathbf{D}^n$ and the diagonal action on $\mathbf{S}^{n-1} \times \mathbf{D}^n$.

To apply the remarks of the previous section, we wish to find an equivariant map

$$\theta\colon \mathbf{S}^{n-1} \to \mathbf{O}(n)$$

(taking $x$ into $\theta_x$), where $\mathbf{O}(n)$ acts on itself by conjugation. It is easy to characterize such maps as follows. If $x \in \mathbf{S}^{n-1}$ and $g(x) = x$, then $\theta_x = \theta_{gx} = g\theta_x g^{-1}$. Thus $\theta_x$ must commute with each element of the isotropy group $\mathbf{O}(n)_x$ [which is conjugate to $\mathbf{O}(n-1)$]. It follows easily from this (consider the generic case for which $x$ is a basis vector) that there are precisely four possibilities for $\theta_x$: either $I$, $-I$, the reflection through the hyperplane perpendicular to $x$, or the reflection through the line $\mathbf{R}x$ (which is equal to minus the preceding case). The continuity of $\theta$ implies that these cases cannot vary as one varies $x$. Now it is clear that the first two cases will not yield any unfamiliar $\mathbf{O}(n)$-spaces and it is fairly clear that the last two cases will give equivalent examples. Thus we consider only the last case in which $\theta_x$ is the reflection through $\mathbf{R}x$, that is, for $x \in \mathbf{S}^{n-1}$ and $y \in \mathbf{R}^n$,

$$\theta_x(y) = 2(x \cdot y)x - y.$$

In this case note that $\theta_x(x) = x$. By the last section it follows that the map

$$\varphi\colon \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \to \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$$

given by

$$\varphi(x, y) = (\theta_x(y), x)$$

is an equivariant homeomorphism (in fact, diffeomorphism) as are all of its powers (positive and negative).

We shall define

$$\Sigma_k^{2n-1} = \mathbf{S}^{n-1} \times \mathbf{D}^n \bigcup\nolimits_{\varphi^k} \mathbf{S}^{n-1} \times \mathbf{D}^n$$

the $\mathbf{O}(n)$-space resulting by attaching $\mathbf{S}^{n-1} \times \mathbf{D}^n$ to itself by means of the $k$th power of $\varphi$ ($k$ any integer) on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. We shall now study these $\mathbf{O}(n)$-spaces.

Since $\varphi^0$ is the identity, $\Sigma_0^{2n-1} \approx \mathbf{S}^{n-1} \times \mathbf{S}^n$ with the diagonal $\mathbf{O}(n)$-action [$\mathbf{O}(n)$ acting on $\mathbf{S}^n$ via the standard inclusion $\mathbf{O}(n) \subset \mathbf{O}(n+1)$].

Interchange of the roles of the two copies of $\mathbf{S}^{n-1} \times \mathbf{D}^n$ shows that

$$\Sigma_k^{2n-1} \approx \Sigma_{-k}^{2n-1} \text{ as } \mathbf{O}(n)\text{-spaces.}$$

Now $\Sigma_1^{2n-1}$ may be described alternatively as the space obtained by attaching $\mathbf{S}^{n-1} \times \mathbf{D}^n$ to $\mathbf{D}^n \times \mathbf{S}^{n-1}$ by means of

$$\varphi'\colon (x, y) \mapsto (x, \theta_x(y))$$

on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ (which is $\varphi$ composed with interchange of factors). But $\varphi'$ clearly makes sense for $y \in \mathbf{D}^n$ and hence extends to an equivariant homeomorphism of $\mathbf{S}^{n-1} \times \mathbf{D}^n$ onto itself. Thus, as noted in the last section, $\Sigma_1^{2n-1}$ is equivalent to the union $\mathbf{S}^{n-1} \times \mathbf{D}^n \cup \mathbf{D}^n \times \mathbf{S}^{n-1} \approx \mathbf{S}^{2n-1} \subset \mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$, where $\mathbf{O}(n)$ acts diagonally. Thus the case $k = 1$ gives only this familiar linear $\mathbf{O}(n)$ action on $\Sigma_1^{2n-1} \approx \mathbf{S}^{2n-1}$. However, we shall soon see that the cases $k \geq 2$ are much less familiar $\mathbf{O}(n)$-spaces.

To study the underlying manifold of $\Sigma_k^{2n-1}$ ($n \geq 2$) we shall compute its homology. Let $\star$ denote a base point in $\mathbf{S}^{n-1}$ and let $\alpha$ and $\beta$ denote the classes in $H_{n-1}(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})$ represented by $\mathbf{S}^{n-1} \times (\star)$ and $(\star) \times \mathbf{S}^{n-1}$, respectively. The homeomorphism $\varphi$ induces the automorphism $\varphi_*$ of $H_{n-1}(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})$ and, for example, $\varphi_*(\alpha) = a_1\alpha + a_2\beta$, where $a_i$ is the degree of the composition $\mathbf{S}^{n-1} \times (\star) \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \xrightarrow{\varphi} \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \xrightarrow{p_i} \mathbf{S}^{n-1}$ where $p_i$ is the projection on the $i$th factor ($i = 1, 2$). Similarly for $\varphi_*(\beta)$. Now

$$\varphi(x, \star) = (\theta_x(\star), x),$$
$$\varphi(\star, y) = (\theta_\star(y), \star),$$

and from this it is not hard to see that

$$\varphi_*(\alpha) = (1 + (-1)^n)\alpha + \beta,$$
$$\varphi_*(\beta) = (-1)^{n-1}\alpha,$$

so that

$$\varphi_* = \begin{bmatrix} 1 + (-1)^n & (-1)^{n-1} \\ 1 & 0 \end{bmatrix}.$$

By taking powers we easily compute that

$$\varphi_*{}^k = \begin{bmatrix} 1 + k & -k \\ k & 1 - k \end{bmatrix} \quad \text{for } n \text{ even},$$

$$\varphi_*^{\text{even}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } n \text{ odd},$$

$$\varphi_*^{\text{odd}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{for } n \text{ odd}.$$

Now from the Mayer–Vietoris sequence ($i$ is the inclusion)

$$\cdots \to H_{j+1}(\Sigma_k^{2n-1}) \to H_j(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})$$
$$\xrightarrow{(i_*, \varphi_*{}^k)} H_j(\mathbf{S}^{n-1} \times \mathbf{D}^n) \oplus H_j(\mathbf{S}^{n-1} \times \mathbf{D}^n) \to H_j(\Sigma_k^{2n-1}) \to \cdots$$

we see that $H_p(\Sigma_k^{2n-1}) = 0$ for $p \neq 0,\ n-1,\ n,\ 2n-1$ (for $n \geq 2$). For $j = n-1$, $(i_*, \varphi_*{}^k)$ is a homomorphism between two free abelian groups of rank 2 and, using the obvious bases, it is represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1+k & -k \end{bmatrix} \quad \text{for } n \text{ even},$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{for } n \text{ odd}, \ k \text{ even},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } n \text{ odd}, \ k \text{ odd}.$$

Putting $n = 2m$ or $n = 2m+1$ it follows that (for $m \geq 1$ and $k \neq 0$)

$$H_p(\Sigma_k^{4m-1}) = \begin{cases} \mathbf{Z} & \text{for } p = 0,\ 4m-1, \\ \mathbf{Z}_k & \text{for } p = 2m-1, \\ 0 & \text{otherwise}. \end{cases}$$

$$H_p(\Sigma_{\text{even}}^{4m+1}) = \begin{cases} \mathbf{Z} & \text{for } p = 0,\ 2m,\ 2m+1,\ 4m+1, \\ 0 & \text{otherwise}. \end{cases}$$

$$H_p(\Sigma_{\text{odd}}^{4m+1}) = \begin{cases} \mathbf{Z} & \text{for } p = 0,\ 4m+1, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\Sigma_{\text{even}}^{4m+1}$ is a homology $\mathbf{S}^{2m} \times \mathbf{S}^{2m+1}$ and $\Sigma_{\text{odd}}^{4m+1}$ is a homology $\mathbf{S}^{4m+1}$.

For $n \geq 3$, $\mathbf{S}^{n-1} \times \mathbf{D}^n$ is simply connected and thus the Van Kampen Theorem implies that $\Sigma_k^{2n-1}$ is simply connected.

In particular, $\Sigma_{\text{odd}}^{4m+1}$ is a homotopy sphere for all $m$. Since it is a differentiable manifold, it follows from Smale's proof of the generalized Poincaré conjecture (see Milnor [5]) that

$$\Sigma_{\text{odd}}^{4m+1} \text{ is homeomorphic to } \mathbf{S}^{4m+1}.$$

(The differentiable structure on $\Sigma_{\text{odd}}^{4m+1}$ is not always standard. We shall discuss this in Chapter VI, Section 6. Of course $\Sigma_{\text{odd}}^5$ is diffeomorphic to $\mathbf{S}^5$ since there is only one differentiable structure on the 5-sphere, up to diffeomorphism; see Kervaire and Milnor [1].)

Now we shall show that the $\mathbf{O}(n)$-actions on $\Sigma_k^{4m+1}$ (for *odd* $k > 1$) are *not* equivalent to orthogonal actions on $\mathbf{S}^{4m+1}$. For this we note that if $\mathbf{D}^r \subset \mathbf{D}^n$ (in the standard way), then $\varphi^k$ preserves $\mathbf{S}^{r-1} \times \mathbf{S}^{r-1}$ and restricts to the $\varphi^k$ defined from this $\mathbf{O}(r)$-action. (This is clear if one notes that the two components $\theta_x(y)$ and $x$ of $\varphi(x, y)$ are obtained from $x$ and $y$ by rotating the plane containing $x$ and $y$ from $y$ toward $x$ through an angle equal to that between $x$ and $y$. Thus $\varphi^k(x, y)$ depends only on the linear span of $x$ and $y$.) Thus if $H$ is any subgroup of $\mathbf{O}(n)$ and if

$$F(H, \mathbf{D}^n) = \mathbf{D}^r,$$

then

$$F(H, \Sigma_k^{2n-1}) = \Sigma_k^{2r-1}.$$

This yields directly some unusual features of these examples.

For instance, let $n = 3$ and let $\mathbf{Z}_2 \approx \mathbf{O}(1) \subset \mathbf{O}(3)$ be the standard inclusion. Then $F(\mathbf{O}(1), \Sigma_k{}^5) = \Sigma_k{}^3$ whose first homology group is $\mathbf{Z}_k$. This gives an involution (a homeomorphism of period 2) on $\mathbf{S}^5$ whose fixed point set $F$ has $H_1(F) \approx \mathbf{Z}_k$ for any *odd* $k$.

In fact, the space $\Sigma_k{}^3$ is just the lens space $\mathbf{L}(k, 1)$. We see this by noting that if we lift $\varphi\colon \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{S}^1 \times \mathbf{S}^1$ to $\tilde{\varphi}\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$, then

$$\tilde{\varphi}(s, t) = (2s - t,\ s)$$

(that is, $\theta_{e^{2\pi is}}(e^{2\pi it}) = e^{2\pi i(2s-t)}$). Thus $\tilde{\varphi}$ is linear and is represented by the matrix

$$\tilde{\varphi} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Clearly $\tilde{\varphi}^k$ covers $\varphi^k$ and is the linear map

$$\tilde{\varphi}^k = \begin{bmatrix} 1 + k & -k \\ k & 1 - k \end{bmatrix}.$$

Now one of the standard definitions of the lens space $\mathbf{L}(k, q)$ is that it is the space obtained by attaching $\mathbf{S}^1 \times \mathbf{D}^2$ to itself by the self-homeomorphism of $\mathbf{S}^1 \times \mathbf{S}^1$ covered by the map represented by the matrix

$$\begin{bmatrix} q' & k \\ n & q \end{bmatrix},$$

where $n$ and $q'$ are integers such that

$$qq' - kn = \pm 1.$$

Moreover, $\mathbf{L}(k, q)$ depends only on $|\, k \,|$ and on the congruence class of $\pm q \pmod{k}$. Thus

$$\Sigma_k{}^3 = \mathbf{L}(k, 1 - k) \approx \mathbf{L}(k, 1).$$

*Remark* We may also describe $\mathbf{L}(k, q)$ as the orbit space of the free action of $\mathbf{Z}_k$ on $\mathbf{S}^3 \subset \mathbf{C}^2$ generated by the matrix

$$\begin{bmatrix} e^{2\pi i/k} & 0 \\ 0 & e^{2\pi iq/k} \end{bmatrix}.$$

Thus $\Sigma_k{}^5$ gives an $\mathbf{O}(3)$-action on $\mathbf{S}^5$ such that $F(\mathbf{O}(1), \mathbf{S}^5) \approx \mathbf{L}(k, 1)$ for any *odd k*.

More generally, for $i \geq 1$, the $\mathbf{O}(2m+1)$-action on $\Sigma_{2i+1}^{4m+1} \approx \mathbf{S}^{4m+1}$ (homeomorphic, but not generally diffeomorphic) is not equivalent to an orthogonal action for $m \geq 1$ and the restricted action to $\mathbf{O}(1)$, and hence to $\mathbf{O}(r)$ for $1 \leq r \leq 2m+1$, is also nonorthogonal since $F(\mathbf{O}(1), \Sigma_{2i+1}^{4m+1}) = \Sigma_{2i+1}^{4m-1}$ has homology $\mathbf{Z}_{2i+1}$ in degree $2m-1$. Similarly, the restriction to $\mathbf{SO}(3)$, and hence to $\mathbf{SO}(r)$ for $3 \leq r \leq 2m+1$, is nonorthogonal *provided that* $m \geq 2$ since then $F(\mathbf{SO}(3), \Sigma_{2i+1}^{4m+1}) = \Sigma_{2i+1}^{4m-5}$ has homology $\mathbf{Z}_{2i+1}$ in degree $2m-3$. (However, we shall see later that the $\mathbf{SO}(3)$-action on $\Sigma_{2i+1}^5$ *is* equivalent to an orthogonal action! Note the interesting fact that the $\mathbf{O}(3)$-action on $\Sigma_{2i+1}^5$ has exactly the same orbits as does its restriction to $\mathbf{SO}(3)$, and yet it is nonorthogonal while the restriction is orthogonal.)

*Remark* These examples were first constructed in Bredon [12] (also see Bredon [11, 19]) by essentially the same method used here. Subsequently, two other methods were developed (Brieskorn [1], Hirzebruch and Mayer [1], and Bredon [19]) which arrive at the same examples, generalize in other directions, and have certain advantages over the present construction. We shall discuss these later. The present method appears to be the simplest from the point of view of computing the homology of the spaces involved.

For later reference we shall discuss the isotropy groups and orbit space of the $\mathbf{O}(n)$-space $\Sigma_k^{2n-1}$ $(n \geq 2)$. Let $G = \mathbf{O}(n)$. Clearly the isotropy groups of $G$ on $\Sigma_k^{2n-1}$ are just those of $G$ on $\mathbf{S}^{n-1} \times \mathbf{D}^n$. However, $G_{(x,y)} = G_x \cap G_y$ and this is just the subgroup of $G = \mathbf{O}(n)$ leaving the linear span of $x$ and $y$ stationary. Thus if $x$ and $y$ are dependent, we have that $G_{(x,y)}$ is conjugate to $\mathbf{O}(n-1)$ and if $x$ and $y$ are independent, then $G_{(x,y)}$ is conjugate to $\mathbf{O}(n-2)$. Thus $\mathbf{O}(n)/\mathbf{O}(n-1) \approx \mathbf{S}^{n-1}$ and $\mathbf{O}(n)/\mathbf{O}(n-2) \approx \mathbf{V}_{n,2}$ are precisely the orbit types which occur in the $\mathbf{O}(n)$-space $\Sigma_k^{2n-1}$.

Next we investigate the orbit space of $G = \mathbf{O}(n)$ on $\mathbf{S}^{n-1} \times \mathbf{D}^n$ $(n \geq 2)$. Let $\mathbf{D}^2 \subset \mathbf{D}^n$ be the disk spanned by the first two coordinate axes, let $\star = (1, 0) \in \mathbf{S}^1 \subset \mathbf{D}^2$ and let $\mathbf{D}^2{}_+ = \{(a, b) \in \mathbf{D}^2 \mid b \geq 0\}$. Any point of $\mathbf{S}^{n-1} \times \mathbf{D}^n$ can be moved by an element of $\mathbf{O}(n)$ to a point in $(\star) \times \mathbf{D}^n$. Then moving this by an element of the isotropy group $G_\star = \mathbf{O}(n-1)$

we can bring this into a point $(\star, y) \in (\star) \times \mathbf{D}^2_+$. Since $y$ is determined completely by its length and by the angle it makes with $\star$, and since these numbers are unchanged under operation by $G$ on $(\star, y)$, it follows that $(\star) \times \mathbf{D}^2_+$ touches each orbit in $\mathbf{S}^{n-1} \times \mathbf{D}^n$ exactly once. Thus $(\star) \times \mathbf{D}^2_+$ is a cross section for the $G$-space $\mathbf{S}^{n-1} \times \mathbf{D}^n$. Consequently $(\mathbf{S}^{n-1} \times \mathbf{D}^n)/\mathbf{O}(n) \approx \mathbf{D}^2_+$ and we also see that $(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})/\mathbf{O}(n)$ corresponds to the arc $\mathbf{D}^2_+ \cap \mathbf{S}^1$. Since $\varphi^k$ must induce a homeomorphism of $(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})/\mathbf{O}(n)$ onto itself, if follows that $\Sigma_k^{2n-1}/\mathbf{O}(n)$ is the union of two 2-disks along an arc in their boundaries. Thus

$$\Sigma_k^{2n-1}/\mathbf{O}(n) \approx \mathbf{D}^2.$$

(This can also be seen by some general facts which we shall discuss in a later chapter.)

Recall that for $n \geq 2$, and the subgroup $\mathbf{O}(n-2)$ of $\mathbf{O}(n)$ we have

$$F\big(\mathbf{O}(n-2),\ \Sigma_k^{2n-1}\big) = \Sigma_k^3 = \mathbf{L}(k, 1).$$

We will also need to know about $F\big(\mathbf{O}(n-1),\ \Sigma_k^{2n-1}\big) = \Sigma_k^1$. Now $\Sigma_k^1$ results from adjoining $\mathbf{S}^0 \times \mathbf{D}^1$ to itself by the map $\varphi^k\colon \mathbf{S}^0 \times \mathbf{S}^0 \to \mathbf{S}^0 \times \mathbf{S}^0$ which is the restriction of the map $\mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ given by

$$\varphi^{\text{even}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \varphi^{\text{odd}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly this shows that

$$\Sigma^1_{\text{even}} \approx \mathbf{S}^0 \times \mathbf{S}^1 = \text{two circles},$$

$$\Sigma^1_{\text{odd}} \approx \mathbf{S}^1.$$

## 8. TWO FURTHER EXAMPLES

Even though we gave some examples, in the last section, of nonorthogonal actions, these actions had some points of similarity with certain orthogonal actions. In this section we shall discuss two examples which show that, in general, actions of compact groups on euclidean spaces or disks can differ very substantially from orthogonal actions.

**(A) The Floyd–Richardson Example** For this example we must give some background material concerning the orthogonal group $\mathbf{SO}(3)$. Let $\mathbf{Q}$ denote the quaternions and let $\mathbf{S}^3$ stand for the group of quaternions of norm 1. Then $\mathbf{S}^3$ acts on $\mathbf{Q}$ by conjugation: $(q, q') \mapsto qq'q^{-1}$ and this action is orthogonal, regarding $\mathbf{Q}$ as euclidean 4-space, since norms are preserved. The subspace $\mathbf{R} \subset \mathbf{Q}$ spanned by 1 is clearly left fixed by this action, so that its orthogonal complement (that is, the subspace spanned by $i$, $j$, and $k$) is invariant under this action. This provides a 3-dimensional orthogonal representation of $\mathbf{S}^3$ and thus gives the well-known homomorphism

$$f\colon \mathbf{S}^3 \to \mathbf{SO}(3).$$

The kernel of $f$ is just $\{1, -1\} \approx \mathbf{Z}_2$, and, since $\mathbf{S}^3$ and $\mathbf{SO}(3)$ both have dimension 3, the invariance of domain implies that $f$ is onto. Hence $f$ is a covering map and identifies $\mathbf{S}^3$ with the universal covering group of $\mathbf{SO}(3)$.

Now let $I$ denote the icosahedral subgroup of $\mathbf{SO}(3)$ (that is, the subgroup carrying a regular icosahedron, or equivalently the dual dodecahedron, centered at the origin into itself). A crucial fact that we will need is that the normalizer of $I$ is $I$ itself. This follows from the explicit knowledge of the subgroups of $\mathbf{SO}(3)$ (see Wolf [1], for example) which shows that there are no groups between $I$ and $\mathbf{SO}(3)$.

We will also need to know that $I$ is perfect (that is, that $I$ abelianized is trivial, which means that $I = [I, I]$, its commutator subgroup). One way to see this is simply to write generators and relations for $I$, and verify it; another way is to note that one can inscribe five regular tetrahedra in the dodecahedron and that the action of $I$ on these five objects represents $I$ *isomorphically* as the alternating group on five letters (Coxeter [1]) and then use the universally known fact that the latter group is simple.

Also note that a cube may be inscribed in the dodecahedron, so that we may assume that $I$ contains the 180° rotations about the coordinate axes spanned by $i$, $j$, and $k$.

We put $I' = f^{-1}(I)$, a subgroup of $\mathbf{S}^3$ of order 120. Note that since $iii^{-1} = i$, $iji^{-1} = -j$, and $iki^{-1} = -k$ we have that $f(i)$ is the 180° rotation about $\mathbf{R}i$. Similarly for $f(j)$ and $f(k)$. Thus $i$, $j$, and $k$ are in $I'$ and

$$-1 = iji^{-1}j^{-1} \text{ is in } [I', I'].$$

Consequently $I'/[I', I'] \approx I/[I, I]$ is the trivial group.

Consider now the coset space $\Sigma^3 = \mathbf{SO}(3)/I \approx \mathbf{S}^3/I'$. Since $\pi_1(\Sigma^3) \approx I'$ we have that

$$H_1(\Sigma^3) \approx I'/[I', I'] = 0.$$

Since $\Sigma^3$ is a 3-manifold, Poincaré duality implies that $\Sigma^3$ is a homology 3-sphere over the integers.

We shall be concerned with the action of $I$ on $\Sigma^3$. It is clear that $I'$ (and hence $I$) preserves the metric induced from that of $\mathbf{S}^3$ via the covering map $\mathbf{S}^3 \to \mathbf{S}^3/I' = \Sigma^3$. Now

$$F(I, \Sigma^3) = N(I)/I = I/I,$$

by Section 5, is precisely one point. Since $I$ preserves the metric, there is an open 3-disk $U$ about this point invariant under $I$. Then $I$ acts *without stationary points* on the *acyclic* 3-manifold $\Sigma^3 - U$ with boundary. (Also note that the action is differentiable.)

Now one can show that this action of $I$ on $\Sigma^3 - U$ is simplicial in some simplicial structure on $\Sigma^3 - U$. For instance, there is a general theorem of Yang [4] about differentiable actions which implies this. It can also be seen from the fact that $\mathbf{S}^3$ may be given the structure of a regular polytope (Coxeter [1]) such that the left and right translations by $I'$ are all cellular (and simplicial in a subdivision), which clearly implies the desired fact. Possibly the simplest way, however, is to use the classical description of the "spherical dodecahedral space" $\Sigma^3$ in Seifert and Threlfall [1] as the regular dodecahedron, with its interior, on which opposite faces are identified by a coherent twist of $\pi/5$ radians (and the reflection through the central plane parallel to them), and use the obvious action of $I$ on this (rather than the one we have described). (The last two methods are essentially equivalent since, in fact, the dodecahedron is the top dimensional cell of the polytope $\mathbf{S}^3$ and is a fundamental domain for the right translations by $I'$, resulting in the given identifications on its boundary.)

Now consider the join $A = I \star (\Sigma^3 - U)$ with the induced diagonal action of $I$ (that is, the union of 60 cones over $\Sigma^3 - U$). Since $A$ is simply connected and acyclic, it is *contractible*. Also, the $I$-action on $A$ has no fixed points. Let $K$ be any finite complex with trivial $I$-action ($K$ may be empty.) Then $I$ acts on the *contractible finite complex* $B = K \star A$ with fixed set $K$.

Any such example automatically induces a similar example of an action on a disk as follows. First embed $B$ as a subcomplex of some simplicial structure on euclidean space or a sphere with a linear and simplicial action of $I$. For example we can consider $B$ as a subcomplex of the boundary of the full simplex $\Delta$ on the vertices of $B$. Let $D$ be the second regular neighborhood of $B$ in $\partial\Delta$. Then, by a famous result of Whitehead [1], $D$ is an $n$-disk for some $n$. Moreover, $F(I, D)$ is a regular neighborhood of $K$ in $F(I, \partial\Delta)$ and hence has $K$ as a deformation retract. Thus we have the following theorem.

**8.1. Theorem** *There are actions of the icosahedral group $I$ on disks $\mathbf{D}^n$ ($n$ large) such that $F(I, \mathbf{D}^n)$ has the homotopy type of any given finite complex, including the case $F(I, \mathbf{D}^n) = \varnothing$.* ▌

This example, in the case $F = \varnothing$, is due to Floyd and Richardson [1].

*Remark* This, together with trivial modifications, is the only known example of an action of a compact group on a disk without stationary points. One might try to construct other such examples by using other coset spaces $G/H$, $G$ compact, which are integral homology spheres. However, it was proved in Bredon [4] that $\mathbf{SO}(3)/I$ is the *only* such space which is not a sphere, and it is also known, by Montgomery and Samelson [1], Borel [3], and Poncet [2], that when $G/H$ is a sphere then the $G$-action is equivalent to an orthogonal one. (The proof consists of *classifying* the pairs $G, H$ for which $G/H$ is a sphere.)

*Remark* One should note that even though the fixed point set has the homotopy type of an arbitrary finite complex, its homeomorphism type is not so arbitrary. For example, each component of the fixed set in this example has the same dimension and one may well conjecture that this holds generally for compact group actions on disks, spheres, and euclidean spaces. In the case of differentiable actions, there is a tangential representation of $G$ at each fixed point, and it was conjectured by Smith [10] that these representations are equivalent at any two fixed points. (The latter conjecture is known to hold for differentiable actions of *connected* compact groups.)

**(B) The Conner–Floyd Example** We shall indicate how to construct an action of a cyclic group $\mathbf{Z}_r$ on euclidean space without stationary points. (Note that the Brouwer Fixed Point Theorem shows that such examples cannot exist on disks.) Later we shall prove a theorem of Smith which implies that such actions do not exist when $r$ is a prime power. Thus we only consider the case

$$r = pq \text{ where } p > 1 \text{ and } q > 1 \text{ are relatively prime.}$$

Let us digress briefly to discuss a construction on joins. Recall that the join $A \star B$ is defined as

$$A \star B = \frac{A \times B \times \mathbf{I}}{\mathscr{R}},$$

where $\mathscr{R}$ is the relation which identifies $A \times B \times \{0\}$ to $A$ and $A \times B \times \{1\}$ to $B$. Thus the points of $A \star B$ have "coordinates" $(a, b, t)$ where $a \in A$, $b \in B$, and $t \in \mathbf{I} = [0, 1]$, and where $a = (a, b, 0)$ is independent of $b$ and $b = (a, b, 1)$ is independent of $a$.

Now there is a map

$$\lambda\colon A \star B \to A \star B \cup_B B \star A \cup_A A \star B$$

defined by

$$\lambda(a, b, t) = \begin{cases} (a, b, 3t) & \text{for } 0 \leq t \leq \frac{1}{3}, \\ (b, a, 3t - 1) & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ (a, b, 3t - 2) & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

If $f\colon A \to A$ and $g\colon B \to B$ are any maps, then one may form the union

$$f \star 1 \cup \tau \cup 1 \star g\colon A \star B \cup_B B \star A \cup_A A \star B \to A \star B,$$

where $\tau\colon B \star A \to A \star B$ is the canonical map $\tau(b, a, t) = (a, b, 1 - t)$. Put

$$f \,\square\, g = (f \star 1 \cup \tau \cup 1 \star g) \circ \lambda.$$

Note that if $A$ and $B$ are spheres, then $\deg(f \star 1) = \deg(f)$ and $\deg(1 \star g) = \deg(g)$ so that

$$\deg(f \,\square\, g) = \deg(f) + \deg(g) - 1.$$

(The $-1$ comes from the change in direction of $t$ in the middle third; note that $a$, $b$ are reversed twice there.)

Now let $A$ and $B$ be copies of the unit circle in the complex plane and let $T = \omega_p \star \omega_q$, where $\omega_p$ is multiplication by $e^{2\pi i/p}$. Then $T$ generates an action of $\mathbf{Z}_{pq}$ on $A \star B \approx \mathbf{S}^3$ *without stationary points.*

Since $p$ and $q$ are relatively prime we can find $n$, $m$ with

$$np + mq = -1.$$

Then let

$$f\colon A \to A \qquad \text{and} \qquad g\colon B \to B$$

be defined by

$$f(z) = z^{np+1} \qquad \text{and} \qquad g(z) = z^{mq+1}.$$

Note that

$$f\omega_p = \omega_p f \qquad \text{and} \qquad g\omega_q = \omega_q g,$$

which clearly implies that

$$(f \square g)T = T(f \square g)$$

so that $f \square g: \mathbf{S}^3 \to \mathbf{S}^3$ is an equivariant map of degree $(np + 1) + (mq + 1) - 1 = np + mq + 1 = 0$. Let $h = f \square g$.

From the existence of such an equivariant map $h: \mathbf{S}^3 \to \mathbf{S}^3$ of degree 0 one may proceed in two directions.

First, note that the inverse limit space of

$$\cdots \to \mathbf{S}^3 \xrightarrow{h} \mathbf{S}^3 \xrightarrow{h} \mathbf{S}^3$$

inherits an action of $\mathbf{Z}_{pq}$ without stationary points and is acyclic with respect to Čech homology or cohomology. Thus we have the following theorem.

**8.2. Theorem** *If r is not a prime power, then there is a compact, acyclic, finite-dimensional space which admits a self-homeomorphism of period r without fixed points.* ▮

*Remark* Note the consequence that the Lefschetz Fixed Point Theorem does not hold in this generality.

Second, there is a $\mathbf{Z}_{pq}$-action on the union $Y$ of the mapping cylinders of the maps

$$\mathbf{S}^3 \xrightarrow{h} \mathbf{S}^3 \xrightarrow{h} \mathbf{S}^3 \xrightarrow{h} \cdots.$$

Since $h$ is homotopically trivial, $Y$ has the homotopy type of the union of mapping cylinders of a constant map and hence is *contractible.*

We shall take a closer look at the latter example. If $A$ and $B$ are triangulated as regular polygons of $p$ and $q$ sides, respectively, and $\mathbf{S}^3 = A \star B$ is given the join triangulation, then $T$ is simplicial. Moreover $h$ can be assumed to be simplicial with respect to the $n$th barycentric subdivision of its domain, for some $n$. (This can be deduced from the Equivariant Simplicial Approximation Theorem of Exercise 6 at the end of the chapter.) Then the mapping cylinder of $h$ can be replaced with the "simplicial mapping cylinder." (We do not give the details of this, which can be found in Conner and Floyd [3].) If we then take the $n$th barycentric subdivision of this mapping cylinder *modulo* the domain of $h$, then the mapping cylinder has the same triangulation on both ends. Thus if we let $C_1, C_2, \ldots$ be copies of this mapping cylinder and if we let $K$ be the infinite complex obtained by identifying the end of $C_i$ with the beginning of $C_{i+1}$, then $K$ is a 4-dimensional, locally finite, contractible complex on which $\mathbf{Z}_r$ $(r = pq)$ acts without stationary points.

To improve on this, we note that $K$ admits an equivariant simplicial embedding in some euclidean space on which $\mathbf{Z}_r$ acts orthogonally. One such embedding of $K$ can be obtained as follows. Let $f: K \to \mathbf{R}^k$ be any embedding of $K$ as a subcomplex of a triangulation of $\mathbf{R}^k$. Let $\mathbf{Z}_r$ act on $\mathbf{R}^{kr}$ by

$$T(x_1, \ldots, x_r) = (x_2, \ldots, x_r, x_1),$$

where $x_i$ is in $\mathbf{R}^k$, and let $\mathbf{R}^{kr}$ be triangulated as the barycentric subdivision of the product cellular structure on $\mathbf{R}^k \times \mathbf{R}^k \times \cdots \times \mathbf{R}^k$. Embed $g: K \to \mathbf{R}^{kr}$ by putting

$$g(x) = (x, Tx, T^2x, \ldots, T^{r-1}x).$$

Then $g$ is equivariant and it is not hard to verify that $g$ is simplicial on the first barycentric subdivision of $K$ to $\mathbf{R}^{kr}$ (see Floyd [9] for the details of this). We let $U$ be the second regular neighborhood of $g(K)$ in $\mathbf{R}^{kr}$ and note that $T(U) = U$. Then $U$ is a contractible open subset of $\mathbf{R}^{kr}$ on which $\mathbf{Z}_r$ acts (differentiably) without stationary points, and the same is true of $U \times \mathbf{R}^i$ for any $i$, where the action on $\mathbf{R}^i$ is trivial. Now it is known that $U \times \mathbf{R}$ is homeomorphic to euclidean space (see McMillan and Zeeman [1] and Stallings [1]).

If $L$ is an arbitrary finite complex, we could replace $\mathbf{S}^3$ by $\mathbf{S}^3 * L$ in all the constructions and would end up with an action of $\mathbf{Z}_r$ on euclidean space with fixed point set of the homotopy type of $L$. Thus we have the following theorem.

**8.3. Theorem** *If $r$ is not a prime power and $L$ is a finite complex, then there is an integer $m$ such that $\mathbf{R}^n$ admits a self-homeomorphism of period $r$ whose fixed set has the homotopy type of $L$, for any $n \geq m$ (including the case $L = \varnothing$).* ▮

*Remarks* Kister [1] has improved on the embedding of the above construction and has shown that $m$ can be taken to be 8 when $L = \varnothing$. It is known (Smith [10]) that, in the differentiable case, $m$ must be at least 7 for $L = \varnothing$. Conner and Montgomery [2] have modified these constructions to produce an action of $\mathbf{SO}(3)$ on some euclidean space with no stationary points. They use a self-map of the 5-dimensional representation of $\mathbf{SO}(3)$, discussed at the end of Section 4, having degree 0 (originally found by Floyd [12]). The same construction was generalized to actions of $\mathbf{SO}(2n+1)$, $n \geq 1$, by Hsiang and Hsiang [4] and, by an examination of the actions of subgroups, they showed that the same is true of any compact, connected, nonabelian, Lie group (but is false for tori, as we shall see later). Again by joining a finite complex $L$ to the construction, one obtains actions whose fixed point set is homotopically equivalent to $L$. The end result, which we state without proof, is the following theorem.

**8.4. Theorem** *If $G$ is any compact, connected, nonabelian, Lie group, then there is an action of $G$ on any euclidean space of sufficiently high dimension for which the fixed point set has the homotopy type of any given finite complex (including $F = \varnothing$).* ▮

*Remark* These examples surely show that, without some restrictions, there is little one can say about a transformation group. However, under certain restrictions, one can prove some strong theorems. For example, we will obtain some important results about actions of $p$-groups and of tori. Also one can prove strong theorems about actions in which the orbits have low codimension, or the fixed set has low codimension, or the number of orbit types is small, and so on. We shall study some such situations in later chapters.

## 9. COVERING ACTIONS

Let $X$ be a $G$-space and suppose we are given a covering space $p: X' \to X$ of $X$. It is natural to ask if we can lift the $G$-action on $X$ to an action on $X'$, perhaps of a covering group of $G$.

We shall assume throughout that $G$ is a Lie group. (We use only the fact that the identity component $G_0$ of $G$ has a simply connected covering space and is open in $G$.) Also we assume that $X$ is connected and locally arcwise connected.

First, we shall discuss the case in which $G$ is connected and we let $G^*$ be the universal covering group of $G$ with projection $\pi: G^* \to G$. Let $e$ and $e^*$ be the identities of $G$ and $G^*$, respectively, and let $x_0'$ be a point in $X'$ with $p(x_0') = x_0$. Let $\Theta: G \times X \to X$ be a given $G$-action on $X$.

Since the inclusion $\{e^*\} \times X' \subset G^* \times X'$ induces an isomorphism of fundamental groups, we see that

$$(\Theta \circ (\pi \times p))_{\#}(\pi_1(G^* \times X', e^* \times x_0')) = p_{\#}(\pi_1(X', x_0'))$$

in $\pi_1(X, x_0)$. By a fundamental theorem of covering space theory this implies that there is a unique map

$$\Theta^*\colon G^* \times X' \to X'$$

which covers $\Theta$ and takes $e^* \times x_0'$ to $x_0'$. We claim that $\Theta^*$ is an action.

To see this, note that $\Theta^*(e^*, \cdot)\colon X' \to X'$ covers the identity on $X$ and takes $x_0'$ to $x_0'$. Hence it is the identity. Also the maps $G^* \times G^* \times X' \to X'$ defined by $(g^*, h^*, x') \mapsto \Theta^*(g^*, \Theta^*(h^*, x'))$ and $\Theta^*(g^*h^*, x')$, respectively, both cover the same map $G \times G \times X \to X$ and both take $(e^*, e^*, x_0')$ to $x_0'$. Hence they are equal and $\Theta^*$ is an action.

Moreover, $\Theta^*$ does not depend on the choice of $x_0'$ since for *any* covering action and *any* point $x_0'$ we must have that $\Theta^*(e^*, x_0') = x_0'$.

We claim that the action $\Theta^*$ commutes with each deck transformation $T$ of $X'$ (a map $X' \to X'$ covering the identity). To see this, consider the map

$$(g^*, x') \mapsto T^{-1}\Theta^*(g^*, Tx')$$

of $G^* \times X' \to X'$. This covers $\Theta$ and takes $(e^*, x')$ into $T^{-1}\Theta^*(e^*, Tx') = T^{-1}Tx' = x'$. Thus this map must be $\Theta^*$, that is,

$$\Theta^*(g^*, Tx') = T\Theta^*(g^*, x')$$

as claimed.

Now suppose that $G$ acts *effectively* (which is no loss of generality) and let $G'$ be the effective factor group of $G^*$ for the action on $X'$. Then $G'$ covers $G$. Clearly, the kernel of $G' \to G$ consists of deck transformations of $X'$.

If there is a stationary point $x$ for $G$ on $X$, then the fiber $p^{-1}(x)$ is invariant under $G'$. Since $G'$ is connected and $p^{-1}(x)$ is discrete, $G'$ leaves $p^{-1}(x)$ pointwise fixed. This would imply that an element of $\ker(G' \to G)$ is the *trivial* deck transformation, and hence that $G' = G$. We have proved the following theorem.

**9.1. Theorem** *Let $G$ be a connected Lie group acting effectively on a connected, locally arcwise connected space $X$ and let $X'$ be any covering space of $X$. Then there is a covering group $G'$ of $G$ with an effective action of $G'$ on $X'$ covering the given action. Moreover, $G'$ and its action on $X'$ are unique.*

*The kernel of $G' \to G$ is a subgroup of the group of deck transformations of $X' \to X$. (In particular, if $X' \to X$ has finitely many sheets, then so does $G' \to G$.) If $G$ has a stationary point in $X$, then $G' = G$ and $F(G, X')$ is the full inverse image of $F(G, X)$.* ∎

Now we shall discuss actions of *disconnected* Lie groups $G$. Let $G_0$ denote the identity component of $G$. First, we consider the case in which $G$ has a stationary point $x_0$ in $X$. Let $x_0'$ be a point of $X'$ over $x_0$ and consider the question of lifting the action so as to leave $x_0'$ fixed.

Note that $G$ acts as a group of automorphisms of $\pi_1(X, x_0)$ and that $G_0$ acts trivially. Let

$$J = p_{\#}(\pi_1(X', x_0')) \subset \pi_1(X, x_0).$$

Then, given $g$ in $G$, covering space theory shows that $\theta_g$: $X \to X$ lifts to a map $\theta_g'$: $X' \to X'$ such that $\theta_g'(x_0') = x_0'$ iff $(\theta_g)_{\#}(J) \subset J$. Thus $\theta_g'$ exists for all $g$ iff $J$ is stable under the $G$-action on $\pi_1(X, x_0)$. Supposing this to be the case, we define

$$\Theta' \colon G \times X' \to X'$$

by $\Theta'(g, x') = \theta_g'(x')$. The uniqueness of the $\theta_g'$ imply that this is an action once we check continuity. But it is clear that the restriction to $G_0 \times X' \to X'$ is the map given by 9.1, and hence is continuous. Now, for any $g \in G$, $\Theta'$ can be written as the composition

$$G \times X' \xrightarrow{L_g^{-1} \times 1} G \times X' \xrightarrow{\Theta'} X' \xrightarrow{\theta_g'} X'$$

and this implies that $\Theta'$ is continuous on $gG_0 \times X'$, and hence everywhere. Thus we have the following theorem.

**9.2. Theorem** *Let $G$ be a Lie group (not necessarily connected) acting on a connected and locally arcwise connected space $X$, and let $p$: $X' \to X$ be a covering space of $X$. Let $x_0' \in X'$ project to $x_0 \in X$ and suppose that $G$ leaves $x_0$ stationary. Then there exists a (unique) $G$-action on $X'$ leaving $x_0'$ stationary and covering the given action on $X$ iff the subgroup $p_{\#}(\pi_1(X', x_0'))$ is invariant under the action of $G$ on $\pi_1(X, x_0)$.* ∎

Now we shall consider the case in which $G$ is disconnected and does not necessarily have stationary points in $X$. Since a general discussion would be difficult we shall assume that $X$ is also semilocally 1-connected, so that

$X$ has a universal covering $p$: $X^* \to X$, and we shall first discuss lifting the action to $X^*$. We also assume that $G$ acts *effectively* on $X$.

For any $g$ in $G$, $\theta_g$ can be covered by a homeomorphism of $X^*$, since $X^*$ is simply connected, and any two such liftings differ by a deck transformation. Clearly, all such liftings, for all $g$, form a subgroup $G'$ of **Homeo**$(X^*)$ and there is a natural surjection $\pi$: $G' \to G$ whose kernel is precisely the group $\Delta$ of deck transformations. [Note that $\Delta \approx \pi_1(X)$.] We must topologize $G'$. (Of course, for $G$ discrete this is trivial.) For this, note that 9.1 provides a canonical subgroup, say $G_0'$, projecting to $G_0$ and carrying a topology under which it covers $G_0$. We shall show that $G_0'$ is normal in $G'$. To see this, let $g \in G'$ and consider the subgroup $gG_0'g^{-1}$ of $G'$. With the topology on this induced from that on $G_0'$ it is clear that $gG_0'g^{-1}$ *covers* $G_0$. By the uniqueness part of 9.1, the transformations of $X^*$ in $gG_0'g^{-1}$ are identical with those in $G_0'$. Hence $G_0'$ is normal in $G'$. Clearly we may topologize $G'$ by taking $G_0'$ to be open and having the above topology. Then $G'$ is a Lie group, with identity component $G_0'$, and

$$1 \to \Delta \to G' \to G \to 1$$

is an exact sequence of topological groups, with $\Delta \approx \pi_1(X)$ discrete.

The more general case of a *regular* covering space $X' \to X$ is also amenable to study. Recall that a regular covering space is a covering space $X' \approx X^*/N$ associated with a *normal* subgroup $N \approx \pi_1(X')$ of $\Delta \approx \pi_1(X)$, and that its group of deck transformations is $\Delta' = \Delta/N$. (Also, regularity is equivalent to transitivity of the group of deck transformations on the fiber.) If $N$ also happens to be normal in $G'$, then we have an exact sequence

$$1 \to \Delta/N \to G'/N \to G \to 1.$$

Moreover $G'/N$ acts on $X' = X^*/N$ and covers the action of $G$. Conversely, if a transformation group exists on $X'$ covering $G$ on $X$ (with the group a covering group of $G$ in the nonconnected sense), then the deck transformations can be thrown in with it. Then $\Delta'$ is the kernel of the projection of this enlarged group $G''$ onto $G$ and thus there is an exact sequence

$$1 \to \Delta' \to G'' \to G \to 1.$$

Then, applying the above remarks to $X^* \to X'$, there is an exact sequence

$$1 \to N \to G' \to G'' \to 1$$

(where it is clear that $G'$ is the original covering group defined above, since it acts effectively on $X^*$, covering the $G$-action, and contains $\Delta$). Thus $G'' \approx G'/N$ and the situation arises as above.

Now the normality of $N$ in $G'$ may be interpreted geometrically as follows, Let $\Delta^c$ denote the set of conjugacy classes of elements of $\Delta$, which can be identified with the set $[\mathbf{S}^1, X]$ of *free* homotopy classes. Similarly $N^c \approx [\mathbf{S}^1, X']$ and the inclusion $N \subset \Delta$ induces a map $N^c \to \Delta^c$ which corresponds to the canonical map $[\mathbf{S}^1, X'] \to [\mathbf{S}^1, X]$. Now $G$ acts on $[\mathbf{S}^1, X]$ in the obvious way and it is clear from standard covering space theory that the corresponding $G$-action on $\Delta^c$ is given by lifting elements of $G$ back to $G'$ and conjugating $\Delta$ by these (well defined on $\Delta^c$). Since $N$ is normal in $\Delta$ this clearly implies that $N$ is normal in $G'$ iff the action of $G$ on $[\mathbf{S}^1, X] \approx \Delta^c$ preserves the image of $[\mathbf{S}^1, X'] \approx N^c$ (not necessarily pointwise). We sum up our remarks in the following theorem.

**9.3. Theorem** *Let $G$ be a Lie group (not necessarily connected) acting effectively on a connected, locally arcwise connected, and semilocally 1-connected space $X$. Let $X'$ be a regular covering space of $X$ with group $\Delta'$ of deck transformations. A necessary and sufficient condition for there to exist a covering group $G'$ of $G$ (in the nonconnected sense) and an effective action of $G'$ on $X'$ covering that of $G$ on $X$, is that the action of $G$ on the set* $[\mathbf{S}^1, X]$ *(free homotopy classes) preserves the image of* $[\mathbf{S}^1, X'] \to [\mathbf{S}^1, X]$. *[Note that if $\pi_1(X)$ is abelian, then the condition is that $G$ preserves the image of $H_1(X') \to H_1(X)$.] By possibly enlarging $G'$, there is an exact sequence*

$$1 \to \Delta' \to G' \to G \to 1. \quad \blacksquare$$

(The reader should note the case of finite groups $G$. It is definitely false that $G'$ can be taken to be $G$ and the reader should be able to find simple counterexamples of this.)

There are some cases in which one can find a canonical lifting of homeomorphisms and thus provide a covering action by the original group; 9.2 gave one such situation. Another is the case of the orientable double covering of a nonorientable manifold. Suppose that $M$ is a nonorientable manifold and that $M'$ is its orientable double cover. Let $G$ be a Lie group (not necessarily connected) acting on $M$. We apply 9.3 by noting that the image of $[\mathbf{S}^1, M'] \to [\mathbf{S}^1, M]$ consists exactly of those classes represented by *orientation preserving loops*. Since any homeomorphism clearly preserves the set of such classes, 9.3 implies that there is a covering action by a Lie group $G'$ and an exact sequence

$$1 \to \mathbf{Z}_2 \to G' \to G \to 1.$$

But there is a splitting homomorphism $G' \to \mathbf{Z}_2$ taking $g'$ to the identity iff $g'$ preserves orientation. (Since the component of the identity is in the kernel, this is continuous.) Thus the subgroup of $G'$ consisting of orientation preserving homeomorphisms maps isomorphically onto $G$, and we have the following corollary.

**9.4. Corollary** *If $G$ is a Lie group (not necessarily connected) acting on a nonorientable manifold $M$, then there is a unique covering action of the same group $G$ by orientation preserving homeomorphisms on the orientable double covering of $M$.* ▌

*Remarks* It is certainly possible to weaken the Lie condition in some of these theorems, at some expense of efficiency in the proofs. We do not know the most general possible conditions. We regard 9.1 and 9.2 as folk theorems. In the special case of compact connected Lie $G$ and finite coverings of locally compact spaces, 9.1 was proved in Kister and Mann [1] by a different method. The case of 9.2 for the universal covering was proved in Conner and Montgomery [1], and the general case can be found in Conner and Raymond [1]. A more general version of 9.4 can be found in Bredon [3]. Theorem 9.3 does not seem to have been noticed.

## EXERCISES FOR CHAPTER I

**1.** Let $X$ be a $G$-space and let $N$ be a closed normal subgroup of $G$. Show that there is a canonically induced action of $G/N$ on $X/N$ and a natural homeomorphism

$$\frac{X/N}{G/N} \approx \frac{X}{G}.$$

**2.** If $X$ is a $G$-space and $A \subset X$ let $G_A = \{g \in G \mid g(a) = a \text{ for all } a \in A\}$. Show that:

(i) $A \subset B \Rightarrow G_A \supset G_B$,
(ii) $H \subset K \Rightarrow X^H \supset X^K$.

Also put $A' = X^{G_A}$ and show that

(iii) $A' \supset A$,
(iv) $A'' = A'$,
(v) $(A \cup B)' = A' \cup B'$,

and hence that $A \mapsto A'$ is a closure operator, when $G$ acts nontrivially.

**3.** In the examples of Section 7 show that the orbits corresponding to boundary points of $\mathbf{D}^2 \approx \Sigma_k^{2n-1}/\mathbf{O}(n)$ have the orbit type of $\mathbf{S}^{n-1} = \mathbf{O}(n)/\mathbf{O}(n-1)$ and those corresponding to interior points have the orbit type of $\mathbf{V}_{n,2} = \mathbf{O}(n)/\mathbf{O}(n-2)$.

**4.** Let $G$ be a compact group and let $\mathscr{E}_G$ be the set of $G$-orbit types. Topologize $\mathscr{E}_G$ by taking the smallest topology for which the set of all types less than or equal to type$(G/H)$ is closed (for each $H \subset G$). If $X$ is a $G$-space, show that the canonical function $\tau\colon X/G \to \mathscr{E}_G$, assigning to an orbit its type, is continuous. (*Hint*: Consider the images of $X^H$ in $X/G$ for each $H$.)

**5.** We define a $G$-**orbit structure** to be a Hausdorff space $Y$ together with a continuous map $Y \to \mathscr{E}_G$ (see Exercise 4), and **the orbit structure** of a $G$-space $X$ to be the canonical map $\tau\colon X/G \to \mathscr{E}_G$. If $f\colon Y \to X/G$ is any map, show that the orbit structure of the $G$-space $f^*X$ is (canonically equivalent to) $\tau \circ f\colon Y \to \mathscr{E}_G$.

A map $Y_1 \to Y_2$ of spaces with given $G$-orbit structures is said to **preserve the orbit structures** if the diagram

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_2 \\ & \searrow \quad \swarrow & \\ & \mathscr{E}_G & \end{array}$$

commutes. An equivariant map $f\colon X_1 \to X_2$ of $G$-spaces is said to **preserve the orbit structure** if its induced map $X_1/G \to X_2/G$ does so. Show that an equivariant map preserves orbit structure iff its restriction to each orbit in $X_1$ is a homeomorphism to its image in $X_2$.

**6.** Let $K$ be a simplicial complex and let $G$ be a discrete group acting simplicially on $K$. Show that the first barycentric subdivision $K'$ of $K$ has the following property (where $g \in G$ and $\sigma$ is a simplex of $K'$):

($\star$) If $g(\sigma) = \sigma$, then $g$ leaves $\sigma$ *pointwise* fixed.

Also prove the following *Equivariant Simplicial Approximation Theorem*: Let $K$ and $L$ be $G$-complexes (as above) and assume that $L$ satisfies ($\star$) (where $\sigma$ is now a simplex of $L$). Let $f\colon |K| \to |L|$ be an equivariant map on the polyhedra associated with $K$ and $L$. Then, for suitable $r$, there is a simplicial approximation $g\colon K^{(r)} \to L$ of $f$ which is *equivariant* ($K^{(r)}$ being the $r$th barycentric subdivision of $K$) and such that $|g|\colon |K| \to |L|$ is *equivariantly* homotopic to $f$. (*Hint*: Choose $g$ suitably on one vertex out of each *orbit* of $G$ on the vertices of $K^{(r)}$ and extend by equivariance.)

**7.** Consider the 5-dimensional representation of $\mathbf{SO}(3)$ given at the end of Section 4. Show how to construct an equivariant proper map $f$: $\mathbf{R}^5 \to \mathbf{R}^5$ such that $f^{-1}(0) = 0$ and such that the degree of $f$ is 0. (*Hint*: Consider $F(\mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{R}^5)$ and, inside this, the cross section of the action given in Section 4. Use Corollary 3.4.)

**8.** Show that the Tietze-Gleason Theorem 2.3 also holds for completely regular spaces $X$ and *compact* invariant subspaces $A$. (*Hint*: Use the Stone-Čech compactification of $X$.)

**9.** Show that each neighborhood of an orbit in a $G$-space, with $G$ compact, contains an *invariant* neighborhood. (*Hint*: Use 3.1.)

**10.** Let $G$ be compact and let $f$: $X \to Y$ be an equivariant orbit structure preserving map between $G$-spaces with $f'$: $X/G \to Y/G$ the induced map of orbit spaces. Show that $f$ is open iff $f'$ is open.

**11.** Let $G$ be compact and $X$ be a $G$-space. If $f$: $X \to X$ is an equivariant map such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ & \searrow \quad \swarrow & \\ & X/G & \end{array}$$

commutes, show that $f$ is an equivalence of $G$-spaces.

**12.** Let $H, K \subset G$ be compact groups. Show that $a^{-1}K \mapsto R_a^{K,H}$ gives a homeomorphism

$$(G/K)^H = F(H, G/K) \xrightarrow{\approx} \mathbf{Map}^G(G/H, G/K)$$

of the space of fixed points of $H$ on $G/K$ onto the space of equivariant maps $G/H \to G/K$ in the compact-open topology.

**13.** If $G$ is a compact group and $X$ is a metric $G$-space, show that there is an equivalent metric which is invariant under the action. Also give an example showing that this is generally *not* possible *uniformly* (that is, so that the two metrics give the same uniform structure).

# CHAPTER II

## GENERAL THEORY OF *G*-SPACES

In this chapter we shall study actions of compact Lie groups from a general point of view, as distinguished from the study of actions on manifolds or of actions of special types of compact Lie groups.

The chapter contains four main theorems. The first is the existence of tubes, or slices, due to Montgomery and Yang [1] and to Mostow [1]. This fundamental result concerns the study of the nature of the neighborhood of an orbit in a $G$-space. This topic is treated in Sections 4 and 5 after some preliminary background material on bundles and twisted products in the first three sections. It is applied in Section 6 to the comparison of the fundamental group of a $G$-space with that of its orbit space.

The second main theorem is the Covering Homotopy Theorem of Palais [3]. It is studied in Section 7 and applied in Section 8 to the situation of a $G$-space with conical orbit structure.

The third main theorem is the Classification Theorem of Palais [3] which is treated in Section 9 and then applied in Section 10 to prove the fourth main theorem, the Equivariant Embedding Theorem of Mostow [1]. The latter states that every separable metric $G$-space of finite dimension and finitely many orbit types, with $G$ compact Lie, can be embedded in an orthogonal representation of $G$.

From Section 5 on, our attention is restricted to actions of compact Lie groups.

### 1. FIBER BUNDLES

In this section we define the notion of a fiber bundle and make some elementary remarks about them. We shall not go far into the *theory* of fiber bundles in this book, but rather refer the reader to Steenrod [1] and Husemoller [1] for this. (But note that the use of "effective" to mean "free" in the latter reference is incorrect.)

Let $K$ be a topological group and let $F$ be an effective *right* $K$-space.

Let $X$ and $B$ be Hausdorff spaces. By a **fiber bundle** over $B$ (the **base space**) with **total space** $X$, **fiber** $F$, and **structure group** $K$, we mean a map

$$p\colon X \to B$$

together with a collection $\Phi$ of homeomorphisms $\varphi\colon F \times U \xrightarrow{\approx} p^{-1}(U)$ for $U$ open in $B$, called **charts over** $U$, such that

(1) The diagram

$$\begin{array}{ccc} F \times U & \xrightarrow{\varphi} & p^{-1}(U) \\ & {\scriptstyle \text{proj}}\searrow \quad \swarrow{\scriptstyle p} & \\ & U & \end{array}$$

commutes for each chart $\varphi \in \Phi$ over $U$.

(2) Each point of $B$ has a neighborhood over which there is a chart in $\Phi$.

(3) If $\varphi\colon F \times U \to p^{-1}(U)$ is in $\Phi$ and $V \subset U$ is open, then the restriction of $\varphi$ to $F \times V \to p^{-1}(V)$ is in $\Phi$.

(4) If $\varphi, \psi \in \Phi$ are charts over $U$, then there is a (continuous) map $\theta\colon U \to K$ such that

$$\psi(f, u) = \varphi(f \cdot \theta(u), u)$$

for all $f \in F$ and $u \in U$. This map $\theta$ is called the **transition function** for the charts $\varphi$ and $\psi$.

(5) The set $\Phi$ is maximal among collections satisfying the preceding conditions.

*Remark* Since $K$ acts effectively on $F$, it is clear that the map $\theta$ in (4) is uniquely determined by $\varphi^{-1}\psi$. Also note that $\varphi^{-1}\psi\colon F \times U \to F \times U$ induces a function

$$U \to \mathbf{Map}(F, F \times U) = \mathbf{Map}(F, F) \times \mathbf{Map}(F, U)$$

which takes $u$ to $(\theta(u), c_u)$, where $c_u$ is the constant map of $F$ to $u$. It is well known (see Dugundji [1], for example) that this map is continuous if these mapping sets are given the compact-open topology (and the "equality" is a homeomorphism). Thus, $\theta$ is *automatically continuous when $K$ has the compact-open topology*. (Also see Arens [1] with regard to "good" topologies on $K$.)

**1.1. Theorem** *Suppose that $p\colon X \to B$ is a bundle with fiber $F$ and structure group $K$. Suppose that $F$ is also a left $G$-space and that the actions of $G$ and*

*$K$ commute [that is, $(gf)k = g(fk)$]. Then there is a unique $G$-action on $X$ covering the trivial action on $B$ and such that each chart $\varphi\colon F \times U \to p^{-1}(U)$ is equivariant [where $G$ acts on $F \times U$ by $(g, (f, u)) \mapsto (gf, u)$].*

***Proof*** The action is to be defined by the equivariance of the charts and it clearly suffices to prove that it is independent of the choice of a chart over $U$. That is, it suffices to show that each $\varphi^{-1}\psi\colon F \times U \to F \times U$ is equivariant. But

$$\begin{aligned} g(\varphi^{-1}\psi(f, u)) &= g(f \cdot \theta(u), u) \\ &= (g(f \cdot \theta(u)), u) \\ &= ((gf) \cdot \theta(u), u) \\ &= \varphi^{-1}\psi(gf, u) = \varphi^{-1}\psi(g(f, u)). \quad \blacksquare \end{aligned}$$

By a **principal $G$-bundle** we mean a bundle with fiber $G$ and structure group $G$ acting by right translation (that is, $g$ in the structure group takes $g'$ in the fiber to $g'g$; giving a *right* action).

Since the usual left translation action of $G$ on itself commutes with right translation, 1.1 shows the following.

**1.2. Corollary** *If $p\colon X \to B$ is a principal $G$-bundle, then there is a canonical free $G$-action on $X$, which covers the identity on $B$ (and is left translation in the fibers). The map $p\colon X \to B$ induces a homeomorphism $X/G \xrightarrow{\approx} B$, and thus may be regarded as the orbit map for this action.* $\blacksquare$

***Remark*** In Section 5 we shall prove a theorem of Gleason [1] which implies that *every* free action comes in this way from a principal bundle *when* $G$ is compact Lie and $X$ is completely regular. Thus, in this case, the notions of a principal $G$-bundle and of a free $G$-action are canonically equivalent.

## 2. TWISTED PRODUCTS AND ASSOCIATED BUNDLES

Suppose that $X$ is a *right* $G$-space and that $Y$ is a *left* $G$-space. Then a left $G$-action on $X \times Y$ is given by letting $g$ take $(x, y)$ to $(xg^{-1}, gy)$. We define the twisted product of $X$ and $Y$ to be the orbit space

$$X \times_G Y$$

of this action. That is, $X \times_G Y$ is the quotient space of $X \times Y$ under the equivalence relation which relates $(xg, y)$ to $(x, gy)$ for all $x \in X$, $y \in Y$, and $g \in G$. The equivalence class (orbit) of $(x, y)$ is denoted by $[x, y]$, so that $[x, y] = [x', y']$ iff there is a $g \in G$ with $x' = xg^{-1}$ and $y' = gy$. Note that $[xg, y] = [x, gy]$.

This construction is functorial. That is, if $f\colon Y \to Y'$ is an equivariant map of left $G$-spaces, then we have the induced map

$$X \times_G f\colon X \times_G Y \to X \times_G Y'$$

given by $[x, y] \mapsto [x, f(y)]$. Similarly for equivariant maps of the left-hand factor.

**2.1. Proposition** *If $f$ is open, then $X \times_G f$ is open.*

*Proof* This is immediate from the commutative diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{X \times f} & X \times Y' \\ \downarrow & & \downarrow \\ X \times_G Y & \xrightarrow{X \times_G f} & X \times_G Y' \end{array}$$

since the vertical maps are open. ∎

In particular, the projection of $Y$ to a point induces an *open* map

$$X \times_G Y \to X \times_G \star \approx X/G.$$

This may also be described as the map on orbit spaces induced by the equivariant projection of $X \times Y$ to $X$. Similarly we have the open map $X \times_G Y \to Y/G$.

If $X$ and $Y$ are right and left $K$-spaces, respectively, and $X$ is a left $G$-space (with right and left operations always commuting), then $X \times_K Y$ is a left $G$-space by $g[x, y] = [gx, y]$. Similarly, if $Y$ is a right $H$-space, then so is $X \times_K Y$ by $[x, y]h = [x, yh]$.

**2.2. Proposition** *The map $X \times_G G \to X$ taking $[x, g] \mapsto xg$ is an equivariant homeomorphism of right $G$-spaces. The inverse is given by $x \mapsto [x, e]$.*

*Proof* Both maps are clearly well defined and inverse to one another. Continuity of the first map follows from that of the action map $X \times G \to X$

and from openness of $X \times G \to X \times_G G$. Continuity of the second map is clear. ▮

**2.3. Proposition** *Let $X$ be a right $H$-space, $Y$ a left $H$- and right $K$-space, and $Z$ a left $K$-space. Then there is the canonical homeomorphism*

$$(X \times_H Y) \times_K Z \xrightarrow{\approx} X \times_H (Y \times_K Z)$$

*given by* $[[x, y], z] \mapsto [x, [y, z]]$.

***Proof*** The map is well defined since $[[xh^{-1}, hy]k^{-1}, kz] = [[xh^{-1}, hyk^{-1}], kz]$ goes to $[xh^{-1}, [hyk^{-1}, kz]] = [xh^{-1}, h[yk^{-1}, kz]] = [x, [y, z]]$. Continuity follows from the fact that the composition $(X \times Y) \times Z \to (X \times_H Y) \times Z \to (X \times_H Y) \times_K Z$ is open, since all orbit maps are open. Clearly the inverse has the same properties. ▮

Recall from 1.2 that a *principal* $K$-bundle $p\colon X \to B$ has a canonical free left $K$-action whose orbit space $X/K$ is homeomorphic to $B$, via $p$.

**2.4. Theorem** *Let $p\colon X \to B$ be a principal $K$-bundle and let $F$ be a right $K$-space. Then*

$$\pi\colon F \times_K X \to B$$

*defined by $\pi[f, x] = p(x)$ is a bundle with fiber $F$ and structure group $K$ and is called the $F$-bundle associated with this principal $K$-bundle. If $\varphi\colon K \times U \to p^{-1}(U)$ is a chart of the principal bundle over $U$, then the composition*

$$\bar{\varphi}\colon F \times U \xrightarrow{\approx} (F \times_K K) \times U \xrightarrow{\approx} F \times_K (K \times U)$$
$$\xrightarrow{F \times_K \varphi} F \times_K p^{-1}(U) \xrightarrow{\approx} \pi^{-1}(U)$$

*is taken to be a chart of the associated bundle. (Note that $\bar{\varphi}(f, u) = [f, \varphi(e,u)]$.)*

***Proof*** The first map in the above composition is that of 2.2, the second is that of 2.3, and the third map is a homeomorphism by 2.1. The last map is induced by inclusion $p^{-1}(U) \to X$, is an embedding in $F \times_K X$ by 2.1, and its image is $\pi^{-1}(U)$ since all these maps commute with the obvious projections to $U$.

If $\varphi$ and $\psi$ are charts over $U$ of the principal bundle and if $\theta\colon U \to K$ is the transition function (so that $\psi(k, u) = \varphi(k\theta(u), u)$), then

$$\begin{aligned}\bar{\psi}(f, u) &= [f, \psi(e, u)] \\ &= [f, \varphi(e \cdot \theta(u), u)] \\ &= [f, \varphi(\theta(u) \cdot e, u)] \\ &= [f,\ \theta(u)\varphi(e, u)] \\ &= [f \cdot \theta(u), \varphi(e, u)] = \bar{\varphi}(f \cdot \theta(u), u)\end{aligned}$$

(by definition of the left $K$-action on $X$) and this shows $\theta$ to be the transition function for $\bar{\varphi}$ and $\bar{\psi}$ as well.

Since any collection of charts satisfying conditions (1)–(4) of the definition in Section 1 is easily seen to be contained in a unique maximal such collection (see Steenrod [1]), the theorem follows. ▮

**2.5. Lemma** *In the situation of* 2.4, *the diagram*

$$\begin{array}{ccc} F \times X & \xrightarrow{\text{proj}} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ F \times_K X & \xrightarrow{\pi} & B \end{array}$$

*is a pull-back diagram.*

*Proof* The given maps induce a map of $F \times X$ into the fibered product of $(X, p)$ and $(F \times_K X, \pi)$ and it is easily checked that this is one–one onto (since $K$ acts freely on $X$). It remains to show that it is open. To do this, we may look at the diagram *locally* in $B$. Then if $U \subset B$ is small, the diagram becomes

$$\begin{array}{ccc} F \times K \times U & \xrightarrow{\text{proj}} & K \times U \\ \downarrow & & \downarrow{\scriptstyle \text{proj}} \\ F \times_K (K \times U) \approx F \times U & \xrightarrow{\text{proj}} & U \end{array}$$

where the left-hand map is $(f, k, u) \mapsto [f, (k, u)] \mapsto (fk, u)$. Then the fibered product is just $F \times K \times U$ and the map in question is $(f, k, u) \mapsto (fk, k, u)$. The inverse of this is $(f, k, u) \mapsto (fk^{-1}, k, u)$ and is continuous. ▮

***Remark*** When $K$ is compact, another proof is available using Chapter I, Exercise 10.

The following theorem is a basic result in bundle theory.

**2.6. Theorem** *Let $p\colon X \to B$ be a principal $K$-bundle and let $Y$ be a left $K$-space, Let $K$ act on the right of $Y$ by putting $yk = k^{-1}y$. Then there is a natural one–one correspondence between the $K$-equivariant maps $f\colon X \to Y$ and cross sections $\bar{f}$ of the associated $Y$-bundle*

$$\pi\colon Y \times_K X \to B.$$

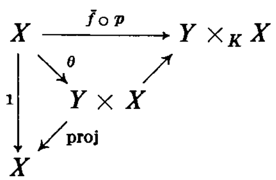
*The correspondence is characterized by the equation $\bar{f}(p(x)) = [f(x), x]$.*

***Proof*** If $f\colon X \to Y$ is equivariant, then $x \mapsto [f(x), x]$ gives a map $f'\colon X \to Y \times_K X$, and since

$$f'(kx) = [f(kx), kx] = [kf(x), kx] = [f(x)k^{-1}, kx] = [f(x), x] = f'(x),$$

we see that $f'$ induces a map $\bar{f}\colon B = X/K \to Y \times_K X$ which is clearly a cross section and is characterized by the equation $\bar{f}(p(x)) = [f(x), x]$.

If $\bar{f}$ is a cross section, then $\pi(\bar{f} \circ p(x)) = p(x)$. It follows from 2.5 that there is a unique equivariant map $\theta\colon X \to Y \times X$ such that the diagram




commutes. Thus $\theta$ has the form $\theta(x) = (f(x), x)$ where $f\colon X \to Y$ satisfies $\bar{f}(p(x)) = [f(x), x]$. ∎

The remainder of this section will be devoted to some material that will not be used until Chapter V, but which will be important there.

Let $S$ and $T$ be topological groups. Let $A$ be a *right* $S$-space and let $A'$ be a *right* $T$-space. Assume that $A$ is locally compact. Let **Map**$(A, A')$ be the space of continuous maps from $A$ to $A'$ with the compact-open topology and let $S \times T$ act on the left of **Map**$(A, A')$ by putting

$$((s, t)f)(a) = f(as)t^{-1},$$

where $f \in$ **Map**$(A, A')$.

Now suppose that $p_X\colon X \to B$ is a *principal* $S$-bundle and that $p_Y\colon Y \to B$ is a *principal* $T$-bundle. The product $X \times Y \to B \times B$ is then a principal $S \times T$-bundle. The restriction of this to the diagonal of $B \times B$ is a principal $S \times T$-bundle over $B$ whose total space will be denoted by

$$\Delta = \{(x, y) \in X \times Y \mid p_X(x) = p_Y(y)\}.$$

We put $p(x, y) = p_X(x) = p_Y(y)$ for $(x, y) \in \Delta$. As usual, by 1.2, $\Delta$ has a canonical left $S \times T$-action which is given by $(s, t)(x, y) = (sx, ty)$. We shall consider the associated bundles $A \times_S X$ and $A' \times_T Y$ over $B$. By a **map** $\varphi$: $A \times_S X \to A' \times_T Y$ **over** $B$ we simply mean a map which commutes with the projections onto $B$.

**2.7. Theorem** *In the above situation, there is a canonical one–one correspondence between maps* $\varphi$: $A \times_S X \to A' \times_T Y$ *over* $B$ *and* $S \times T$-*equivariant maps* $\hat{\varphi}$: $\Delta \to \mathbf{Map}(A, A')$. *The correspondence is characterized by the equation*

$$\varphi[a, x] = [\hat{\varphi}(x, y)(a), y].$$

*Thus these are also in one–one correspondence with the cross sections* $\bar{\varphi}$ *of the associated bundle*

$$\mathbf{Map}(A, A') \times_{S\times T} \Delta \to B,$$

*given by* $\bar{\varphi}(p(x, y)) = [\hat{\varphi}(x, y), (x, y)]$.

*Proof* Note that

$$\begin{array}{ccc} \Delta & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[p_X]{} & B \end{array}$$

is a pull-back diagram, so that the projection $\Delta \to X$ is the bundle induced from the bundle $p_Y$ by the map $p_X$. In particular, $\Delta \to X$ is *open*. Thus also $A \times \Delta \to A \times X$ and the composition $\mu$: $A \times \Delta \to A \times X \to A \times_S X$ (where $\mu(a, (x, y)) = [a, x]$) are *open maps*.

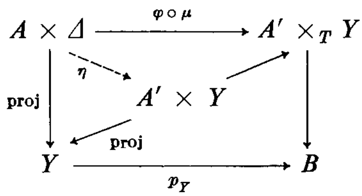
Suppose that $\hat{\varphi}$: $\Delta \to \mathbf{Map}(A, A')$ is given. Map $A \times \Delta \to A' \times_T Y$ by $(a, (x, y)) \mapsto [\hat{\varphi}(x, y)(a), y]$. If $(x, y)$ and $(x, y')$ are both in $\Delta$, then $y' = ty$ for some $t \in T$. Since

$$[\hat{\varphi}(x, ty)(a), ty] = [\hat{\varphi}(x, y)(a)t^{-1}, ty] = [\hat{\varphi}(x, y)(a), y],$$

by equivariance of $\hat{\varphi}$, we see that this map factors through $A \times X$. Since

$\hat{\varphi}(s^{-1}x, y)(as) = \hat{\varphi}(x, y)(a)$ we see that the above map factors through $A \times_S X$. That is, there is a function $\varphi$: $A \times_S X \to A' \times_T Y$ given by $\varphi[a, x] = [\hat{\varphi}(x, y)(a), y]$ [where $p_Y(y) = p_X(x)$]. Since $\mu$: $A \times \Delta \to A \times_S X$ is an open map it follows that $\varphi$ is continuous. Since $p_X(x) = p_Y(y)$, $\varphi$ is a map over $B$.

Now suppose that $\varphi$ is given and consider the diagram




Since $\varphi$ is a map over $B$, $(\varphi \circ \mu)(a, (x, y)) = \varphi[a, x]$ projects to $p_X(x) = p_Y(y)$ in $B$, which means that the square commutes. The lower right part of the diagram is a pull-back diagram by 2.5 and thus there exists a unique map $\eta$ as indicated, completing the commutative diagram. By commutativity, $\eta$ has the form

$$\eta(a, (x, y)) = (\eta'(a, (x, y)), y),$$

where $\eta'$: $A \times \Delta \to A'$. By the "exponential law" the function $\hat{\varphi}$: $\Delta \to \mathbf{Map}(A, A')$ given by $\hat{\varphi}(x, y)(a) = \eta'(a, (x, y))$ is continuous. Commutativity of the top part of the above diagram becomes

$$\varphi[a, x] = [\hat{\varphi}(x, y)(a), y].$$

The above diagram can be considered as a diagram of $S \times T$-spaces with the obvious actions and thus $\eta$ is equivariant, that is,

$$\begin{aligned}(\hat{\varphi}(sx, ty)(as^{-1}), ty) &= \eta(as^{-1}, (sx, ty)) \\ &= \eta((s, t)(a, (x, y))) \\ &= t(\eta(a, (x, y))) = (\hat{\varphi}(x, y)(a)t^{-1}, ty).\end{aligned}$$

This is equivalent to the equivariance of $\hat{\varphi}$ and finishes the proof of the first part of the theorem. The second part follows from 2.6. ▮

It is edifying to see what the correspondence between $\varphi$ and $\bar{\varphi}$ becomes "locally." Thus if both bundles are trivial over $U \subset B$, then (for given charts over $U$) $\varphi$ can be regarded as a map $\varphi$: $A \times U \to A' \times U$ commuting with projection to $U$. Thus $\varphi$ has the form $\varphi(a, u) = (\bar{\varphi}(u)(a), u)$, where $\bar{\varphi}$: $U$

$\to$ **Map**$(A, A')$ (by the exponential law). Thus $\bar{\varphi}$ can be regarded as a cross-section of the product bundle **Map**$(A, A') \times U \to U$. The reader may check that this is, indeed, the correspondence given by 2.7 (forgetting now about $\hat{\varphi}$). Note that a homotopy of maps over $B$ corresponds to a homotopy of cross sections (by 2.7) and that continuity of the homotopies is easily checked by looking at them locally for trivial bundles. We shall leave the details of this to the reader.

Also note that if $W \subset$ **Map**$(A, A')$ is any subspace invariant under the $S \times T$-action (that is, invariant under composition with the $S$-action on $A$ and the $T$-action on $A'$), then the maps $\varphi$ over $B$ which are *fiberwise in* $W$ correspond to cross sections of the subbundle

$$W \times_{S\times T} \Delta \to B.$$

In particular, suppose that $A$ and $A'$ are left $G$-spaces with the $G$-actions commuting with the right $S$ and $T$ actions. Then $A \times_S X$ and $A' \times_T Y$ are canonically left $G$-spaces by 1.1. The space $\mathbf{Map}^G(A, A') \subset$ **Map**$(A, A')$ of *G-equivariant* maps is $S \times T$-invariant (since the $S$ and $T$ actions are $G$-equivariant). Moreover, a map $A \times_S X \to A' \times_T Y$ over $B$ is fiberwise in $\mathbf{Map}^G(A, A')$ iff it is $G$-equivariant. Thus we have the following corollary.

**2.8. Corollary** *If, in* 2.7, *A and A′ are left G-spaces with the G-actions commuting with the right S- and T-actions, then* 2.7 *gives a one–one correspondence between G-equivariant maps* $\varphi$: $A \times_S X \to A' \times_T Y$ *over B and cross sections* $\bar{\varphi}$ *of the associated bundle*

$$\mathbf{Map}^G(A, A') \times_{S\times T} \Delta \to B. \quad ▮$$

Since we shall eventually make important use of it, we remark again that the correspondence of 2.8 induces a correspondence between *G-equivariant* homotopies of maps $A \times_S X \to A' \times_T Y$ over $B$ and homotopies of cross sections of the bundle

$$\mathbf{Map}^G(A, A') \times_{S\times T} \Delta \to B.$$

## 3. TWISTED PRODUCTS WITH A COMPACT GROUP

Throughout this section we let $G$ be a *compact* group and we shall specialize our considerations to the case of twisted products

$$G \times_H A,$$

where $H$ is a closed subgroup of $G$ and $A$ is a left $H$-space. This case was noted briefly in Chapter I, Section 6. Left translation makes $G \times_H A$ into a left $G$-space: $(g, [g', a]) \mapsto [gg', a]$.

**3.1. Proposition** *If $K \subset H \subset G$ and $B$ is a (left) $K$-space, then the map $G \times_K B \to G \times_H (H \times_K B)$ taking $[g, b] \mapsto [g, [e, b]]$ is a $G$-equivariant homeomorphism.*

***Proof*** The map is the composition

$$G \times_K B \xrightarrow{\approx} (G \times_H H) \times_K B \xrightarrow{\approx} G \times_H (H \times_K B)$$

of 2.2 and 2.3. ▌

Recall the map $i_e$: $A \to G \times_H A$ given by $i_e(a) = [e, a]$, which is an $H$-equivariant embedding since $H$ is compact. (This is also an embedding when $G \to G/H$ is a bundle mapping, since then $G \times_H A$ is an associated bundle by 2.4. Thus $i_e$ is an embedding when $G$ is any Lie group (not necessarily compact) and $H$ is closed. Perhaps it is always an embedding.)

Note that the map $G \times_H A \to G/H$ given by $[g, a] \mapsto gH$ is $G$-equivariant and that the inverse image of the point $H/H$ is $i_e(A) = \{[e, a] \mid a \in A\}$. The following proposition is a converse to this.

**3.2. Proposition** *Let $X$ be a $G$-space and suppose that $f$: $X \to G/H$ is equivariant. Put $A = f^{-1}(eH)$. Then $A$ is invariant under $H$ and the map $\varphi$: $G \times_H A \to X$ defined by $\varphi[g, a] = g(a)$ is an equivariant homeomorphism.*

***Proof*** Clearly $\varphi$ is well defined, equivariant, and is continuous by the diagram

$$\begin{array}{ccc} G \times A & \searrow & \\ \downarrow & & X \\ G \times_H A & \nearrow_{\varphi} & \end{array}$$

To see that $\varphi$ is onto, let $x$ be in $X$ with $f(x) = gH$ and then note that $f(g^{-1}x) = eH$ so that $(g^{-1}x) \in A$ and $x = g(g^{-1}x) = \varphi[g, g^{-1}x]$.

To see that $\varphi$ is one–one, suppose that $\varphi[g, a] = \varphi[g', a']$. Then $ga = g'a'$ so that

$$gH = g(f(a)) = f(ga) = f(g'a') = g'(f(a')) = g'H$$

whence $h = g^{-1}g'$ is in $H$. Thus $ga = g'a' = gha'$ so that $a = ha'$ and $[g', a'] = [g'h^{-1}, ha'] = [g, a]$ as claimed.

To complete the proof is suffices to show that $\varphi$ is closed. But this follows from the above diagram, since $A$ is closed in $X$ and $G \times A \to X$ is a closed map by I.1.2. ∎

**3.3. Proposition** *The inclusion $i_e$: $A \to G \times_H A$ induces a homeomorphism $A/H \to (G \times_H A)/G$* (i.e., $H(a) \mapsto G[e, a]$).

*Proof* Recall that the embedding $i_e$: $A \to G \times_H A$ is $H$-equivariant. Thus every $H$-orbit in $A$ is mapped into an $H$-orbit (and hence in a $G$-orbit) of $G \times_H A$, so that the map is defined. It is continuous by the diagram

$$\begin{array}{ccc} A & \longrightarrow & G \times_H A \\ \downarrow & & \downarrow \\ A/H & \longrightarrow & (G \times_H A)/G. \end{array}$$

The projection $G \times A \to A$ is $H$-equivariant and thus induces a map

$$G \times_H A \to A/H.$$

This is given by $[g, a] \mapsto H(a)$ and hence clearly factors as

$$\begin{array}{ccc} G \times_H A & & \\ \downarrow & \searrow & \\ (G \times_H A)/G & \longrightarrow & A/H. \end{array}$$

The diagram shows the horizontal map $G[g, a] \mapsto H(a)$ to be continuous and it is clearly the inverse of the original map. ∎

*Remark* Clearly if $f$: $G^* \to G$ is an isomorphism and $H = f(H^*)$ and if $A^* \to A$ is a homeomorphism, then an $H$-action on $A$ pulls back to an $H^*$-action on $A^*$ and $G^* \times_{H^*} A^*$ is equivalent to $G \times_H A$. Applying this to an inner automorphism by some given $g \in G$ we see that, given an $H$-action on $A$, there is an action of $gHg^{-1}$ on a copy $A^*$ of $A$ and an equivalence

$$G \times_H A \approx G \times_K A^* \qquad (\text{where} \quad K = gHg^{-1})$$

of $G$-spaces.

We wish to find the isotropy group at a point of the twisted product $G \times_H A$. It suffices to treat points of the form $[e, a]$. Clearly $g \in G_{[e,a]}$ iff $[e, a] = g[e, a] = [g, a]$, which means that, for some $h \in H$, $(h^{-1}, h(a)) = (g, a)$, that is $g \in H_a$. Thus

**(3.4)** $$G_{[e,a]} = H_a \quad \text{in} \quad G \times_H A.$$

## 4. TUBES AND SLICES

Let $X$ be a $G$-space with $G$ *compact* and let $P \subset X$ be an *orbit of type* $G/H$. By a **tube about** $P$ (or a $G$-tube about $P$) we mean a $G$-equivariant embedding (homeomorphism into)

$$\varphi\colon G \times_H A \to X$$

onto an *open neighborhood* of $P$ in $X$, where $A$ is some space on which $H$ acts.

In this situation, note that every $G$-orbit in $G \times_H A$ passes through a point of the form $[e, a]$. Thus let $a \in A$ be such that $\varphi[e, a] \in P$ and put $x = \varphi[e, a]$, so that $P = G(x)$. Then, by (3.4), $G_x = G_{[e,a]} = H_a \subset H$. Since $G_x$ is conjugate to $H$, by assumption, this implies, by 0.1.9, that

$$G_x = H_a = H.$$

Thus such a point $a \in A$ is *stationary* under $H$.

Since $i_e\colon A \to G \times_H A$ is an $H$-embedding, the composition $\varphi \circ i_e\colon A \to X$ is also an $H$-embedding when $\varphi$ is a tube. Clearly, it is no loss of generality to suppose that $A \subset X$. This gives rise to the following definition.

**4.1. Definition** Let $x \in X$, a $G$-space. Let $x \in S \subset X$ be such that $G_x(S) = S$. Then $S$ is called a **slice** at $x$, if the map

$$G \times_{G_x} S \to X,$$

taking $[g, s] \mapsto g(s)$, is a tube about $G(x)$.

**4.2. Theorem** *Let $X$ be a $G$-space, let $x \in S \subset X$, and put $H = G_x$. Then the following statements are equivalent:*

(i) *There is a tube $\varphi\colon G \times_H A \to X$ about $G(x)$ such that $\varphi[e, A] = S$.*

(ii) *S is a slice at x.*

(iii) *$G(S)$ is an open neighborhood of $G(x)$ and there is an equivariant retraction $f\colon G(S) \to G(x)$ such that $f^{-1}(x) = S$.*

*Proof* That (i) implies (ii) is clear since $A$ can be replaced with $S$. To show that (ii) implies (iii) let $S$ be a slice and define $f\colon G(S) \to G(x)$ by commutativity of

$$\begin{array}{ccc} G \times_H S & \xrightarrow{\approx} & G(S) \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ G/H & \xrightarrow{\approx} & G(x). \end{array}$$

(Recall that $H = G_x$.) Then $f(g(s)) = g(x)$ and $f$ clearly has the desired properties. Finally, if $f$ is as in (iii), then $\varphi\colon G \times_H S \to X$, defined by $\varphi[g, s] = g(s)$, is a tube by Proposition 3.2. ∎

From part (iii) the following fact is clear.

**4.3. Corollary** *If $S$ is a slice at $x$, then $g(S)$ is a slice at $g(x)$.* ∎

The next theorem gives another characterization of a slice.

**4.4. Theorem** *Let $X$ be a $G$-space and let $x \in S \subset X$. Suppose that:*

(i) *$S$ is closed in $G(S)$.*

(ii) *$G(S)$ is an open neighborhood of $G(x)$.*

(iii) *$G_x(S) = S$.*

(iv) *$(gS) \cap S \neq \varnothing \Rightarrow g \in G_x$.*

*Then $S$ is a slice at $x$. Conversely, every slice satisfies these conditions.*

*Proof* Because of (i) and (iv) the constant map $S \to \{x\} \subset G(x)$ satisfies the conditions of I.3.3 and hence extends uniquely to an equivariant map $f\colon G(S) \to G(x)$, necessarily a retraction. If $x = f(gs) = g(f(s)) = g(x)$, then $g \in G_x$ and $gs \in S$ by (iii). Thus $f^{-1}(x) = S$ and $S$ is a slice. For the converse, suppose $S = f^{-1}(x)$ for $f\colon G(S) \to G(x)$ an equivariant retraction. Then $gs \in S$ iff $x = f(gs) = g(f(s)) = gx$ which holds iff $g \in G_x$. ∎

Recall that for an $H$-equivariant map $f\colon A \to A'$ there is an induced $G$-equivariant map

$$G \times_H f\colon G \times_H A \to G \times_H A'$$

and that this is open when $f$ is open, by 2.1.

**4.5. Theorem** *Let $X$ be a $G$-space and let $\varphi\colon G \times_H A \to X$ be a tube about $G(x)$. Let $a \in A$, put $\varphi[e, a] = y$, and let $\psi\colon H \times_K B \to A$ be a tube about $H(a)$ in $A$. Then the composition*

$$\theta\colon G \times_K B \xrightarrow{\approx} G \times_H (H \times_K B) \xrightarrow{G \times_H \psi} G \times_H A \xrightarrow{\varphi} X$$

*is a tube about $G(y)$ in $X$. (Here the first map is that of* 3.1.)

*Proof* By 2.1, $\theta$ is open and is clearly an embedding. Also $G_y = G_{[e,a]}$, since $\varphi$ is an equivalence, and $G_{[e,a]} = H_a$ which is conjugate to $K$ [by definition of a tube about $H(a)$], so that $G(y)$ has type $G/K$. ∎

**4.6. Corollary** *If $S$ is a slice at $x$ in the $G$-space $X$ and if $S'$ is a slice at a point $s \in S$ for the $G_x$-space $S$, then $S'$ is a slice at $s$ for the $G$-space $X$.* ∎

The following fact is immediate from 3.3.

**4.7. Proposition** *If $S$ is a slice at $x$ in the $G$-space $X$, then the natural map*

$$S/G_x \to X/G$$

*is a homeomorphism onto the open subspace $G(S)/G$.* ∎

## 5. EXISTENCE OF TUBES

Let $G$ be a *compact Lie group.* In the present section we shall prove the existence of a tube about each orbit of a completely regular $G$-space. The proof will require a knowledge of some elementary differential geometry, but this has been reduced to an absolute minimum. We remark that throughout a major part of this book we shall be working under hypotheses (locally smooth actions) which assure the existence of tubes of an especially nice type.

The first piece of information we need is the following lemma.

**5.1. Lemma** *Let $G$ be a compact Lie group acting orthogonally on $\mathbf{R}^n$ and let $v_0$ be a point with isotropy group $H$. Let $V \subset \mathbf{R}^n$ be the normal space to the orbit $G(v_0)$ at $v_0$, and note that this is a vector subspace of $\mathbf{R}^n$ since it passes through the origin. Then there exists a neighborhood $U$ of $eH$ in $G/H$,* *a local cross section* $\sigma: U \to G$, *and a number* $\varepsilon > 0$ *such that the restriction of* $G \times \mathbf{R}^n \to \mathbf{R}^n$ *to* $\sigma(U) \times V_\varepsilon \to \mathbf{R}^n$ *is a homeomorphism onto an open neighborhood of* $v_0$ *in* $\mathbf{R}^n$, *where* $V_\varepsilon$ *is the open* $\varepsilon$*-ball in* $V$ *about* $v_0$.

*Proof* Let $\sigma$ be any differentiable local cross section at $eH$ with $\sigma(eH) = e$. Since $G \times \mathbf{R}^n \to \mathbf{R}^n$ is differentiable, it follows from Chapter 0, Section 5 that $G/H \to G(v_0)$ is a diffeomorphism. Thus the action map $\sigma(U) \times \{v_0\} \to G(v_0) \approx G/H$, being the inverse of $\sigma$, is a diffeomorphism onto $U$ and hence its differential at $(e, v_0)$ is an isomorphism onto the tangent space of $G(v_0) \subset \mathbf{R}^n$ at $v_0$. Also, the differential of $\{e\} \times V \to V \subset \mathbf{R}^n$ is an isomorphism onto the normal space $V$ of $G(v_0)$ at $v_0$. It follows that the differential of $\sigma(U) \times V \to \mathbf{R}^n$ is an isomorphism from the tangent space of $\sigma(U) \times V$ at $(e, v_0)$ to that of $\mathbf{R}^n$ at $v_0$. By the Implicit Function Theorem it follows that $\sigma(U) \times V \to \mathbf{R}^n$ is a diffeomorphism on some neighborhood of $(e, v_0)$ to a neighborhood of $v_0$. ∎

**5.2. Corollary** *In the situation of* 5.1, *the map*

$$G \times_H V \to \mathbf{R}^n$$

*given by* $[g, v] \mapsto g(v)$, *induces a homeomorphism of* $G \times_H V_\varepsilon$ *onto the open neighborhood* $G(V_\varepsilon)$ *of* $G(v_0)$ *in* $\mathbf{R}^n$ *for* $\varepsilon$ *sufficiently small.*

*Proof* Let $U \subset G/H$ be as in 5.1 and let $K$ be the *compact* set $G - \sigma(U)H$. Then $K(v_0) \subset \mathbf{R}^n - \{v_0\}$. Now $K(v_0) = \bigcap K(C)$, where $C$ ranges over the compact neighborhoods of $v_0$ in $\mathbf{R}^n$. As is well known, any neighborhood of $K(v_0)$ must contain one of the $K(C)$, since the $K(C)$ are all compact. This implies that, for sufficiently small $C$, we have $K(C) \cap C = \varnothing$. In particular, $K(V_\varepsilon) \cap V_\varepsilon = \varnothing$ for $\varepsilon$ sufficiently small. Now suppose that $g(v) = g'(v')$ for some $v$ and $v'$ in $V_\varepsilon$. Then $g^{-1}g'(v') = v$, showing that $g^{-1}g' \notin K$. Thus $g^{-1}g' \in \sigma(U)H$, that is, $g' = g\sigma(u)h$ for some $u \in U$ and $h \in H$. Then $g\sigma(u)h(v') = g(v)$, so that $\sigma(u)(h(v')) = v$. Now $H(V_\varepsilon) = V_\varepsilon$ since $H = G_{v_0}$ acts orthogonally on $V$. Thus the equation $\sigma(u)(h(v')) = e(v)$ implies, by 5.1, that $\sigma(u) = e$ and $h(v') = v$. Thus

$$[g, v] = [g\sigma(u), h(v')] = [g\sigma(u)h, v'] = [g', v']$$

showing that the given map is one–one for $\varepsilon$ small. But $G \times_H V_\varepsilon \to G(V_\varepsilon)$ is continuous and closed since $G \times V_\varepsilon \to G(V_\varepsilon)$ is. Since $G(V_\varepsilon)$ is the saturation of the open set $\sigma(U)(V_\varepsilon)$, by 5.1, it is open in $\mathbf{R}^n$. ∎

**5.3. Corollary** *In the situation of 5.2, the map $G(V_\varepsilon) \to G(v_0)$ defined by $g(v) \mapsto g(v_0)$ is a well-defined equivariant retraction for $\varepsilon$ small.* ▮

We now prove the main result on the existence of tubes.

**5.4. Theorem** *There is a tube about any orbit of a completely regular $G$-space, where $G$ is a compact Lie group.*

*Proof* Let $X$ be a completely regular $G$-space and let $x_0 \in X$ have isotropy group $H$. By 0.5.2 there is an orthogonal representation of $G$ on $\mathbf{R}^n$ and a point $v_0$ in $\mathbf{R}^n$ with $G_{v_0} = H$. Map $\varphi$: $G(x_0) \xrightarrow{\approx} G(v_0) \subset \mathbf{R}^n$ by $g(x_0) \mapsto g(v_0)$. Then by the Tietze–Gleason Theorem I.2.3 and Chapter I, Exercise 8, there is an equivariant extension $\psi$: $X \to \mathbf{R}^n$ of $\varphi$. With $\varepsilon$ as in 5.3, put $W = \psi^{-1}(G(V_\varepsilon))$. Then $G(W) = W$ is open in $X$ and the composition

$$W \xrightarrow{\psi} G(V_\varepsilon) \to G(v_0) \xrightarrow{\varphi^{-1}} G(x_0)$$

is clearly an equivariant retraction. ▮

*Remarks* Gleason [1] first proved the existence of tubes in the important special case of free actions. The general case was first proved by Montgomery and Yang [1], under slightly more restrictive conditions, by using a selection theorem of Michael. Shortly thereafter, Mostow [1] gave essentially the present proof. The idea of using equivariant retractions is due to Palais [3, 4].

As a consequence of the existence of tubes we have the following Corollary.

**5.5. Corollary** *If $P$ is any orbit in a completely regular $G$-space, $G$ compact Lie, then there is a neighborhood of $P$ such that* type$(Q) \geq$ type$(P)$ *for any orbit $Q$ in this neighborhood. More precisely, for any given neighborhood $U$ of $e$ in $G$ and any point $x$ in $X$, there is a neighborhood $V$ of $x$ such that for any $y$ in $V$, there is a $u$ in $U$ with $u^{-1}G_y u \subset G_x$.*

*Proof* If $P = G(x)$ and if $f$: $G(S) \to G(x)$ is an equivariant retraction with $S = f^{-1}(x)$, then $f$ takes any orbit $G(y)$ in $G(S)$ to $G(x)$, whence type$(G(y)) \geq$ type$(G(x))$. If $y = us \in US$, then $G_y = uG_s u^{-1}$ and $G_s \subset G_{f(s)} = G_x$, so that $u^{-1}G_y u \subset G_x$. ▮

**5.6. Corollary** *Let $G$ be a compact Lie group and let $H$ be a closed subgroup of $G$. Then for any neighborhood $U$ of $e$ in $G$ there is a neighborhood $W \subset U$ of $e$ such that if $K$ is any subgroup of $G$ with $K \subset WH$, then there is an element $u$ of $U$ such that $u^{-1}Ku \subset H$.*

*Proof* Let $X$ be the space of all closed subsets of $G/H$, [with the Hausdorff metric $d(A, B) = \max d(a, B) + \max d(A, b)$] and let $x$ be the point $\{eH\}$ in $X$. Then $G$ acts on $X$ via left translation on $G/H$ and clearly $G_x = H$. Let $U$ be given and let $V$ be a neighborhood of $x$ satisfying the conclusion of 5.5. Clearly we may choose $W \subset U$ so small that any closed set contained in $(WH)/H \subset G/H$ is a point of $V$. Then, if $K \subset WH$, it follows that $y = KH/H$ is an element of $V$ and obviously $K \subset G_y$. Thus $u^{-1}Ku \subset u^{-1}G_yu \subset G_x = H$ for some $u$ in $U$. ∎

*Remark* Corollary 5.6 also holds for noncompact Lie $G$ and compact $H$. The proof of this more general result, which makes clever use of riemannian geometry, can be found in Montgomery and Zippin [4]. The present proof for compact $G$ was given by Mostow [1]. A similar proof for noncompact $G$ was later given by Palais [1].

Let $G$ be compact Lie and $K \subset H \subset G$ be closed subgroups. Recall from Chapter I, Section 5 that the action of $N(K)$ on $(G/H)^K$ is not necessarily transitive. However, the following corollary shows that this is nearly the case.

**5.7. Corollary** *Let $K \subset H \subset G$ be compact Lie. Then the orbit space $(G/H)^K/N(K)$ of the left translation action of $N(K)$ on $(G/H)^K$ is finite.*

*Proof* Suppose that $KgH = gH$, that is $g^{-1}Kg \subset H$. By 5.6 there is a neighborhood $W$ of $e$ in $G$ such that if

$$L \subset (W \cap H)(g^{-1}Kg) = W(g^{-1}Kg) \cap H,$$

then $h^{-1}Lh \subset g^{-1}Kg$ for some $h$ in $H$. By continuity of multiplication and compactness of $g^{-1}Kg$, we can find a neighborhood $V$ of $e$ such that

$$V^{-1}g^{-1}KgV \subset Wg^{-1}Kg.$$

Let $g' = gv \in gV$ and suppose that $Kg'H = g'H$, that is, $v^{-1}g^{-1}Kgv \subset H$. Then, for some $h$ in $H$, we have that

$$h^{-1}v^{-1}g^{-1}Kgvh \subset g^{-1}Kg$$

which means that $gvhg^{-1}$ is in $N(K)$. Thus

$$g'H = gvH \subset N(K)gH,$$

which means that $g'H$ is in the $N(K)$-orbit of $gH$ on $(G/H)^K$. Since $gVH/H$ is a neighborhood of $gH$ in $G/H$, this shows that the $N(K)$-orbits in $(G/H)^K$ are open (and closed). By compactness, they are finite in number. ∎

An important consequence of the existence of tubes is that the orbit map of a $G$-space with orbits all of the same type is a fiber bundle projection map.

**5.8. Theorem** *Suppose $X$ is a completely regular $G$-space, $G$ compact Lie, and that all orbits have type $G/H$. Then the orbit map $X \to X/G$ is the projection in a fiber bundle with fiber $G/H$ and structure group $N(H)/H$ (acting by right translation on $G/H$). Conversely every such bundle comes from such an action.*

***Proof*** The converse follows from 1.1. For the first part, we note that a tube in $X$ is of the form $G \times_H A$. But $G_{[e,a]} = H_a \subset H$ and the fact that $G_{[e,a]}$ is conjugate to $H$ imply that $H_a = H$ for all $a$ in $A$. That is, $A$ has trivial $H$-action. In this case we have $G \times_H A \approx (G/H) \times A$ (equivariantly) and, identifying $A$ with its homeomorphic image $A/H \approx (G \times_H A)/G$ in the orbit space, this gives a product representation

$$\begin{array}{ccc} \varphi_A\colon (G/H) \times A & \xrightarrow{\approx} & \pi^{-1}(A) \\ \searrow & & \swarrow_{\pi} \\ & A & \end{array}$$

If $\varphi_A$ and $\varphi_B$ are two such product representations, then

$$\begin{array}{ccc} \varphi_B^{-1}\varphi_A\colon (G/H) \times (A \cap B) & \longrightarrow & (G/H) \times (A \cap B) \\ \searrow & & \swarrow \\ & A \cap B & \end{array}$$

gives a map $\theta\colon A \cap B \to \mathbf{Homeo}^G(G/H)$ [where $\varphi_B^{-1}\varphi_A(gH, x) = (\theta(x)(gH), x)$] into the group of self-equivalences of $G/H$ in the compact-open topology. By I.4.3, $\mathbf{Homeo}^G(G/H) \approx N(H)/H$. ∎

Borel [5] noticed that a $G$-space $X$ all of whose orbits have type $G/H$ also fibers in another manner with structure group $N(H)/H$. This fibration is given by the following theorem.

**5.9. Theorem** *Let $G$ be compact and let $X$ be a $G$-space all of whose orbits have type $G/H$. Then the map*

$$G \times_N X^H \to X$$

*[taking $[g, x] \mapsto g(x)$] is a homeomorphism, where $N = N(H)$ is the normalizer of $H$ in $G$. Thus $X$ is equivalent to the bundle over $G/N$ with fiber $X^H$ associated with the principal bundle $G \to G/N$ when $G$ is Lie (so that $G \to G/N$ is a bundle).*

*Proof* The map is continuous, onto, and closed (since $X^H$ is closed in $X$ and $G \times X \to X$ is closed). Thus we need only show that it is one–one. For this, suppose that $g(x) = g'(x')$, where $x$ and $x'$ are in $X^H$ (so that $G_x = H = G_{x'}$). Let $n = g^{-1}g'$ so that $n(x') = x$. Then

$$H = G_x = G_{n(x')} = nG_{x'}n^{-1} = nHn^{-1}$$

so that $n \in N$. Thus $[g, x] = [gn, n^{-1}x] = [g', x']$. ∎

**5.10. Corollary** *With the hypotheses of 5.9, the inclusion $X^H \subset X$ induces a homeomorphism*

$$X^H/N \xrightarrow{\approx} X/G.$$

*Proof* By 3.3, $X^H/N \to (G \times_N X^H)/G$ is a homeomorphism and also $(G \times_N X^H)/G \approx X/G$ by 5.9. ∎

**5.11. Corollary** *With the hypotheses of 5.9, let $K = N/H$. Then the map*

$$(G/H) \times_K X^H \to X,$$

*defined by $[gH, x] \mapsto g(x)$, is an equivalence of $G$-spaces.*

*Proof* Since $H$ acts trivially on $X^H$ we have

$$G \times_N X^H \approx \frac{(G \times X^H)/H}{N/H} \approx \frac{(G/H) \times X^H}{N/H}. \quad ∎$$

**5.12. Corollary** *With the hypotheses of 5.11, let $Y$ be any $G$-space. Then restriction to $X^H$ gives a one–one correspondence between $G$-equivariant maps $X \to Y$ and $K$-equivariant maps $X^H \to Y^H$.*

*Proof* If $f$: $X^H \to Y^H$ is $K$-equivariant, let $f'$: $X \to Y$ be defined as the composite

$$X \approx (G/H) \times_K X^H \to (G/H) \times_K Y^H \to Y$$

[the latter map being $[gH, y] \mapsto g(y)$]. Then for $x \in X^H$, $f'(gx) = gf(x)$, so that $f' \mid X^H = f$. ∎

Note that, in the above situation, $K = N(H)/H$ acts *freely* on $X^H$ and the orbit space is $X^H/K \approx X/G$ by 5.10. Thus, by 5.8 applied to this action, we have the *principal $K$-bundle* $\pi_X$: $X^H \to X/G$ (the restriction to $X^H$ of the orbit map $X \to X/G$). Combining 5.12 and 2.6 we obtain the following theorem.

**5.13. Theorem** *Let $X$ be a completely regular $G$-space with all orbits of type $G/H$, and with $G$ compact Lie. Let $K = N(H)/H$. If $Y$ is any other $G$-space, then there is a natural one–one correspondence between the $G$-equivariant maps $f$: $X \to Y$ and cross sections $\bar{f}$ of the $Y^H$-bundle*

$$Y^H \times_K X^H \to X/G$$

*associated with the principal $K$-bundle $\pi_X$: $X^H \to X/G$. The correspondence is characterized by the equation*

$$\bar{f}(\pi_X(x)) = [f(x), x] \qquad \textit{for} \quad x \in X^H. \quad ∎$$

The reader should note that 5.13 is also an easy direct corollary of 2.8.

## 6. PATH LIFTING

In this section we shall use slices to prove a result of Montgomery and Yang [1] that paths in $X/G$ can be lifted to $X$ when $X$ is any $G$-space, $G$ compact Lie. First we consider the case in which $X/G$ is an arc.

**6.1. Lemma** *If $X$ is a $G$-space, $G$ compact Lie, and if $X/G$ is homeomorphic to* $\mathbf{I} = [0, 1]$, *then there is a global cross section for the orbit map $\pi$: $X \to X/G$.*

*Proof* Since $X$ is compact, by I.3.1, it is completely regular and hence has a slice at each point. First we note that it suffices to prove that $\pi$ has a local cross section near each point of $X/G$. This is true since, if $\sigma_i$: $[i/n, (i+1)/n]$

$\to X$ is a cross section for $i = 0, 1, \ldots, n-1$, and if $g_i \in G$ is such that $g_0 = e$ and $g_i\sigma_i(i/n) = \sigma_{i-1}(i/n)$ for $1 \leq i \leq n-1$, then the map $\sigma\colon \mathbf{I} \to X$, defined by $\sigma(t) = g_0 g_1 \cdots g_i \sigma_i(t)$ for $i/n \leq t \leq (i+1)/n$, is a global cross section. Similarly, if $J \subset \mathbf{I}$ is an open subset, and if local cross sections exist near all points of $J$, then a cross section over $J$ exists.

Now by double induction over the dimension and number of components of $G$, we can assume that the lemma is true for actions of any proper subgroup of $G$.

Consider $F = X^G$ and its image $F^* \subset \mathbf{I} = X/G$, a *closed* subset. Then $G$ acts on $X - F$ with no stationary points and with orbit space $\mathbf{I} - F^*$. If $y \in X - F$, let $S$ be a slice at $y$. Since $G_y \neq G$ and since $S/G_y \approx G(S)/G$ (by 4.7) is an arc near $y^*$, the inductive assumption, applied to the $G_y$-action on $S$, yields a local cross section at $y$ for the orbit map $S \to S/G_y$ and hence for $X - F \to \mathbf{I} - F^*$. As shown above, the existence of these local cross sections implies the existence of a global cross section, say $C' \subset X - F$, of $X - F \to \mathbf{I} - F^*$. Then $C'$ is closed in $X - F$ and thus $C = C' \cup F$ is closed in $X$. Since $C$ clearly touches each orbit of $X$ exactly once, it is a cross section by I.3.2. ▮

**6.2. Theorem** *Let $X$ be a $G$-space, $G$ compact Lie, and let $f\colon \mathbf{I} \to X/G$ be any path. Then there exists a lifting $f'\colon \mathbf{I} \to X$, $\pi_X \circ f' = f$.*

*Proof* Consider the pull-back $f^*X$ (see Chapter I, Section 6) and recall that $f^*X/G \approx \mathbf{I}$. Let $\sigma\colon \mathbf{I} \to f^*X$ be a cross section. Then, from the diagram

$$\begin{array}{ccc} f^*X & \xrightarrow{\ f_1\ } & X \\ {\scriptstyle\sigma}\uparrow\downarrow & & \downarrow{\scriptstyle\pi_X} \\ \mathbf{I} & \longrightarrow & X/G \end{array}$$

it is clear that $f' = f_1 \circ \sigma$ is a lifting of $f$.

**6.3. Corollary** *If $X$ is an arcwise connected $G$-space, $G$ compact Lie, and if there is an orbit which is connected* (e.g., $G$ *connected or* $X^G \neq \varnothing$), *then the fundamental group of $X$ maps onto that of $X/G$. Thus if $X$ is simply connected, then so is $X/G$.*

*Proof* Suppose that $G(x)$ is connected and put $x^* = \pi(x)$. If $f\colon \mathbf{I} \to X/G$ is a loop at $x^*$, that is $f(0) = f(1) = x^*$, let $f'\colon \mathbf{I} \to X$ be a lifting of $f$. By composing $f'$ by an element of $G$ we may suppose that $f'(0) = x$. Let

$k$: $\mathbf{I} \to G(x)$ be a path from $f'(1)$ to $x$. Then the path composite $f' \circ k$ ($f'$ then $k$) is a loop at $x$ projecting to $f \circ 1_{x^*}$ which is homotopic to $f$. ∎

**6.4. Corollary** *If $X$ is an arcwise connected, locally simply connected, completely regular $G$-space, $G$ compact Lie, then $X/G$ is locally simply connected.*

***Proof*** Let $x \in X$, let $V^*$ be a neighborhood of $x^*$ in $X/G$ and put $V = \pi^{-1}(V^*)$. Let $U \subset V$ be a neighborhood of $x$ such that $\pi_1(U, x) \to \pi_1(V, x)$ is trivial. Let $S$ be a slice at $x$ and assume that $S \subset U$. Since $x$ is fixed for $G_x$ on $S$, and hence is a connected orbit of $G_x$ in $S$, we have that $\pi_1(S, x) \to \pi_1(S/G_x, x^*) = \pi_1(G(S)/G, x^*)$ is onto. By the diagram

$$\begin{array}{ccc} \pi_1(S, x) & \xrightarrow{0} & \pi_1(V, x) \\ \downarrow & & \downarrow \\ \pi_1(G(S)/G, x^*) & \longrightarrow & \pi_1(V^*, x^*) \end{array}$$

the bottom map is trivial. ∎

**6.5. Corollary** *Let $X$ be an arcwise connected $G$-space, $G$ compact Lie. Then $H_1(X; \mathbf{Q}) \to H_1(X/G; \mathbf{Q})$ is onto (rational singular homology).*

***Proof*** By the Universal Coefficient Theorem, $H_1(X; \mathbf{Q}) \approx H_1(X; \mathbf{Z}) \otimes \mathbf{Q}$. Let $\alpha \in H_1(X/G; \mathbf{Z})$. By the Hurewicz Theorem, $\alpha$ can be represented by a *loop* $f$: $\mathbf{I} \to X/G$. Then we can lift $f$ to a *path* $f'$: $\mathbf{I} \to X$. If $f'(0)$ and $f'(1)$ are not in the same component of their orbit, then let $f'(1) = gf'(0)$ for some $g$ in $G$. Suppose that $g^n$ is in the identity component of $G$. Then $g^{n-1}f'(1) = g^n f'(0)$ and $f'(0)$ are in the same component of $G(f'(0))$ and there is a path $k$ in this orbit from $g^{n-1}f'(1)$ to $f'(0)$. Then the path composite $f' \circ gf' \circ g^2 f' \circ \cdots \circ g^{n-1}f' \circ k$ is a *loop* in $X$ projecting to $f \circ f \circ \cdots \circ f \circ 1$ which is homotopic to $f^n$. Thus $n\alpha$ is the image of a class in $H_1(X; \mathbf{Z})$, so that $\alpha \otimes q = n\alpha \otimes (q/n)$ is in the image of $H_1(X; \mathbf{Z}) \otimes \mathbf{Q} \to H_1(X/G; \mathbf{Z}) \otimes \mathbf{Q}$, whence this homomorphism is onto. ∎

## 7. THE COVERING HOMOTOPY THEOREM

This section is devoted to the proof of a type of covering homotopy theorem for orbit maps $X \to X/G$, due to Palais [3]. First we prove the following result about $G$-spaces whose orbit structure is a product of an orbit structure with the unit interval $\mathbf{I}$. Recall the notation and terminology of Chapter I, Exercise 5.

**7.1. Theorem** *Let $Y$ be a topological space such that every open subspace is paracompact, and let $G$ be a compact Lie group. Suppose that $W$ is a $G$-space with orbit space $W/G = Y \times \mathbf{I}$ such that the orbit structure is a composition*

$$Y \times \mathbf{I} \to Y \xrightarrow{\tau} \mathscr{C}_G$$

*with the projection on $Y$ (that is, the orbit types are constant on each $\{y\} \times \mathbf{I}$). Let $\pi\colon W \to Y \times \mathbf{I}$ be the orbit map. Then there is a $G$-space $X$ with $X/G \approx Y$ and orbit structure $X/G \approx Y \xrightarrow{\tau} \mathscr{C}_G$, and an equivalence $\varphi\colon W \xrightarrow{\approx} X \times \mathbf{I}$ of $G$-spaces (where $G$ acts trivially on $\mathbf{I}$) such that the diagram*

$$\begin{array}{ccc} W & \xrightarrow[\approx]{\varphi} & X \times \mathbf{I} \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & Y \times \mathbf{I} & \end{array}$$

*commutes.*

*Moreover, $X$ can be taken to be $\pi^{-1}(Y \times \{0\})$ and $\varphi \mid \pi^{-1}(Y \times \{0\})\colon X \to X \times \mathbf{I}$, the inclusion $x \mapsto (x, 0)$.*

*Proof* First we note that the last paragraph follows from the rest, since $\varphi$ induces an equivalence of $\pi^{-1}(Y \times \{0\})$ onto $X \times \{0\}$. We shall identify $Y$ with $Y \times \{0\}$ and $X = \pi^{-1}(Y)$ with $X \times \{0\}$.

We prove the theorem by double induction over the dimension of $G$ and the number of components of $G$. Thus we may assume that it holds for actions of all proper subgroups of $G$. We let $F$ denote the homeomorphic image of $X^G$ in $Y$. Thus $W^G = \pi^{-1}(F \times \mathbf{I})$.

The proof will be given in four parts which we shall first outline.

In part A we shall show that the inductive assumption implies that the result holds for the action above a small product neighborhood of $(y, t)$ when $y \in Y - F$.

In part B we conclude that the result holds for the action over $U \times \mathbf{I}$, where $U$ is a small neighborhood of $y \in Y - F$.

In part C we show that the result holds for the action over $(Y - F) \times \mathbf{I}$.

In part D we finally prove the theorem for the given action.

*Part* A: Let $y \in Y - F$ and $t \in \mathbf{I}$. Let $w \in W$ project to $(y, t) \in Y \times \mathbf{I}$ and let $S$ be a slice at $w$. We may identify $S/G_w$ with $G(S)/G \subset Y \times \mathbf{I}$ by 4.7 and we may take $S$ so that $S/G_w = U \times [a, b]$, where $U$ is a neighborhood of $y$ on $Y$ and $[a, b]$ is a closed interval about $t$ in $\mathbf{I}$. Since $G_w \neq G$, the inductive assumption implies that

$$S \approx T \times [a, b]$$

for some $G_w$-space $T$ with $T/G_w \approx U$ and $S \to S/G_w$ corresponding to the obvious map $T \times [a, b] \to U \times [a, b]$. Then we have the equivalence

$$G(S) \approx G \times_{G_w} S \approx G \times_{G_w} (T \times [a, b]) \approx (G \times_{G_w} T) \times [a, b]$$

of $G$-spaces, which obviously commutes with the projections to $U \times [a, b]$.

*Part* B: Let $y \in Y - F$. Then, by an obvious compactness argument on $\mathbf{I}$, we can cover $\{y\} \times \mathbf{I}$ by $U \times [i/n, (i+1)/n], 0 \leq i \leq n-1$, with $U$ a neighborhood of $y$ in $Y$, such that there are equivalences

$$\pi^{-1}(U \times [i/n, (i+1)/n]) \to \pi^{-1}(U \times \{i/n\}) \times [i/n, (i+1)/n]$$

of $G$-spaces. This shows that the $G$-spaces $\pi^{-1}(U \times \{i/n\})$ and $\pi^{-1}(U \times \{(i+1)/n\})$ are equivalent for all $i$ so that we may replace the above equivalences by

$$\varphi_i\colon \pi^{-1}(U \times [i/n, (i+1)/n]) \xrightarrow{\approx} \pi^{-1}(U) \times [i/n, (i+1)/n]$$

(identifying $U$ with $U \times \{0\}$). Now $\varphi_0\varphi_1^{-1}$ is defined on $\pi^{-1}(U) \times \{1/n\}$ to itself. Let $\psi_1$ on $\pi^{-1}(U) \times [1/n, 2/n]$ to itself be the extension of this which is essentially the product of it with the identity on $[1/n, 2/n]$. Then we may replace $\varphi_1$ by $\varphi_1' = \psi_1 \circ \varphi_1$ and note that $\varphi_0$ agrees with $\varphi_1'$ on their common domain. Continuing this way, the $\varphi_i$ can be modified so that they agree with one another on their common domains and together give an equivalence

$$\varphi_U\colon \pi^{-1}(U \times \mathbf{I}) \xrightarrow{\approx} \pi^{-1}(U) \times \mathbf{I}.$$

Clearly one may assume that $\varphi_U(x) = (x, 0)$ for $x$ in $\pi^{-1}(U \times \{0\})$.

*Part* C: Since $Y - F$ is paracompact, by assumption, we may cover $Y - F$ by a *locally finite* collection $\{V_\alpha \mid \alpha \in A\}$ of open sets, for each of which there exists an equivalence

$$\varphi_{V_\alpha}\colon \pi^{-1}(V_\alpha \times \mathbf{I}) \xrightarrow{\approx} \pi^{-1}(V_\alpha) \times \mathbf{I}$$

normalized by $\varphi_{V_\alpha}(x) = (x, 0)$ for $x$ in $\pi^{-1}(V_\alpha \times \{0\})$ and commuting, as always, with the orbit maps onto $V_\alpha \times \mathbf{I}$.

Well-order the index set $A$ and put $U_\alpha = \bigcup_{\beta<\alpha} V_\beta$. We shall inductively define equivalences $\varphi_{U_\alpha}$ with the property that for $\beta < \alpha$, $\varphi_{U_\beta} = \varphi_{U_\alpha}$ *over* $(U_\beta - \bigcup_{\beta\leq\gamma<\alpha} V_\gamma) \times \mathbf{I}$.

Assume that $\varphi_{U_\beta}$ has been defined for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, then $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$. Now if $y \in U_\alpha$, there is a neighborhood $N$ of $y$ in $Y$ touching only a finite number of the $V_\gamma$ for $\gamma < \alpha$. If $\beta$, $\beta' < \alpha$ are both *larger* than any of *these* $\gamma$, then the assumption on the $\varphi_{U_\beta}$ implies that $\varphi_{U_\beta} = \varphi_{U_{\beta'}}$ on $\pi^{-1}(N \times \mathbf{I})$. Hence we may put $\varphi_{U_\alpha} = \lim_{\beta\to\alpha} \varphi_{U_\beta}$ in the obvious sense, and obtain an equivalence over $U_\alpha \times \mathbf{I}$.

Now suppose that $\alpha$ is the successor of $\alpha'$. Put $U_{\alpha'} = U$ and $V_{\alpha'} = V$, so that $U_\alpha = U \cup V$. Then it clearly suffices to define $\varphi_{U\cup V}$ coinciding with $\varphi_U$ over $(U - V) \times \mathbf{I}$. Let

$$\psi = \varphi_U \circ \varphi_V^{-1}: \pi^{-1}(U \cap V) \times \mathbf{I} \xrightarrow{\approx} \pi^{-1}(U \cap V) \times \mathbf{I},$$

and let $\psi_1: \pi^{-1}(U \cap V) \times \mathbf{I} \to \pi^{-1}(U \cap V)$ be defined by

$$\psi(x, t) = (\psi_1(x, t), t).$$

Note that

$$\psi_1(x, 0) = x.$$

Now, by assumption, $U \cup V$ is normal and thus there is a map

$$f: U \cup V \to \mathbf{I}$$

such that $f = 1$ on a *neighborhood* of $U - V$ and $f = 0$ on a neighborhood of $V - U$.

Put $\psi'(x, t) = (\psi_1(x, f(\pi x)t), t)$ and note that

$$\psi': \pi^{-1}(U \cap V) \times \mathbf{I} \to \pi^{-1}(U \cap V) \times \mathbf{I}$$

is equivariant, continuous, and covers the identity on $(U \cap V) \times \mathbf{I}$. Thus $\psi'$ is an equivalence by Chapter I, Exercise 11. (In fact, the inverse of $\psi'$ is easily seen to arise from $\psi^{-1}$ by the same construction.)

Now consider

$$\psi' \circ \varphi_V: \pi^{-1}((U \cap V) \times \mathbf{I}) \xrightarrow{\approx} \pi^{-1}(U \cap V) \times \mathbf{I}.$$

On the neighborhood of $U - V$ where $f = 1$ we see that $\psi' \circ \varphi_V = \psi \circ \varphi_V = \varphi_U$. Similarly, on the neighborhood of $V - U$, where $f = 0$, we see that $\psi' \circ \varphi_V = 1 \circ \varphi_V = \varphi_V$. Thus we can define

$$\varphi_{U\cup V}(x) = \begin{cases} \varphi_U(x) & \text{if } \pi(x) \in (U - V) \times \mathbf{I}, \\ \psi' \circ \varphi_V(x) & \text{if } \pi(x) \in (U \cap V) \times \mathbf{I}, \\ \varphi_V(x) & \text{if } \pi(x) \in (V - U) \times \mathbf{I}. \end{cases}$$

By exhaustion of ordinals we eventually obtain an equivalence

$$\varphi_{Y-F}\colon \pi^{-1}((Y-F)\times \mathbf{I}) \xrightarrow{\approx} \pi^{-1}(Y-F)\times \mathbf{I}$$

of $G$-spaces commuting with the projections to $(Y-F)\times \mathbf{I}$.

*Part* D: Since $\pi^{-1}(F\times \mathbf{I})$ is precisely the set of fixed points of $G$ on $W$, it maps homeomorphically onto $F\times \mathbf{I}$ via $\pi$. Similarly $\pi^{-1}(F)\xrightarrow{\approx} F$. Thus define $\varphi_F$ by commutativity of

$$\begin{array}{ccc} \pi^{-1}(F\times \mathbf{I}) & \xrightarrow{\varphi_F} & \pi^{-1}(F)\times \mathbf{I} \\ & {\scriptstyle\approx}\searrow \quad \swarrow{\scriptstyle\approx} & \\ & F\times \mathbf{I} & \end{array}$$

and put $\varphi_Y = \varphi_{Y-F}\cup \varphi_F$. That this is bicontinuous, and hence an equivalence, follows from the next lemma, which will thus complete the proof of Theorem 7.1. ∎

**7.2. Lemma** *Let $X$ and $Y$ be $G$-spaces, $G$ compact, and assume that we are given an equivariant map $\varphi\colon X - X^G \to Y$ and a map $\psi\colon X/G\to Y/G$ such that $\psi(X^G/G)\subset Y^G/G$ and*

$$\begin{array}{ccc} X - X^G & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{\psi} & Y/G \end{array}$$

*commutes. Extend $\varphi$ to $X$ by putting $\varphi(x) = \pi_Y^{-1}\psi\pi_X(x)$ for $x$ in $X^G$. Then this extension is continuous.*

***Proof*** Let $x\in X^G$ and put $y=\varphi(x)\in Y^G$. Then a basis for the neighborhoods of $y$ is given by the $\pi_Y^{-1}(N)$, where $N$ ranges over the neighborhoods of $\pi_Y(y)$, by I.3.1. But $\varphi^{-1}(\pi_Y^{-1}(N)) = \pi_X^{-1}\psi^{-1}(N)$ is a neighborhood of $x$ by the continuity of $\psi$, which shows that $\varphi$ is continuous at $x$. ∎

Let $X$ and $Y$ be $G$-spaces and recall that a map $X/G\to Y/G$ is said to "preserve the orbit structure" if it commutes with the orbit structure maps $\tau_{X/G}\colon X/G\to \mathscr{C}_G$ and $\tau_{Y/G}\colon Y/G\to\mathscr{C}_G$ (see Chapter I, Exercise 5). An equivariant homotopy is an equivariant map $X\times \mathbf{I}\to Y$, where $X\times\mathbf{I}$ has the $G$-action $(g,(x,t))\mapsto (gx,t)$. Note that $(X\times \mathbf{I})/G\approx (X/G)\times \mathbf{I}$.

Thus a homotopy $X/G \times \mathbf{I} \to Y/G$ preserves the orbit structure if the diagram

$$\begin{array}{ccc} X/G \times \mathbf{I} & \longrightarrow & Y/G \\ \big\downarrow{\scriptstyle \text{proj}} & & \big\downarrow{\scriptstyle \tau_{Y/G}} \\ X/G & \xrightarrow{\tau_{X/G}} & \mathscr{C}_G \end{array}$$

commutes. The following is the Covering Homotopy Theorem of Palais [3]:

**7.3. Theorem** *Let $G$ be a compact Lie group and let $X$ and $Y$ be $G$-spaces. Assume that every open subspace of $X/G$ is paracompact. Let $f$: $X \to Y$ be equivariant and let $f'$: $X/G \to Y/G$ be the induced map. Let $F'$: $X/G \times \mathbf{I} \to Y/G$ be a homotopy which preserves the orbit structure and starts at $f'$ [that is, $F'(x^*, 0) = f'(x^*)$]. (In particular, $f'$ must preserve orbit structure.) Then there exists an equivariant homotopy $F$: $X \times \mathbf{I} \to Y$ covering $F'$ and starting at $f$. Moreover, any two such liftings of $F'$ differ by composition with a self-equivalence of $X \times \mathbf{I}$ covering the identity on $X/G \times \mathbf{I}$ and equal to the identity on $X \times \{0\}$.*

*Proof* Consider the $G$-space $f'^*Y$, the pull-back of $Y$ by $f'$. By the universal property of pull-backs, there is a unique equivariant map $\varphi$: $X \to f'^*Y$ such that the diagram

$$\begin{array}{ccccc} X & & \xrightarrow{f} & & Y \\ \big\downarrow & \searrow^{\varphi} & & \nearrow & \big\downarrow \\ & & f'^*Y & & \\ \big\downarrow & \swarrow & & & \big\downarrow \\ X/G & & \xrightarrow[f']{} & & Y/G \end{array}$$

commutes. [Precisely, $f'^*Y = \{(x^*, y) \in X/G \times Y \mid f'(x^*) = y^*\}$, where $y^*$ is the orbit of $y$, and $\varphi$ is given by $\varphi(x) = (x^*, f(x))$.] Now $\varphi$ preserves the orbit structure, by Chapter I, Exercise 5, and covers a homeomorphism $X/G \to (f'^*Y)/G$. Hence it is one–one and onto. It is also open by Chapter I, Exercise 10. Thus it is an equivalence. That is,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow & & \big\downarrow \\ X/G & \xrightarrow{f'} & Y/G \end{array}$$

is a pull-back diagram.

Consider the pull-back $W = F'^*Y$. Then $W/G \approx X/G \times \mathbf{I}$ and the inverse image in $W$ of $X/G \times \{0\}$ is the pull-back $f'^*Y \approx X$. By 7.1 we have that $W \approx X \times \mathbf{I}$ and thus have the diagram

$$\begin{array}{ccc} X \times \mathbf{I} & \xrightarrow{F} & Y \\ \big\downarrow & & \big\downarrow \\ X/G \times \mathbf{I} & \xrightarrow{F'} & Y/G \end{array}$$

where $F(x, 0) = f(x)$ by construction.

Any other lifting $F_1\colon X \times \mathbf{I} \to Y$ of $F'$ factors through the pull-back $F'^*Y \approx X \times \mathbf{I}$; that is, $F_1$ is the composition

$$X \times \mathbf{I} \xrightarrow{\psi} X \times \mathbf{I} \xrightarrow{F} Y.$$

Since $\psi$ covers the identity on $X/G \times \mathbf{I}$, it is an equivalence by Chapter I, Exercise 11. ▮

**7.4. Theorem** *Let $G$ be a compact Lie group and let $Y$ be a $G$-space. Let $B$ be a space, every open subspace of which is paracompact, and with a given orbit structure $B \to \mathscr{C}_G$. If $f_0$ and $f_1$ are homotopic maps of $B$ into $Y/G$, by a homotopy which preserves the orbit structure, then the pull-backs $f_0^*X$ and $f_1^*X$ are equivalent by an equivalence which preserves the orbit map onto $B$.*

*Proof* This is an immediate consequence of 7.1 applied to the pull-back $F^*X$, where $F\colon B \times \mathbf{I} \to X/G$ is the given homotopy. (It also follows easily from 7.3.) ▮

**Conjecture** Suppose that $W$ is a compact $G$-space, $G$ compact Lie. Also suppose that $W/G$ has the form of a mapping cylinder with orbit types constant along generators of the cylinder less the base. Then we conjecture that $W$ is equivalent to a mapping cylinder of an equivariant map inducing the given mapping cylinder structure on $W/G$.

## 8. CONICAL ORBIT STRUCTURES

We shall now apply the Covering Homotopy Theorem to investigate $G$-spaces whose orbit structure is conical. We define the *open cone* $C^oB$ on a space $B$ to be the quotient space $C^oB = (B \times \mathbf{R}^+)/(B \times \{0\})$ and will use $b_0$ to denote the vertex of $C^oB$ (the point corresponding to $B \times \{0\}$). Here $\mathbf{R}^+$ denotes the nonnegative reals.

**8.1. Definition** Let $\tau_Y\colon Y \to \mathscr{E}_G$ and $\tau_B\colon B \to \mathscr{E}_G$ be orbit structures and let $y_0 \in Y$. Then $(Y, \tau_Y)$ is said to be conical with base $(B, \tau_B)$ and vertex $y_0$ if there is a *homeomorphism* $h\colon C^oB \xrightarrow{\approx} Y$ with $h(b_0) = y_0$, such that the diagram

$$\begin{array}{ccc} C^oB - \{b_0\} & \xrightarrow{h} & Y - \{y_0\} \\ \downarrow & & \downarrow{\scriptstyle \tau_Y} \\ B & \xrightarrow{\tau_B} & \mathscr{E}_G \end{array}$$

commutes.

Of course, this just means that $Y$ has the structure of an open cone with the orbit types constant along rays (less the vertex). Since $\tau_Y$ is an orbit structure, its continuity implies that the orbit type of $y_0$ is less than or equal to the orbit type of any other point in $Y$.

As an obvious example, we remark that the orbit structure of any orthogonal action on euclidean space is conical.

**8.2. Theorem** *Suppose that $(Y, \tau_Y)$ and $(W, \tau_W)$ are conical orbit structures with vertices $y_0$ and $w_0$ and paracompact bases $B$ and $D$, respectively. Let $f\colon W \to Y$ be an orbit structure preserving homeomorphism onto an open neighborhood $U$ of $y_0$ and taking $w_0$ to $y_0$. Then there is an orbit structure preserving homeomorphism of $W$ onto $Y$ coinciding with $f$ on some neighborhood of $w_0$.*

*Proof* We may take $Y$ to be $C^oB$ and $W$ to be $C^oD$. We regard $B \times (0, \infty)$ as the subspace $C^oB - \{b_0\}$ and shall use the notation $B \times [\star, t]$ for the image of $B \times [0, t]$ in $C^oB$.

First let us assume that we have the following conditions holding:

(i) $B \times [\star, 8] \subset \operatorname{Image}(f)$;

(ii) $f(D \times [\star, 2]) \subset B \times [\star, 8)$;

(iii) $B \times [\star, 5] \subset f(D \times [\star, 2))$;

(iv) $f(D \times [\star, 1]) \subset B \times [\star, 2)$;

(v) $B \times [\star, 1] \subset f(D \times [\star, 1))$;

(see Figure II-1). Since $f$ is a homeomorphism onto an open set containing $B \times [\star, 8]$ we see that $f(D \times \{1\}) \subset B \times (1, 2)$ and $f(D \times \{2\}) \subset B \times (5, 8)$ (since, for example, $f(D \times \{1\})$ consists exactly of the boundary points of $f(D \times [\star, 1])$ in $C^oB$).

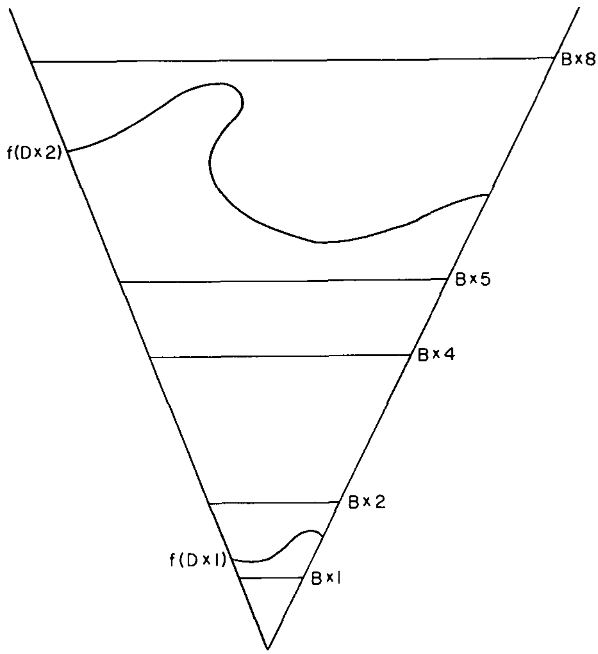



FIGURE II-1

We shall use $sA$ to denote the image of $A \subset C^oB$ under $(b, t) \mapsto (b, st)$. Consider Figure II-2. Let $P = f(D \times [\star, 2]) - 2f(D \times [\star, 1))$ and $Q = 2f(D \times [\star, 1]) - f(D \times [\star, 1))$. Define $\varphi$: $\mathbf{R}^+ \to \mathbf{R}^+$ by

$$\varphi(t) = \begin{cases} t/2 & \text{for } 0 \leq t \leq 4, \\ 3t - 10 & \text{for } 4 \leq t \leq 5, \\ t & \text{for } 5 \leq t. \end{cases}$$

Then $(b, t) \mapsto (b, \varphi(t))$ induces a homeomorphism

$$P \xrightarrow{\approx} Q \cup P = f(D \times [1, 2]) \approx D \times [1, 2].$$

Now, since $f$ is a homeomorphism, $P \cap Q$ has a product neighborhood in $Q$ of the form $D \times \mathbf{I}$. (These homeomorphisms, and those to follow, preserve "ends" and orbit structures.) Hence

$$Q \approx Q \cup (D \times \mathbf{I}) \approx Q \cup P \approx D \times [1, 2].$$

But

$$\begin{aligned} C^oB &= f(D \times [\star, 1]) \cup Q \cup 2Q \cup 4Q \cup \ldots \\ &\approx (D \times [\star, 1]) \cup (D \times [1, 2]) \cup (D \times [2, 4]) \cup (D \times [4, 8]) \cup \cdots \\ &= C^oD. \end{aligned}$$

Since this homeomorphism $C^oD \to C^oB$ preserves orbit structure, by construction, and extends $f|\ (D \times [\star, 1])$ we are done.

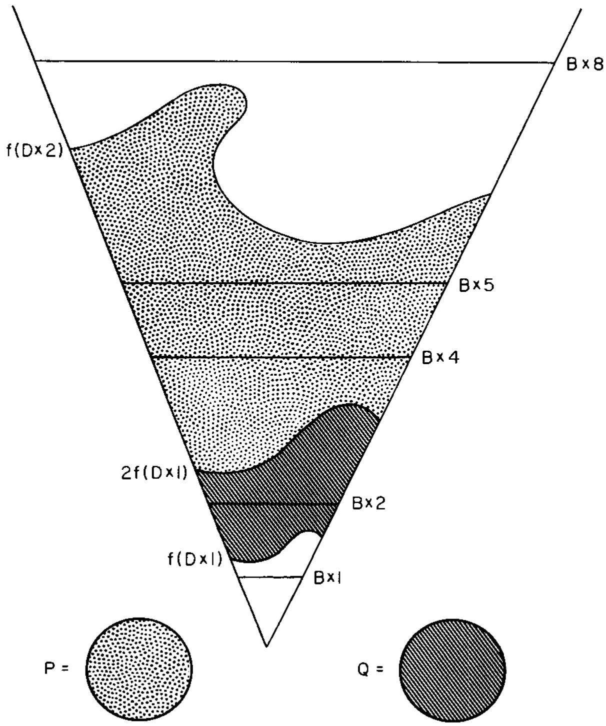



FIGURE II-2

Now we must justify conditions (i)–(v). This will be done by reparametrizing the cone structure of $Y$ and $W$.

Consider the function defined by

$$\bar{a}_8(b) = \sup\{t \mid \{b\} \times [\star, t] \subset \text{Image}(f)\}$$

from $B$ to the extended reals. Since $\bar{\alpha}_8$ is lower semicontinuous there exists a *continuous* $\alpha_8$: $B \to \mathbf{R}$ with $0 < \alpha_8(b) < \bar{\alpha}_8(b)$ for all $b$ (see Dugundji [1, p. 170]). Let

$$B_{\alpha_8} = \{(b, t) \in C^{\circ}B \mid t \leq \alpha_8(b)\}$$

and

$$B^{\circ}_{\alpha_8} = \text{int } B_{\alpha_8}.$$

Then

$$B_{\alpha_8} \subset \text{Image}(f).$$

By using the inverse of $f$ we may similarly define a continuous positive function $\beta_2$: $D \to \mathbf{R}$ with

$$\beta_2(d) < \sup\{t \mid f(\{d\} \times [\star, t]) \subset B^{\circ}_{\alpha_8}\}$$

and define

$$D_{\beta_2} = \{(d, t) \in C^{\circ}D \mid t \leq \beta_2(d)\}$$

and

$$D^{\circ}_{\beta_2} = \text{int } D_{\beta_2}.$$

Thus

$$f(D_{\beta_2}) \subset B^{\circ}_{\alpha_8}.$$

Similarly, we can find $0 < \alpha_2 < \alpha_4 < \alpha_5 < \alpha_8$ with

$$B_{\alpha_5} \subset f(D^{\circ}_{\beta_2})$$

and $0 < \beta_1 < \beta_2$ with

$$f(D_{\beta_1}) \subset B^{\circ}_{\alpha_2}$$

and $0 < \alpha_1 < \alpha_2$ with

$$B_{\alpha_1} \subset f(D^{\circ}_{\beta_1}).$$

It is clear that one may reparametrize $Y$ and $W$ such that the $\alpha_i$ and $\beta_j$ are "level" and, in fact, such that in the new parameters we have $B_{\alpha_i} = B \times [\star, i]$ and $D_{\beta_j} = D \times [\star, j]$, so that conditions (i)–(v) are thereby satisfied. ∎

Note that if $X$ is a $G$-space and $A \subset X$ is an invariant closed set, then $X/A$ is a $G$-space with $\{A\}$ stationary.

The following is the main result of this section. ("Completely paracompact" means that every open subspace is paracompact.)

**8.3. Theorem** *Let $X$ be a $G$-space, with $G$ compact Lie, and let $A \subset X$ be a closed invariant subspace. Suppose that the orbit structure on $(X/A)/G$ is conical with vertex $\{A\}^*$ and paracompact base. Suppose that $U \subset X$ is an invariant open neighborhood of $A$ and that the orbit structure on $(U/A)/G$ is also conical with vertex $\{A\}^*$ and completely paracompact base (unrelated to the cone structure of $(X/A)/G$). Then $U$ and $X$ are equivalent as $G$-spaces via an equivalence which is the identity in some neighborhood of $A$.*

*Proof* By 8.2 there is a homeomorphism $f\colon (U/A)/G \xrightarrow{\approx} (X/A)/G$ preserving orbit structure and which is the identity in some neighborhood of $\{A\}^*$. Because of the latter property, $f$ induces a homeomorphism $F\colon U/G \xrightarrow{\approx} X/G$ preserving orbit structure and equal to the identity on some neighborhood of $A/G$. Moreover, by the proof of 8.2, we may assume that $F$ is the identity for $t < \varepsilon$, where $t$ is the parameter coming from the cone structure of $(U/A)/G$. Let $F_{[0,t]}$ denote the restriction of $F$ to the part of $U/G$ with cone parameter in $[0, t]$. Then $F_{[0,\varepsilon]}$ lifts to the identity $F'_{[0,\varepsilon]}$ on that portion of $U \subset X$. By the Covering Homotopy Theorem 7.3, we can clearly extend this lifting to one, say $F'_{[0,2\varepsilon]}$, of $F_{[0,2\varepsilon]}$. Continuing indefinitely, we finally obtain a lifting $F'$ of $F$. Since $F$ is a homeomorphism and preserves orbit structure, $F'$ must be an equivalence. (The openness of $F'$ follows from Chapter I, Exercise 10.) ▮

**8.4. Corollary** *Let $M$ be a manifold which is a $G$-space with $G$ compact Lie. Suppose that $x_0$ is a stationary point and that the orbit structure on $M/G$ is conical with vertex $x_0^*$. If there is a coordinate system about $x_0$ in which $G$ acts linearly, then the $G$-space $M$ is equivalent to euclidean space with this linear action.* ▮

Somewhat more general than 8.4 is the following corollary.

**8.5. Corollary** *Let $M$ be a $G$-manifold, $G$ compact Lie. Suppose that the orbit structure is conical with vertex $x_0^*$ and that there is a slice $V$ at $x_0$ which has the form of euclidean space with linear $G_{x_0}$-action. Then $M$ is equivalent, as a $G$-space, to*

$$G \times_H V,$$

*where $H = G_{x_0}$.* ▮

*Remark* In a later chapter, we shall see that the hypothesis of a linear coordinate system in 8.4 and that of a linear slice in 8.5 are equivalent to a hypothesis that $G$ acts differentiably in a neighborhood of $x_0$ (in *some* differentiable structure on this neighborhood).

*Remark* Theorem 8.2, without orbit structure, can essentially be found in Kwun [2] and Kwun and Raymond [1]. Obviously any kind of extra structure which is constant on rays can be carried along in the proof. Note the implications of 8.3 for $G$-spaces which are mapping cylinders, with "bottom" $A$, of equivariant maps.

## 9. CLASSIFICATION OF $G$-SPACES

In this section we shall give the construction, due to Palais [3], of a "classifying space" for $G$-actions, $G$ compact Lie. Since the results in this section, and the next, will not be used elsewhere in this book, they may be skipped. However, we feel that this material deserves more attention than it has received and the reader might do well to study it. Assume that $G$ is a compact Lie group throughout this section. "Dimension" refers to covering dimension.

First, we need some preliminary material. The following two lemmas are quite well known, but since there is no adequate reference we shall indicate their proofs.

**9.1. Lemma** *Let $X$ be a paracompact space of dimension $n$ at most. Let $|L|$ be a compact $(n-1)$-connected polyhedron. Then any map of a closed subspace $A \subset X$ into $|L|$ can be extended to $X$.*

*Proof* Let $\varphi: A \to |L|$ be the given map and consider the open covering $\varphi^{-1}(\text{star}(v))$ of $A$, $v$ ranging over the vertices of $L$. Since $\dim X \leq n$ we can find a covering $\mathscr{U} = \{U\}$ of $X$ refining this (that is, $\varphi(U \cap A) \subset \text{star}(v_U)$ for some vertex $v_U$ of $L$) and such that the nerve $\mathbf{K}(\mathscr{U})$ of this covering has dimension $n$ at most. Let $\{f_U\}$ be a partition of unity subordinate to this covering and $g: X \to |\mathbf{K}(\mathscr{U})|$ the corresponding map; that is, $g(x) = \sum f_U(x)U$. Let $\mathbf{K}(\mathscr{U}|A)$ be the subcomplex of $\mathbf{K}(\mathscr{U})$ corresponding to $A$, that is, the simplex $(U_0, \ldots, U_r)$ is in $\mathbf{K}(\mathscr{U}|A)$ iff $U_0 \cap \cdots \cap U_r \cap A \neq \varnothing$. Note that $\varphi(U_0 \cap \cdots \cap U_r \cap A) \subset \text{star}(v_{U_0}) \cap \cdots \cap \text{star}(v_{U_r})$ which is nonempty iff $(v_{U_0}, \ldots, v_{U_r})$ is a simplex of $L$. If $y \in |\mathbf{K}(\mathscr{U}|A)|$, $y = \sum a_U U$, put $\bar{\varphi}(y) = \sum a_U v_U \in |L|$. Then $\bar{\varphi}: |\mathbf{K}(\mathscr{U}|A)| \to |L|$ and $\bar{\varphi}g(a) = \sum f_U(a)v_U$. If $f_U(a) \neq 0$, then $a \in U \cap A$ so that $\varphi(a) \in \text{star}(v_U)$;

that is, $v_U \in \text{support}(\varphi(a))$, the smallest simplex containing $\varphi(a)$. Thus $\bar{\varphi}g(a) \in \text{support}(\varphi(a))$, whence the "line segment" between $\bar{\varphi}g(a)$ and $\varphi(a)$ is in $|L|$. This shows that $\varphi$ is homotopic to $\bar{\varphi}g$ by $(a, t) \mapsto t\varphi(a) + (1 - t)\bar{\varphi}g(a)$. Now $\bar{\varphi}$, being a map of the subpolyhedron $|\mathbf{K}(\mathscr{U}| A)|$ of the $n$-dimensional polyhedron $|\mathbf{K}(\mathscr{U})|$ into the $(n - 1)$-connected polyhedron $|L|$, must extend to a map of $|\mathbf{K}(\mathscr{U})|$ into $|L|$ (by a standard argument). Thus $\bar{\varphi}g\colon A \to |L|$ extends to a map $X \to |L|$. Since $|L|$ is an ANR and $X$ is binormal (because it is paracompact) it follows that the homotopic map $\varphi\colon A \to |L|$ also extends to $X$ (see Spanier [1, p. 57]). ∎

**9.2. Lemma** *Let $B$ be a paracompact space of dimension at most $n$. Let $\xi\colon E \to B$ be a bundle whose fiber $F$ is a compact $(n - 1)$-connected polyhedron. Then every cross section $\sigma$ of $\xi$ over a closed set $A \subset B$ can be extended to a global cross section.*

***Proof*** Cover $B$ by a locally finite collection $\{C_\alpha\}$ of closed sets over each of which $\xi$ is trivial. Well-order the index set and put $A_\alpha = A \cup \bigcup_{\beta<\alpha} C_\beta$ which is closed by local finiteness. Extend $\sigma$ over the $A_\alpha$ inductively as follows: If $\sigma$ is defined on $A_\beta$ for all $\beta < \alpha$ and if $\alpha$ is a limit ordinal, then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, and the continuity of $\sigma$ on this follows from the local finiteness. If $\alpha$ is the successor to $\beta$, then $A_\alpha = A_\beta \cup C_\beta$. Since $\xi$ is trivial over $C_\beta$, $\sigma | C_\beta \cap A_\beta$ extends to $C_\beta$ by 9.1 and this extends $\sigma$ to $A_\alpha$. ∎

Let us say that a $G$-space is of type $G/H$ if all orbits have type $G/H$. More generally, if $\Sigma \subset \mathscr{E}_G$ is any collection of types, we say that a $G$-space $X$ has type $\Sigma$ if each orbit type occuring in $X$ is in $\Sigma$. We shall confine our attention to the case in which $\Sigma$ is *finite* and in this case will write $\Sigma = (H_1, \ldots, H_k)$, where the $H_i$ are representatives of the (distinct) corresponding isotropy types. We put

$$X_{(H)} = \{x \in X \mid \text{type } G(x) = \text{type } G/H\}.$$

**9.3. Theorem** *Let $H \subset G$, $K = N(H)/H$, and let $\xi\colon E \to B$ be a principal $K$-bundle whose total space $E$ is a compact $(n - 1)$-connected polyhedron. Then the associated $G/H$-bundle*

$$Y = (G/H) \times_K E \to B$$

*has the following property: Let $X$ be a paracompact $G$-space of type $G/H$*

*with* $\dim X/G \leq n$. *Then any equivariant map* $\varphi\colon A \to Y$, $A \subset X$ *closed and invariant, extends to an equivariant map* $\bar{\varphi}\colon X \to Y$.

***Proof*** By 5.13 the equivariant maps $\varphi\colon A \to Y$ are in natural correspondence with the cross sections over $A/G$ of the $Y^H$-bundle

$$Y^H \times_K X^H \to X/G$$

associated with the principal bundle $X^H \to X/G$. However, it is clear that $Y^H = K \times_K E \approx E$. Thus such a cross section extends to $X/G$ by 9.2. ▌

Now we shall note the existence of a special type of bundle space $E$ satisfying the hypotheses of 9.3. Since any compact Lie group may be embedded as a subgroup of an orthogonal group, we can assume that

$$K = N(H)/H \subset \mathbf{O}(r)$$

for some $r$. Let $\mathbf{O}(r) \times \mathbf{O}(n) \subset \mathbf{O}(r+n)$ in the standard way. Then the map (of *right* coset spaces)

$$E = (\{e\} \times \mathbf{O}(n))\backslash\mathbf{O}(r+n) \to (K \times \mathbf{O}(n))\backslash\mathbf{O}(r+n) = B$$

is a principal $K$-bundle and, as is well known, $E$ is $(n-1)$-connected. (One proves this by considering the fibrations $\mathbf{O}(i+1+n)/\mathbf{O}(n) \to \mathbf{O}(i+n)/\mathbf{O}(n)$ with fiber $\mathbf{O}(i+1+n)/\mathbf{O}(i+n) = \mathbf{S}^{i+n}$, for $i = 0, 1, \ldots, r-1$.)

Note that, for this $E$, $G$ operates on the left of $Y = (G/H) \times_K E$ and $\mathbf{O}(r+n)$ on the right. Moreover, the action of $G \times \mathbf{O}(r+n)$ is *transitive* on $Y$. Thus, this particular space $Y$ satisfying 9.3, is a *coset space* of the compact Lie group $G \times \mathbf{O}(r+n)$. By 5.3, $Y$ embeds in an orthogonal action of $G \times \mathbf{O}(r+n)$, and hence of $G$, on some euclidean space as a *neighborhood equivariant retract*. This fact will be important below.

We denote the closed cone over a $G$-space $Y$ by $CY$ and define it to be the quotient $[0, 1] \times Y/\{0\} \times Y$, where $G$ acts trivially on $[0, 1]$. The image of $(t, y)$ in $CY$ will be denoted by $ty$, and $0y$ will be written simply as 0, the vertex of the cone.

The following is a basic lemma for the Classification Theorem.

**9.4. Lemma** *Let* $H \subset G$, $n$ *and* $Y$ *be as in* 9.3 *and such that* $Y$ *embeds in an orthogonal* $G$*-action as a neighborhood equivariant retract. Let* $X$ *be a* $G$*-space such that* $X/G$ *is metrizable and has* $\dim X/G \leq n$. *Let* $A \subset X$

*be an invariant closed set and let*

$$\varphi\colon A \to CY$$

*be equivariant and such that* $\varphi^{-1}(0) \cap X_{(H)} = \emptyset$. *Then* $\varphi$ *extends to*

$$\psi\colon X \to CY$$

*with* $\psi^{-1}(0) \cap X_{(H)} = \emptyset$.

*Proof* Using the cone coordinates, we can write

$$\varphi(a) = f(a)\theta(a),$$

where

$$f\colon A \to [0, 1],$$
$$\theta\colon A - Z \to Y \qquad \text{where} \quad Z = f^{-1}(0).$$

Conversely, such a pair of equivariant maps gives a map $\varphi$. If type $G(a)$ $<$ type $G/H$, then we must have $f(a) = 0$, since $Y$ only has orbits of type $G/H$. Since the set of orbits of type less than type $G/H$ is closed (the complement is open by 5.5) $\varphi$ extends by 0 to this. Thus we may *assume* that $A$ contains all orbits of type less than type $G/H$. Since the set of orbits of type less than or equal to type $G/H$ is also closed (by Chapter I, Exercise 4) we then see that $A \cup X_{(H)}$ is closed. Also $A \cap X_{(H)} \subset A - Z$ by assumption.

Now $(\overline{X_{(H)}} - X_{(H)}) \subset Z$ since it consists of orbits of smaller type. Thus $\theta \mid A - Z$ extends to $(A - Z) \cup X_{(H)}$ by 9.3 (applied to $(A \cap \overline{X_{(H)}}) - Z$ $= (A \cap X_{(H)})$ and $\overline{X_{(H)}} - Z = X_{(H)}$). Also $f$ extends to $A \cup X_{(H)}$ by the classical Tietze Theorem applied to the induced map on the orbit space, and, to this extension, we may add a real (equivariant) function vanishing *exactly* on $A$ (e.g., the distance function from $A/G$ in some metric on $X/G$) and then bound the sum by 1. Thus we may extend $f$ so that the extension is nonzero on $X_{(H)}$. These extensions of $f$ and $\theta$ show that we may as well *assume* that

$$A \supset X_{(H)}.$$

By assumption, $Y$ embeds in an orthogonal $G$-action on some euclidean space $\mathbf{R}^m$ and has a neighborhood in $\mathbf{R}^m$ retracting equivariantly onto $Y$. Thus $\theta\colon A - Z \to Y$ extends to $X - Z \to \mathbf{R}^m$ by the Tietze–Gleason Theorem I.2.3. Following this by the retraction, we see that $\theta$ extends to

$$\theta'\colon W \to Y,$$

where $W$ is some open neighborhood of $A - Z$ in $X - Z$. Now $f$ is 0 on $Z \supset A \cap (\overline{X - W})$. Thus $f$ extends by 0 to $A \cup (\overline{X - W})$ and then extends equivariantly to

$$f'\colon X \to [0, 1]$$

by the classical Tietze Theorem applied to the induced map on the orbit space (or directly by the Tietze–Gleason Theorem). Then

$$\psi\colon X \to CY$$

defined by

$$\psi(x) = f'(x)\theta'(x)$$

is the desired extension. ∎

We shall now digress to discuss the join of several $G$-spaces. If $Y_1, \ldots, Y_k$ are $G$-spaces, then

$$CY_1 \times CY_2 \times \cdots \times CY_k$$

is a $G$-space. The canonical maps $CY_i \to [0, 1] = \mathbf{I}$ induce a map

$$CY_1 \times \cdots \times CY_k \to \mathbf{I} \times \cdots \times \mathbf{I} = \mathbf{I}^k$$

given by

$$(t_1y_1, \ldots, t_ky_k) \mapsto (t_1, \ldots, t_k).$$

The **join** $Y_1 \star \cdots \star Y_k$ of $Y_1, \ldots, Y_k$ is the inverse image of the standard simplex $\Delta^{k-1} \subset \mathbf{I}^k$; that is, it consists of those points $(t_1y_1, \ldots, t_ky_k)$ with $\sum t_i = 1$. Clearly, the isotropy group of a point in the join is just

$$G_{(t_1y_1,\ldots,t_ky_k)} = \bigcap_i G_{t_iy_i} = \bigcap \{G_{y_i} \mid t_i \neq 0\}.$$

It will be necessary to consider a certain subspace of the join. Thus we define the **Palais join**

$$Y_1 \circledast \cdots \circledast Y_k$$

to be the subspace of $Y_1 \star \cdots \star Y_k$ consisting of those points with

$$G_{(t_1y_1,\ldots,t_ky_k)} = G_{y_i} \quad \text{for some } i \text{ with } t_i \neq 0.$$

Note that then, $G_{(t_1y_1,\ldots,t_ky_k)} \subset G_{y_j}$ for all $j$ with $t_j \neq 0$. Also note that the Palais join is invariant, and hence is a $G$-space.

*Remark* Palais [3] calls this the *reduced* join, but we feel that that terminology is unfortunate since it has another established meaning.

**9.5. Theorem** *Let $\Sigma = (H_1, \ldots, H_k)$, the $H_i$ being mutually nonconjugate closed subgroups of $G$, and put $K_i = N(H_i)/H_i$. Let*

$$Y_i = (G/H_i) \times_{K_i} E_i,$$

*where $E_i$ is the total space of a principal $K_i$-bundle, with $E_i$ a compact $(n-1)$-connected polyhedron. Also assume, as we may, that $Y_i$ embeds as a neighborhood equivariant retract in some orthogonal $G$-action. Put $Y = Y_1 \circledast \cdots \circledast Y_k$. Let $X$ be any metrizable $G$-space of type $\Sigma$ such that* $\dim X/G \leq n$. *Then any orbit structure preserving equivariant map $\varphi$: $A \to Y$ on a closed invariant subspace $A$ of $X$, can be extended to an orbit structure preserving equivariant map $\psi$: $X \to Y$.*

*Proof* Consider

$$\varphi: A \to Y = Y_1 \circledast \cdots \circledast Y_k \subset CY_1 \times \cdots \times CY_k,$$

let $\varphi_i$: $A \to CY_i$ be the projection to $CY_i$, and put

$$\varphi_i(a) = f_i(a)\theta_i(a), \qquad \text{where} \quad f_i\colon\ A \to [0, 1] \quad \text{and} \quad \theta_i\colon\ A - f_i^{-1}(0) \to Y_i.$$

Let $a$ be in $A \cap X_{(H_i)}$. We claim that then $f_i(a) > 0$, for if not, then there is an index $j \neq i$ with $f_j(a) > 0$ and with

$$G_a = G_{\varphi(a)} = G_{\theta_j(a)} \sim H_j$$

contradicting the assumption that $G_a \sim H_i \not\sim H_j$ for $j \neq i$. Thus $f_i$ is *positive* on $A \cap X_{(H_i)}$. By 9.4 we conclude that $\varphi_i$ extends to an equivariant map map

$$\begin{cases} \varphi_i'\colon\ X \to CY_i, \\ \varphi_i'(x) = f_i'(x)\theta_i'(x), \\ f_i' > 0 \quad \text{on} \quad X_{(H_i)}. \end{cases}$$

Put

$$\begin{cases} h_i = f_i'/\sum f_i', \\ \psi_i(x) = h_i(x)\theta_i'(x), \\ \psi = (\psi_1, \ldots, \psi_k)\colon\ X \to CY_1 \times \cdots \times CY_k. \end{cases}$$

Then $\sum h_i = 1$ so that $\psi$: $X \to Y_1 \star \cdots \star Y_k$. For $x \in X_{(H_i)}$ we have

$$G_x \subset G_{\psi(x)} \subset G_{\psi_i(x)} \sim H_i \qquad (\text{since} \quad h_i(x) \neq 0)$$

from which we conclude that $G_{\psi(x)} = G_{\psi_i(x)}$ and that $G_x = G_{\psi(x)}$. The first equation implies that $\psi(x)$ is in $Y_1 \circledast \cdots \circledast Y_k$, and the second equation implies that $\psi$ preserves orbit structure. ∎

**9.6. Definition** A $G$-space $Y$ of type $\Sigma = (H_1, \ldots, H_k)$ which satisfies the conclusion of 9.5 is called **$n$-universal for $G$-structures of type $\Sigma$.**

**9.7. Theorem** *Let $\Sigma = (H_1, \ldots, H_k)$ and let $Y$ be $n$-universal for $G$-structures of type $\Sigma$. Then $Y/G$ is $(n-1)$-classifying for $G$-structures of type $\Sigma$, by which we mean the following:*

(i) *If $X$ is a metrizable $G$-space of type $\Sigma$ and with* $\dim X/G \leq n$, *then there is an orbit structure preserving map $f\colon X/G \to Y/G$ such that $X$ is equivalent over $X/G$ to $f^*Y$ ("over $X/G$" means commuting with the projections to $X/G$).*

(ii) *If $W$ is a metrizable space with* $\dim W \leq n-1$ *and if $f_0, f_1\colon W \to Y/G$ are maps inducing the same orbit structure on $W$, then $f_0^*Y$ and $f_1^*Y$ are equivalent over $W$ if and only if $f_0$ is homotopic, preserving orbit structure, to $f_1$.*

*Proof* For (i) note that, from 9.5, there is an equivariant map $X \to Y$ preserving orbit structure. This induces a map $\bar{f}\colon X \to f^*Y$, where $f\colon X/G \to Y/G$ is the induced map on orbit spaces. Then $\bar{f}$ preserves orbit types and

$$\begin{array}{ccc} X & \longrightarrow & f^*Y \\ & \searrow \quad \swarrow & \\ & X/G & \end{array}$$

commutes, so that $\bar{f}$ is an equivalence (openness following from Chapter I, Exercise 10).

For part (ii), suppose that $h\colon f_1^*Y \to f_0^*Y$ is an equivalence over $W$. Let $X = M_h$ be the mapping cylinder of $h$ and $A$ the union of the ends. By 9.5, the canonical map $f_0^*Y \to Y$ together with $f_1^*Y \to Y$ extends to $X \to Y$ preserving orbit structure. The induced map $W \times \mathbf{I} \approx X/G \to Y/G$ clearly gives a homotopy between $f_0$ and $f_1$. The converse follows directly from 7.4. ∎

## 10. LINEAR EMBEDDING OF $G$-SPACES

In this section, we shall use the results of the last section to prove a famous theorem of Mostow [1] to the effect that any finite-dimensional, separable metric $G$-space with only finitely many orbit types may be embedded, equivariantly, in an orthogonal action of $G$ on some euclidean space, where $G$ is compact Lie. (The converse is also true by Exercise 2.) We shall not use this material elsewhere in this book. (However, when we study differentiable actions, we will treat this subject again in that context, since there are important consequences of the smooth case.)

**10.1. Theorem** *If $X$ is a finite-dimensional, separable metric $G$-space, $G$ compact Lie, and if there are only a finite number of orbit types, then there is an equivariant embedding of $X$ as an invariant subspace of some euclidean space with an orthogonal $G$-action.*

*Proof* Let $X$ have type $\Sigma = (H_1, \ldots, H_k)$ as in 9.5. By Exercise 3, $X/G$ is also finite-dimensional. Let $n = \dim X/G$. Then, as is well known, there is an embedding

$$f_1\colon X/G \to \mathbf{R}^q$$

for some $q$. (It suffices for $q = 2n + 1$; see Hurewicz and Wallman [1].)

Now let us recall from the construction in Section 9, that the $G$-space $Y$, constructed below 9.3, was a coset space of $G \times \mathbf{O}(r+n)$ and hence was an orbit of some orthogonal action of $G \times \mathbf{O}(r+n)$. Consequently, we may regard the $Y_i$ of 9.5 to be embedded in the unit sphere $\mathbf{S}^{m_i-1}$ of an orthogonal $G$-action on $\mathbf{R}^{m_i}$. Then $CY_i \subset \mathbf{D}^{m_i}$, and

$$Y = Y_1 \circledast \cdots \circledast Y_k \subset CY_1 \times \cdots \times CY_k \subset \mathbf{D}^{m_1} \times \cdots \times \mathbf{D}^{m_k} \subset \mathbf{R}^m,$$

where $m = \sum m_i$, and where $G$ acts diagonally.

By 9.5 there is an orbit structure preserving, equivariant map

$$f_2\colon X \to Y \subset \mathbf{R}^m.$$

We define

$$f\colon X \to \mathbf{R}^q \times \mathbf{R}^m = \mathbf{R}^{q+m}$$

by $f(x) = \big(f_1(x^*), f_2(x)\big)$, and let $G$ act trivially on $\mathbf{R}^q$. Then $f$ is equivariant, and the induced orbit space map

$$f/G = (f_1, f_2/G)$$

is an embedding, since $f_1$ is an embedding. Since $f$ preserves orbit types, it is one–one. Thus $f$ is a homeomorphism onto its image by Chapter I, Exercise 10. ∎

**10.2. Corollary** *If $X$, as in* 10.1, *is also locally compact, then the embedding may be taken to be as a closed subspace.*

*Proof* The one-point compactification $X \cup \{\star\}$ satisfies the conditions and hence may be embedded equivariantly as a subspace of $\mathbf{R}^p$, say. The ideal point $\star$ is fixed under the action, and it follows easily that the translation taking $\star$ to 0 is equivariant. Thus $\star$ may be taken to be the origin 0. Then the map

$$y \mapsto y/\| y \|^2$$

of $\mathbf{R}^p - \{0\}$ to itself, is equivariant and takes $X = \bar{X} - \{0\}$ to a closed subspace of $\mathbf{R}^p$. ∎

*Remark* The method of proof given here differs considerably from Mostow's original proof in Mostow [1]. For two expositions of Mostow's method, see Palais [3, 4].

It should be noted that the orthogonal representation of $G$ in which the embedding of 10.1 occurs, depends only on $\dim X/G$ and on the orbit types which occur in $X$.

## EXERCISES FOR CHAPTER II

**1.** Let $X$ be a simply connected $G$-space, $G$ compact Lie. If there is an orbit with $n$ components, show that $n$ bounds the order of each element of $\pi_1(X/G)$.

**2.** Show that an orthogonal action on $\mathbf{R}^n$ has only a finite number of orbit types. (*Hint*: Consider the unit sphere and the action of the isotropy group at each point.)

**3.** If $X$ is a separable metric $G$-space, $G$ compact Lie, show that $\dim X/G \leq \dim X$.

**4.** Let $X$ be a completely regular $G$-space, $G$ compact Lie, and assume that all orbits have type $G/H$. Let $T$ be a maximal torus of $H$ and let $N = N(T)$ be the normalizer of $T$ in $G$. Show that the map

$$G \times_N X^T \to X$$

[taking $[g, x] \mapsto g(x)$] is the projection of a fiber bundle with fiber $H/(N \cap H)$. Also show that the canonical map

$$X^T/N \to X/G$$

is a homeomorphism.

**5.** Let $G \supset K, H$ be compact. Let $\xi\colon X \to B$ be a bundle with fiber $G/H$ and structure group $N(H)/H$ and let $\eta\colon Y \to B$ be a bundle with fiber $G/K$ and structure group $N(K)/K$. Consider the canonical left actions of $G$ on $X$ and $Y$ whose orbits are the fibers of $\xi$ and $\eta$; see 1.1. Let $\mathbf{Map}^G(\xi, \eta)$ be the set of equivariant maps $X \to Y$ over $B$ (i.e., commuting with the projections to $B$). Show how to construct a fiber bundle $\mathcal{M}ap^G(\xi, \eta)$ over $B$ with fiber $(G/K)^H$ and structure group $N(H)/H \times N(K)/K$ (made effective) such that

$$\Gamma(\mathcal{M}ap^G(\xi, \eta)) = \mathbf{Map}^G(\xi, \eta),$$

where $\Gamma$ is the cross section functor and equality means a canonical correspondence.

**6.** Let $\mathbf{Z}_2$ act on a separable metric space $X$ of dimension $n$. Show that $X$ embeds equivariantly in $\mathbf{R}^{3n+2}$ with the representation of $\mathbf{Z}_2$ which is $-1$ on the first $n + 1$ coordinates and 1 on the last $2n + 1$ coordinates.

**7.** Let $X$ be the total space of a principal $G$-bundle over $B$. Consider $X$ as a free (left) $G$-space and $X \to B$ as the orbit map. Let $H \subset G$ be a closed subgroup. If $Y$ is the orbit space of the action of $H$ on $X$, show that the induced map $Y \to B$ is the projection in the associated $H\backslash G$-bundle where $H\backslash G$ denotes the space of *right* cosets of $H$ in $G$, and the structure group $G$ acts by right translation on $H\backslash G$.

**8.** Let $p\colon W \to B$ be a principal $G$-bundle and let $A$ and $A'$ be right $G$-spaces. Then $G$ acts on $A \times W$ and $A' \times W$ as in Section 2. Show that passage to orbit spaces gives a one–one correspondence between equivariant maps $\psi\colon A \times W \to A' \times W$ commuting with projection onto $W$ and maps $\psi'\colon A \times_G W \to A' \times_G W$ *over* $B$. Also, for locally compact $A$, show that the exponential law gives a one–one correspondence of the $\psi$ with the equivariant maps $\psi''\colon W \to \mathbf{Map}(A, A')$ [where $G$ acts on $\mathbf{Map}(A, A')$ by putting $(g(f))(a) = f(ag)g^{-1}$] and that 2.6 gives a correspondence of these with the cross sections $\bar{\psi}$ of the associated bundle $\mathbf{Map}(A, A') \times_G W \to B$. Use this to give another derivation of 2.7. (*Hint*: In 2.7 put $G = S \times T$ and $W = \Delta$.)

**9.** Let $X$ be a $G$-space and let $H \subset G$ be a closed subgroup. Show that the $G$-space $G \times_H X$ is equivalent to the $G$-space $(G/H) \times X$ with diagonal $G$-action.

# CHAPTER III

## HOMOLOGICAL THEORY OF FINITE GROUP ACTIONS

In this chapter we shall study the homological relationships among the total space, orbit space, and fixed point set of an action of a finite group. For the total space and the orbit space, the basic tool is the transfer map which is a homomorphism in homology or cohomology going in the opposite direction from that of the homomorphism induced by the orbit map. For the fixed point set, the basic tools are the Smith exact sequences defined for actions of cyclic groups of prime order.

Our approach to this material, in this chapter, will be to present it in the most elementary way possible. We will return to the subject in a later chapter, assuming more background in algebraic topology, and shall obtain more powerful methods to deal with such matters.

We begin by presenting the transfer and Smith theory for simplicial actions on abstract simplicial complexes. This material is elementary enough to be included in a first course in homology, and we believe that it should be included in every topologist's tool kit. In Sections 6 and 7, the Čech method is then used (as was done by Smith) to generalize these results to a wide class of topological spaces.

The last three sections contain some divergent topics, all of which make use of the theory developed in the first seven sections. Section 8 contains results of Smith and Milnor concerning groups acting freely on spheres. Section 9 contains a theorem of Newman, as treated by Smith, showing that a compact Lie group action on a manifold cannot have uniformly small orbits. Section 10 contains Smith-type theorems for actions of toral groups.

### 1. SIMPLICIAL ACTIONS

Let us establish our terminology and notational conventions, which, with minor exceptions, will be those of Spanier [1]. By a **simplicial complex** $K$ we mean an abstract simplicial complex; that is, $K$ is a set (possibly infinite), whose elements are called **vertices**, together with a collection of finite nonempty subsets, called **simplices**, of the set of vertices such that

(1) Every vertex is contained in some simplex.
(2) Every nonempty subset of a simplex is a simplex.

A nonempty subset of a simplex is called a **face** of that simplex. A **simplicial map** from one simplicial complex to another is a map of the set of vertices which carries simplices into simplices.

There is a **topological realization**, or **polyhedron**, associated with a simplicial complex $K$, which is a topological space denoted by $|K|$. For its definition and properties, we refer to Spanier [1]. We always use the *coherent* (or *weak*) topology for $K$; that is, a subset of $|K|$ is closed iff its intersection with each $|s|$ is closed, where $s$ is a simplex of $K$.

The **barycentric subdivision** (or **first derived** complex) of a simplicial complex $K$ is the simplicial complex $K'$ whose vertices are the simplices of $K$ and whose simplices are the sets $(s_0, \ldots, s_n)$, where the $s_i$ are vertices of $K'$ (i.e., simplices of $K$) such that, *after reordering*,

$$s_0 \subset s_1 \subset \cdots \subset s_n$$

(i.e., each $s_i$ is a face of $s_{i+1}$). There is a canonical homeomorphism

$$|K'| \approx |K|.$$

Now suppose that $G$ is a discrete group. (We shall be concerned almost exclusively with finite groups $G$.) Suppose also that $G$ acts on $K$ simplicially; that is, such that each transformation is a simplicial map. The simplicial complex $K$, together with such an action is called a **simplicial $G$-complex.**

Consider the following statements concerning a simplicial $G$-complex $K$:

(A) For any $g \in G$ and simplex $s$ of $K$, $g$ leaves $s \cap g(s)$ pointwise fixed.

(B) If $g_0, g_1, \ldots, g_n$ are elements of $G$ and $(v_0, \ldots, v_n)$ and $(g_0v_0, \ldots, g_nv_n)$ are both simplices of $K$, then there exists an element $g$ of $G$ such that $g(v_i) = g_i(v_i)$ for all $i$.

Clearly neither of these need hold. For example, neither holds for the cyclic permutation of the three vertices of a 2-simplex, and (B) even fails for the barycentric subdivision of this.

First let us note that (A) is equivalent to

(A′) If $v$ and $g(v)$ belong to the same simplex, then $v = g(v)$.

To see that (A) implies (A′) let $s = (g^{-1}v, v)$, whence $g$ acts trivially on $v \in s \cap gs$. To see that (A′) implies (A) let $v \in s \cap gs$, and note that then $(v, gv) \subset gs$ so that $gv = v$ by (A′).

Moreover, note that (B) implies (A′) and hence (A), since if $v$ and $gv$ belong to some simplex, then $(v, v)$ and $(v, gv)$ are simplices of $K$, whence, for some $g'$, $v = g'v$ and $gv = g'v$, so that $gv = g'v = v$.

**1.1. Proposition** *If $K$ is a simplicial $G$-complex, then the induced action on the barycentric subdivision $K'$ satisfies* (A). *If* (A) *is satisfied for $G$ on $K$, then* (B) *is satisfied for $G$ on $K'$.*

***Proof*** If $s$ is a vertex of $K'$, thus a simplex of $K$, and if $s$ and $gs$ belong to a simplex of $K'$, then $s$ is a face of $gs$ (or vice versa). But $G$ clearly preserves dimension of simplices of $K$, so that $s = gs$, proving the first part.

Now suppose that (A) is satisfied on $K$. We shall prove (B) for $K'$ by induction on $n$. Suppose that $(s_0, s_1, \ldots, s_n)$ is a simplex of $K'$. By reordering, we may assume that

$$s_0 \subset s_1 \subset \cdots \subset s_{n-1} \subset s_n$$

as simplices of $K$. Suppose that $(g_0s_0, \ldots, g_ns_n)$ is also a simplex of $K'$. By the inductive assumption, there is a $g$ in $G$ with $gs_i = g_is_i$ for $0 \leq i < n$. Operation by $g^{-1}$ shows that

$$(s_0, s_1, \ldots, s_{n-1}, g's_n)$$

is a simplex of $K'$, where $g' = g^{-1}g_n$. Since these are ordered by ascending (or rather by nondescending) dimension, we must have that

$$s_0 \subset s_1 \subset \cdots \subset s_{n-1} \subset g's_n.$$

Then $s_{n-1} \subset (s_n \cap g's_n)$ which implies that $g'$ acts trivially on $s_{n-1}$ by (A), hence trivially on $s_i$ for all $i < n$. Since $g' = g^{-1}g_n$ we have $g_is_i = gs_i = g_ns_i$ for $i < n$. Thus $g_ns_i = g_is_i$ for all $i$. ▌

**1.2. Definition** A simplicial action of $G$ on $K$ satisfying (B) for the action of *each subgroup* of $G$ is called **regular**, and $K$ (with the action) is called a **regular $G$-complex.**

Thus any simplicial action becomes regular upon passage to the second barycentric subdivision. Consequently, an assumption of regularity is no loss of generality from the topological viewpoint.

For a *regular* $G$-complex $K$ we define a simplicial complex $K/G$ as follows. The vertices of $K/G$ are just the orbits $v^* = G(v)$ of the action of $G$ on the vertices of $K$, and we take the simplices of $K/G$ to be those simplices of the form

$$(v_0^*, \dots, v_n^*),$$

where $(v_0, \dots, v_n)$ is a simplex of $K$ (that is, there *exist* representatives $v_i$ of $v_i^*$ such that this is the case; it is *not* required for *all* systems of representatives of the orbits $v_i^*$). The simplex $(v_0, \dots, v_n)$ of $K$ is said to be **over** the simplex $(v_0^*, \dots, v_n^*)$ of $K/G$. (We remark that the above definition makes sense for arbitrary simplicial actions, but we shall only rarely consider it for actions which are not regular.)

Now, *by regularity*, if $(v_0, \dots, v_n)$ and $(w_0, \dots, w_n)$ are simplices of $K$ over the *same* simplex $(v_0^*, \dots, v_n^*) = (w_0^*, \dots, w_n^*)$ of $K/G$, then $(w_0, \dots, w_n) = g(v_0, \dots, v_n)$ for some $g$ in $G$. That is, the simplices over a given simplex of $K/G$ form an *orbit* of the action of $G$ on the simplices of $K$.

Clearly $v \mapsto v^*$ defines a simplicial map $K \to K/G$, and hence a continuous map $|K| \to |K/G|$, and $G(\sum \lambda_i v_i) \mapsto \sum \lambda_i G(v_i) = \sum \lambda_i v_i^*$ defines a map $|K|/G \to |K/G|$ giving the factorization

$$\begin{array}{ccc} & |K| & \\ \swarrow & & \searrow \\ |K|/G & \longrightarrow & |K/G|. \end{array}$$

Moreover, the horizontal map is one–one and onto, *by regularity*. Now, $|K| \to |K/G|$ is onto and it is clear from the definition of the coherent topology (Spanier [1]) that $|K/G|$ has the quotient topology defined by this map. (This holds for any surjective simplicial map.) It follows that the horizontal map is a *homeomorphism*

$$|K|/G \approx |K/G|.$$

It is also clear by regularity [or, in fact, by (A)] that

$$|K^G| = |K|^G,$$

where $K^G$ is the subcomplex of $K$ consisting of all simplices which are pointwise fixed under $G$.

## 2. THE TRANSFER

Let $K$ be a regular $G$-complex with $G$ finite, and let $C(K)$ be the oriented chain complex of $K$ (see Spanier [1]). Then $C(K)$ inherits an action of $G$ by chain mappings, by putting

$$g[v_0, \ldots, v_n] = [gv_0, \ldots, gv_n],$$

and hence is a module over the group ring $\mathbf{Z}G$ of $G$ (whose elements are formal sums

$$\sum_{g \in G} n_g g,$$

where the $n_g$ are integers). The norm $\sigma \in \mathbf{Z}G$ is defined to be the sum

$$\sigma = \sum_{g \in G} g$$

of the elements of $G$. Then $\sigma c = \sum gc$ for a chain $c$ in $C(K)$. The image

$$\sigma C(K) \subset C(K)$$

of $\sigma\colon C(K) \to C(K)$ is a subcomplex.

Similarly, if $L \subset K$ is a subcomplex which is invariant under the action of $G$, then $G$ acts on $C(K, L) = C(K)/C(L)$ and $\sigma C(K, L)$ is a subcomplex of $C(K, L)$.

Let $\pi\colon K \to K/G$ be the canonical simplicial map and use $\pi$ to also denote the induced chain map

$$\pi\colon C(K, L) \to C(K/G, L/G).$$

### 2.1. Lemma

$$\ker\{\sigma\colon C(K, L) \to C(K, L)\} = \ker\{\pi\colon C(K, L) \to C(K/G, L/G)\}.$$

***Proof*** Let $s$ be any simplex of $K/G$ and let $s_1, \ldots, s_n$ be the simplices of $K$ over $s$. By regularity, we may orient $s$ and the $s_i$ so that $\pi\colon s_i \to s$ preserves these orientations. Since $\{s_1, \ldots, s_n\}$ is an orbit of the action of $G$, we see that $n = |G| / |G_{s_1}|$, where $|G|$ denotes the order of $G$. Clearly, it will suffice to consider chains of the form

$$c = \sum n_i s_i.$$

Then $\pi c = (\sum n_i)s$, so that $\pi c = 0$ iff $\sum n_i = 0$. Also note that $g\sigma(c) = \sigma(c)$, and, since $G$ permutes the $s_i$ transitively, it follows that

$$\sigma c = m \sum s_i$$

for some integer $m$. By adding the coefficients on both sides of this equation we see that

$$|G|\sum n_i = mn = m|G|/|G_{s_1}|.$$

Consequently, $m = |G_{s_1}|\sum n_i$ and $\sigma c = 0$ iff $m = 0$ iff $\sum n_i = 0$. ▌

(Note that if we work over an arbitrary base ring $\Lambda$ instead of over **Z**, then 2.1 will hold as long as $|G|$ is not a zero-divisor in $\Lambda$.)

From 2.1, it follows that we have the canonical *chain isomorphism*

$$\sigma C(K, L) \approx C(K, L)/\ker \sigma = C(K, L)/\ker \pi \approx C(K/G, L/G)$$

since $\pi$ is clearly onto. In fact, the map $\sigma C(K, L) \to C(K/G, L/G)$, taking $\sigma c$ to $\pi c$, is this isomorphism. The composition

$$\mu\colon\ C(K/G, L/G) \xrightarrow{\approx} \sigma C(K, L) \subset C(K, L)$$

of the inverse of this with the inclusion is given by $c^* = \pi c \mapsto \sigma c$, and induces a homomorphism

$$\mu_*\colon\ H(K/G, L/G) \to H(K, L)$$

called the **transfer**. Of course, $\pi$ induces the usual map

$$\pi_*\colon\ H(K, L) \to H(K/G, L/G).$$

Note that, for $c^* = \pi c$, we have $\pi\mu(c^*) = \pi\sigma(c) = |G|\,\pi(c) = |G|\,c^*$, since $\pi \circ g = \pi$. Also $\mu\pi c = \sigma c$ by definition. Thus

**(2.2)**
$$\pi_*\mu_* = |G|\colon\ H(K/G, L/G) \to H(K/G, L/G)$$
$$\mu_*\pi_* = \sigma_* = \sum_{g\in G} g_*\colon\ H(K, L) \to H(K, L).$$

We may also define $\mu$ for arbitrary coefficient groups, simply by tensoring the chain groups with the coefficient group, and (2.2) clearly holds in this general case.

Also note that the image of $\mu$ consists of chains which are fixed under the action of $G$. Consequently, for the induced action of $G$ on homology, we have

$$\operatorname{Im} \mu_* \subset H(K, L)^G.$$

Clearly the restriction of $\pi_*$ to $H(K, L)^G$ satisfies

**(2.3)**
$$\mu_*\pi_* = |G|\colon\ H(K, L)^G \to H(K, L)^G.$$

Together with (2.2) this gives the following theorem.

**2.4. Theorem** *If $\Lambda$ is a field of characteristic 0 or prime to $|G|$, then*

$$\pi_*\colon H(K, L; \Lambda)^G \to H(K/G, L/G; \Lambda)$$

*is an isomorphism, as is*

$$\mu_*\colon H(K/G, L/G; \Lambda) \to H(K, L; \Lambda)^G.$$

*Moreover, the kernel of $\pi_*$ on $H(K, L; \Lambda)$ is equal to the kernel of $\sigma_*$.* ∎

Let us generalize the above discussion by considering a subgroup $H$ of $G$ and the diagram

$$\begin{array}{ccc} & K & \\ {}^{\pi_H}\swarrow & & \searrow^{\pi_G} \\ K/H & \xrightarrow[\pi_{G/H}]{} & K/G \end{array}$$

of simplicial maps. (We need not have $H$ normal in $G$.) As before we use $\pi_G$, $\pi_G$, and $\pi_{G/H}$ to also stand for the induced chain maps. Also let

$$\sigma_H = \sum_{h\in H} h \qquad \text{and} \qquad \sigma_G = \sum_{g\in G} g.$$

We define a chain map

$$\mu_{G/H}\colon C(K/G,\ L/G) \to C(K/H, L/H)$$

by commutativity of

$$\begin{array}{ccc} C(K/G, L/G) & \xrightarrow{\mu_{G/H}} & C(K/H, L/H) \\ \downarrow{\scriptstyle\approx} & & \uparrow{\scriptstyle\approx} \\ \sigma_G C(K, L) & \xrightarrow{\text{incl}} & \sigma_H C(K, L) \end{array}$$

which, on elements, is

$$\begin{array}{ccc} \pi_G(c) & \longmapsto & \pi_H(c') \\ \downarrow & & \uparrow \\ \sigma_G(c) & \overset{=}{\longmapsto} & \sigma_H(c') \end{array}$$

where $c' = \sum g_i c$, $\{g_i\}$ being any system of representatives of the right cosets of $H$ in $G$.

Now $\pi_{G/H}(\pi_H(c')) = \pi_G(c') = m\pi_G(c)$, where $m = |G|/|H|$. Thus we have a natural homomorphism, the transfer,

**(2.5)** $$(\mu_{G/H})_*\colon H(K/G,\ L/G) \to H(K/H, L/H)$$

such that

**(2.6)** $(\pi_{G/H})_*(\mu_{G/H})_* = |G|/|H|: H(K/G, L/G) \to H(K/G, L/G)$

for any coefficients.

*Remarks* The transfer in this generality is due to Floyd [5] and Conner [1]; also see Floyd [10] and Bredon [13]. The transfer was defined earlier for covering spaces by Eckmann [1,2] and Liao [1]; also see Steenrod and Epstein [1].

It is easy, of course, to dualize these results to cohomology. Thus, for any coefficient group $A$, $\mu$ induces a cochain map

$$\mathrm{Hom}(\mu, A): \mathrm{Hom}(C(K, L), A) \to \mathrm{Hom}(C(K/G, L/G), A)$$

and hence a homomorphism in cohomology

$$\mu^*: H^*(K, L;\ A) \to H^*(K/G, L/G; A).$$

The analog of (2.2) is

$$\mu^*\pi^* = (\pi\mu)^* = |G|,$$
$$\pi^*\mu^* = (\mu\pi)^* = \sigma^* = \textstyle\sum g^*.$$

The obvious analog of (2.4) holds. Similarly, the generalization $\mu_{G/H}$ of $\mu$ dualizes in the obvious way.

Also note that the transfer is natural. That is, if $f$: $(K, L) \to (K', L')$ is an equivariant simplicial map with $\bar{f}$: $(K/G, L/G) \to (K'/G, L'/G)$ the induced map, then

$$\begin{array}{ccc} H(K/G, L/G) & \xrightarrow{\bar{f}_*} & H(K'/G, L'/G) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} \\ H(K, L) & \xrightarrow{f_*} & H(K', L') \end{array}$$

commutes. In fact, this is already the case on the chain level, since $\mu\bar{f}(\pi c) = \mu\pi f(c) = \sigma f(c) = f(\sigma c) = f\mu(\pi c)$. Similar remarks clearly hold for the generalized version (2.5) of the transfer and for the cohomological versions.

## 3. TRANSFORMATIONS OF PRIME PERIOD

We shall now restrict our attention to a (multiplicative) group $G$ of prime order $p$, and shall study homology with coefficients in $\mathbf{Z}_p$. Let $K$ be a regular $G$-complex. To simplify notation slightly we shall use $K^*$ for $K/G$ and will identify $K^G$ with its copy $K^G/G$ in $K^*$.

Let $g$ be a fixed generator of $G$ and put

$$\sigma = 1 + g + g^2 + \cdots + g^{p-1},$$
$$\tau = 1 - g,$$

in the group ring $\mathbf{Z}_pG$. Since $g^p = 1$ we have that

$$\sigma\tau = 0 = \tau\sigma.$$

Since we are working over $\mathbf{Z}_p$, and since

$$(-1)^i \binom{p-1}{i} \equiv 1 \quad (\text{modulo } p),$$

we have that

$$\sigma = \tau^{p-1}.$$

In particular, $\sigma = \tau$ when $p = 2$.

If $\varrho = \tau^i$, we put $\bar{\varrho} = \tau^{p-i}$. Thus $\tau = \bar{\sigma}$ and $\sigma = \bar{\tau}$.

Let $L \subset K$ be an invariant subcomplex and note that $L^G = L \cap K^G$. We shall consider the chain subcomplexes

$$\varrho C(K, L; \mathbf{Z}_p)$$

of $C(K, L; \mathbf{Z}_p)$ for $\varrho = \tau^i$, $1 \le i \le p-1$. Coefficients will always be in $\mathbf{Z}_p$ and will usually be dropped from the notation except when desired for stress.

The basic result is the following theorem.

**3.1. Theorem** *For each $\varrho = \tau^j$, $1 \le j \le p-1$,*

$$0 \to \bar{\varrho}C(K, L) \oplus C(K^G, L^G) \xrightarrow{i} C(K, L) \xrightarrow{\varrho} \varrho C(K, L) \to 0$$

*(coefficients in $\mathbf{Z}_p$) is an exact sequence of chain complexes, where $i$ is the sum of the inclusions and $\varrho$: $c \mapsto \varrho c$.*

***Proof*** It clearly suffices to consider $n$-chains in the orbit of $s$ for each $n$-simplex $s$ of $K$ not in $L$, and there are two cases, depending on whether

$s$ is in $K^G$ or not. If $s$ is in $K^G$, then $\tau s = 0$, so that $\varrho s = 0 = \bar{\varrho} s$, and the sequence is clear.

If $s$ is not in $K^G$, an $n$-chain in $G(s)$ has the form

$$\textstyle\sum n_i g^i s, \qquad n_i \in \mathbf{Z}_p,$$

and corresponds to the unique element $\sum n_i g^i$ of the group ring $\Lambda = \mathbf{Z}_p G$. Thus the sequence reduces to

$$0 \to \bar{\varrho}\Lambda \xrightarrow{i} \Lambda \xrightarrow{\varrho} \varrho\Lambda \to 0.$$

To check the exactness of this, we note that these are vector spaces over $\mathbf{Z}_p$ and that $i$ is injective and $\varrho$ surjective. Thus it suffices to show that $\dim \varrho\Lambda + \dim \bar{\varrho}\Lambda = \dim \Lambda = p$ as vector spaces over $\mathbf{Z}_p$. Consider $\tau^i\Lambda$, $1 \leq i \leq p-1$. The kernel of $\tau\colon \Lambda \to \Lambda$ consists of the elements with constant coefficients and hence is 1-dimensional, generated by $\sigma = 1 + g + \cdots + g^{p-1}$. It follows that $\dim \tau\Lambda = \dim \Lambda - 1$. However, $\ker \tau = \mathbf{Z}_p\sigma \subset \tau^i\Lambda$ for all $i$, since $\sigma = \tau^{p-1} = \tau^i\tau^{p-i-1}$. Thus, more generally, $\dim \tau^{i+1}\Lambda = \dim \tau(\tau^i\Lambda) = \dim \tau^i\Lambda - 1$. Since $\dim \Lambda = p$ we conclude that $\dim \tau^i\Lambda = p - i$. ▮

**3.2. Definition** For $\varrho = \tau^i$, $1 \leq i \leq p-1$, we let

$$H^\varrho(K, L;\ \mathbf{Z}_p) = H\big(\varrho C(K, L;\ \mathbf{Z}_p)\big).$$

This graded group is called the **Smith special homology group.**

*Remark* It is important to note that we are using $\varrho C(K, L;\ \mathbf{Z}_p)$ and *not* $\big(\varrho C(K, L;\ \mathbf{Z})\big) \otimes \mathbf{Z}_p$.

From standard facts, the short exact sequence of 3.1 gives a long exact sequence in homology.

**3.3. Theorem** *For $\varrho = \tau^i$, $1 \leq i \leq p-1$, there is an exact triangle (coefficients in $\mathbf{Z}_p$)*

$$\begin{array}{ccc} & H(K, L) & \\ \swarrow^{\varrho_*} & & \nwarrow^{i_*} \\ H^\varrho(K, L) & \xrightarrow{\ \delta_*\ } & H^{\bar{\varrho}}(K, L) \oplus H(K^G, L^G) \end{array}$$

*where $i_*$ and $\varrho_*$ have degree 0 and $\delta_*$ has degree $-1$.* ▮

These sequences (triangles) are called the **Smith sequences**. It usually suffices to consider the two cases $\varrho = \sigma$ or $\varrho = \tau$.

Although most of these Smith groups have no interpretation in terms of more "familiar" things, the special group $H^\sigma(K, L)$ does. To see this, we note that the map

$$\sigma\colon C(K, L;\ \mathbf{Z}_p) \to C(K, L;\ \mathbf{Z}_p)$$

has the *same kernel* as does the *composition*

$$C(K, L;\ \mathbf{Z}_p) \xrightarrow{j} C(K, K^G \cup L;\ \mathbf{Z}_p) \xrightarrow{\pi} C(K^*, K^G \cup L^*;\ \mathbf{Z}_p).$$

(Contrast this with Section 2!) This follows from the observation that if $s$ is a simplex of $K$ not in $L$, then $\sigma(\sum n_i g^i s) = (\sum n_i)\sigma(s) = 0$ iff $\sum n_i = 0$ *or* $s \subset K^G$.

Thus the images of these maps are isomorphic via $\sigma c \mapsto \pi j(c)$. Passing to homology we have

$$H^\sigma(K, L;\ \mathbf{Z}_p) \approx H(K^*, K^G \cup L^*;\ \mathbf{Z}_p). \tag{3.4}$$

Moreover, it is clear that the diagram

$$\begin{array}{ccccc} 0 \to \tau C(K, L) \oplus C(K^G, L^G) & \to & C(K, L) & \xrightarrow{\sigma} & \sigma C(K, L) \to 0 \\ \big\downarrow{\scriptstyle 0+\mathrm{incl}} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \approx} \\ 0 \to C(K^G \cup L^*, L^*) & \to & C(K^*, L^*) & \xrightarrow{j} & C(K^*, K^G \cup L^*) \to 0 \end{array} \tag{3.5}$$

commutes and hence induces a homology ladder which we shall not display.

Denoting $\varrho C(K, L;\ \mathbf{Z}_p)$ by $C(\varrho)$ for short, we note that, for $p > 2$,

$$\begin{array}{ccccc} 0 \to C(\tau) \oplus C(K^G, L^G) & \to & C(K, L) & \xrightarrow{\sigma} & C(\sigma) \to 0 \\ \big\downarrow{\scriptstyle \tau^{p-2}\oplus 0} & & \big\downarrow{\scriptstyle \tau^{p-2}} & & \big\downarrow{\scriptstyle \mathrm{incl}} \\ 0 \to C(\sigma) \oplus C(K^G, L^G) & \to & C(K, L) & \xrightarrow{\tau} & C(\tau) \to 0 \end{array} \tag{3.6}$$

and

$$\begin{array}{ccccc} 0 \to C(\sigma) \oplus C(K^G, L^G) & \to & C(K, L) & \xrightarrow{\tau} & C(\tau) \to 0 \\ \big\downarrow{\scriptstyle \mathrm{incl}\oplus 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle \tau^{p-2}} \\ 0 \to C(\tau) \oplus C(K^G, L^G) & \to & C(K, L) & \xrightarrow{\sigma} & C(\sigma) \to 0 \end{array} \tag{3.7}$$

commute. These diagrams give homology ladders which yield a relationship between the two Smith sequences for $\varrho = \sigma$ and $\varrho = \tau$, respectively.

Note that it is a consequence of the proof of 3.1 that

$$0 \to \sigma C(K, L) \xrightarrow{\text{incl}} \tau^j C(K, L) \xrightarrow{\tau} \tau^{j+1} C(K, L) \to 0$$

is exact for all $1 \leq j \leq p-1$. Hence there is the *exact triangle*

$$\begin{array}{ccc} & H^{\tau^j}(K, L) & \\ {\scriptstyle\tau_*}\swarrow & & \nwarrow{\scriptstyle i_*} \\ H^{\tau^{j+1}}(K, L) & \longrightarrow & H^{\sigma}(K, L) \end{array} \tag{3.8}$$

where the horizontal map has degree $-1$ and the others have degree 0.

It is clear how to dualize these results to cohomology. Thus, one defines the **Smith special cohomology groups**

$$H_\varrho{}^*(K, L;\ \mathbf{Z}_p) = H^*\big(\varrho C(K, L;\ \mathbf{Z}_p), \mathbf{Z}_p\big);$$

that is, the homology of the cochain complex

$$\mathrm{Hom}\big(\varrho C(K, L;\ \mathbf{Z}_p), \mathbf{Z}_p\big).$$

(We use the notation of Spanier [1, p. 237] here.)

The sequence 3.1 dualizes to an exact sequence of cochain complexes and passage to homology gives the **Smith exact triangle** of cohomology groups (coefficients in $\mathbf{Z}_p$)

$$\begin{array}{ccc} & H^*(K, L) & \\ {\scriptstyle\varrho^*}\nearrow & & \searrow{\scriptstyle i^*} \\ H_\varrho{}^*(K, L) & \xleftarrow{\ \delta^*\ } & H_{\bar\varrho}{}^*(K, L) \oplus H^*(K^G, L^G) \end{array} \tag{3.9}$$

where $i^*$ and $\varrho^*$ have degree 0 and $\delta^*$ has degree $+1$. Similarly, (3.8) dualizes in the obvious way. The isomorphism

$$\sigma C(K, L;\ \mathbf{Z}_p) \approx C(K^*, K^G \cup L^*;\ \mathbf{Z}_p)$$

yields

$$H_\sigma{}^*(K, L;\ \mathbf{Z}_p) \approx H^*(K^*, K^G \cup L^*;\ \mathbf{Z}_p) \tag{3.10}$$

which is dual to (3.4).

Also, the Smith groups are natural. That is, an equivariant simplicial map $f\colon (K, L) \to (K', L')$ induces homomorphisms of the Smith groups

$$f_*\colon H^\varrho(K, L) \to H^\varrho(K', L'),$$
$$f^*\colon H_\varrho{}^*(K', L') \to H_\varrho{}^*(K, L).$$

It is clear that these homomorphisms commute with the homomorphisms in the Smith sequences.

Moreover, let us note, for later reference, that if $f_0$ and $f_1$ are equivariant simplicial maps $(K, L) \to (K', L')$ (regular $G$-complexes) which are *contiguous* [i.e., for any simplex $s$ of $K$ (or $L$) $f_0(s)$ and $f_1(s)$ belong to a common simplex of $K'$ (or $L'$)], then the induced chain maps $f_0, f_1$: $C(K, L) \to C(K', L')$ are *equivariantly* chain homotopic. To see this, choose a simple ordering of the vertices of $K/G$. This induces an ordering of the vertices of any simplex $s$ in $K$ and these orderings are clearly equivariant under action by $G$. Then if $s = [v_0, \ldots, v_n]$ with vertices in the given order, we put

$$Ds = \sum_{i=0}^{n} (-1)^i [f_0 v_0, \ldots, f_0 v_i, f_1 v_i, \ldots, f_1 v_n].$$

It is easy to see that

$$D\partial + \partial D = f_1 - f_0$$

and clearly $Dg = gD$. [These remarks clearly hold for arbitrary $G$, but will be used only for the present case of cyclic $G$ of prime order.]

Since $D$ is equivariant, we have that $D\varrho = \varrho D$ for all $\varrho = \tau^i$, and hence $D$ induces a chain homotopy between the restrictions of $f_0$ and $f_1$ to $\varrho C(K, L)$. Consequently, we conclude that contiguous equivariant maps $f_0$ and $f_1$ induce the *same* map $H^\varrho(K, L) \to H^\varrho(K', L')$ for any $\varrho$ (and similarly in cohomology).

## 4. EULER CHARACTERISTICS AND RANKS

We shall now apply the algebra developed in Section 3 to obtain some more explicit relationships between $H(K)$, $H(K^G)$, and $H(K^*)$ due to Floyd [3]. As in Section 3, $K$ is a regular $G$-complex, with $G$ the cyclic group of prime order $p$, and $K^* = K/G$. Homology is always taken with $\mathbf{Z}_p$ coefficients. rk $H_i(K)$ denotes the rank (i.e., dimension) of $H_i(K)$ as a vector space over $\mathbf{Z}_p$.

**4.1. Theorem** *Suppose that $K$ is a finite-dimensional regular $G$-complex and that $L \subset K$ is an invariant subcomplex. Then, for any integer $n \geq 0$ and any $\varrho = \tau^i$, $1 \leq i \leq p-1$,*

$$\text{rk } H_n{}^\varrho(K, L) + \sum_{i \geq n} \text{rk } H_i(K^G, L^G) \leq \sum_{i \geq n} \text{rk } H_i(K, L).$$

*In particular, if the right-hand side is finite, then so is the left-hand side, and so is* rk $H_i(K^*, L^* \cup K^G)$ *for all* $i \geq n$.

*Proof* The exact sequence (of 3.3)

$$H_{i+1}^{\bar{\varrho}}(K, L) \to H_i^{\varrho}(K, L) \oplus H_i(K^G, L^G) \to H_i(K, L)$$

shows that

$$\text{rk } H_i^{\varrho}(K, L) + \text{rk } H_i(K^G, L^G) \leq \text{rk } H_{i+1}^{\bar{\varrho}}(K, L) + \text{rk } H_i(K, L)$$

(and recall that $\varrho$ and $\bar{\varrho}$ may be interchanged). Let us put

$$a_i = \text{rk } H_i(K^G, L^G), \qquad b_i = \text{rk } H_i(K, L),$$
$$c_i = \text{rk } H_i^{\varrho}(K, L), \qquad \bar{c}_i = \text{rk } H_i^{\bar{\varrho}}(K, L).$$

Then we have

$$\begin{aligned} c_n + a_n &\leq \bar{c}_{n+1} + b_n, \\ \bar{c}_{n+1} + a_{n+1} &\leq c_{n+2} + b_{n+1}, \\ &\vdots \\ \bar{c}_{n+2k-1} + a_{n+2k-1} &\leq c_{n+2k} + b_{n+2k-1}, \\ c_{n+2k} + a_{n+2k} &\leq \bar{c}_{n+2k+1} + b_{n+2k}. \end{aligned} \tag{4.2}$$

Now if $\sum_{i \geq n} b_i$ is infinite there is nothing to prove. Thus assume that $b_i < \infty$ for $i \geq n$. For $n + 2k \geq \dim K$ we have $\bar{c}_{n+2k+1} = 0$, so that the last inequality shows that both sides are finite. Then the next to the last inequality has both sides finite, and so on. Thus everything in (4.2) is finite. Adding these inequalities and canceling gives

$$c_n + \sum_{i \geq n} a_i \leq \sum_{i \geq n} b_i$$

as claimed. The last statement follows from the fact that

$$H_i(K^*, L^* \cup K^G) \approx H_i^{\sigma}(K, L)$$

by (3.4). ∎

**4.3. Theorem** *Suppose that K is a finite-dimensional regular G-complex and* $L \subset K$ *an invariant subcomplex. Assume that* rk $H(K, L; \mathbf{Z}_p) < \infty$ *and let* $\chi(K, L) = \sum(-1)^i$ rk $H_i(K, L; \mathbf{Z}_p)$ *be the euler characteristic of* $(K, L)$. *Then*

$$\chi(K, L) + (p - 1)\chi(K^G, L^G) = p\chi(K^*, L^*).$$

*In particular,*

$$\chi(K^G, L^G) \equiv \chi(K, L) \quad (\textit{modulo } p).$$

***Proof*** First, let us note that these euler characteristics are all defined. From 4.1 we have rk $H(K^G, L^G) < \infty$ and rk $H(K^*, L^* \cup K^G) < \infty$. The exact triangle

$$\begin{array}{ccc} & H(K^*, L^*) & \\ \swarrow & & \nwarrow \\ H(K^*, L^* \cup K^G) & \longrightarrow & H(L^* \cup K^G, L^*) \approx H(K^G, L^G) \end{array}$$

shows that rk $H(K^*, L^*) < \infty$ as claimed.

We shall use the fact that if

$$\begin{array}{ccc} & B & \\ {}^{g}\swarrow & & \nwarrow{}^{f} \\ C & \xrightarrow{\ h\ } & A \end{array}$$

is an exact triangle of graded groups with $\deg f = 0 = \deg g$ and $\deg h = -1$, then $\chi(B) = \chi(A) + \chi(C)$. This well-known fact follows from the similar fact that the euler characteristic is preserved under passage to homology (see Spanier [1, p. 172]) as follows: Consider the triangle as a chain complex. Then its homology is trivial and hence its euler characteristic is 0. Interpreting this for the grading on $A$, $B$, and $C$, we see that $\chi(B) - \chi(A) - \chi(C) = 0$ as claimed.

Thus the above triangle gives

$$\chi(K^*, L^*) = \chi(K^*, L^* \cup K^G) + \chi(K^G, L^G). \tag{1}$$

Putting $\chi(\varrho) = \chi(H^\varrho(K, L))$ we see, from the Smith sequence 3.3 with $\varrho = \sigma$, that

$$\chi(K, L) = \chi(\sigma) + \chi(\tau) + \chi(K^G, L^G). \tag{2}$$

Similarly, from (3.8) we see that

$$\begin{aligned} \chi(\tau) &= \chi(\tau^2) + \chi(\sigma), \\ \chi(\tau^2) &= \chi(\tau^3) + \chi(\sigma), \\ &\vdots \\ \chi(\tau^{p-2}) &= \chi(\tau^{p-1}) + \chi(\sigma), \\ \chi(\tau^{p-1}) &= \chi(\sigma). \end{aligned} \tag{3}$$

Adding (2) and all of (3), and canceling, gives

$$\chi(K, L) = p\chi(\sigma) + \chi(K^G, L^G).$$

Using (1) and the fact that $\chi(\sigma) = \chi(K^*, L^* \cup K^G)$ by (3.4), we have

$$\chi(K, L) = p\big(\chi(K^*, L^*) - \chi(K^G, L^G)\big) + \chi(K^G, L^G). \quad \blacksquare$$

*Remark* The results of this section and their extension to locally compact spaces are due to Floyd [3]. Some further developments were made by Heller [2]; also see Swan [2].

There is no difficulty in proving these results, and those of the next section, using cohomology rather than homology. Readers with minimal background in algebraic topology might well carry this out as practice in the use of cohomology.

## 5. HOMOLOGY SPHERES AND DISKS

We shall now apply the results of the preceding sections to the special cases of actions on homology spheres, disks, and acyclic complexes.

**5.1. Theorem** *If $G$ is a $p$-group ($p$ prime) and if $K$ is a finite-dimensional regular $G$-complex which is a* mod $p$ *homology $n$-sphere* [i.e., $H(K;\ \mathbf{Z}_p) \approx H(\mathbf{S}^n;\ \mathbf{Z}_p)$], *then $K^G$ is a* mod $p$ *homology $r$-sphere for some $-1 \le r \le n$ (where $r = -1$ means that $K^G$ is empty). If $p$ is odd, then $n - r$ is even.*

*Proof* Since a $p$-group is solvable (and, in fact, has a nontrivial center) there is a normal subgroup $G' \neq \{e\}, G$. Since $F(G, K) = F\big(G/G', F(G', K)\big)$, an induction on the order of $G$ shows that it suffices to treat the case in which $G$ is cyclic of order $p$. In this case 4.1 shows that $\operatorname{rk} H(K^G) \le \operatorname{rk} H(K) = 2$. The case $\operatorname{rk} H(K^G) = 1$ is impossible since, then $\chi(K^G) = 1 \not\equiv 0, 2 \pmod p$ contrary to 4.3. If $p$ is odd, then $\chi(K^G) \equiv \chi(K) \pmod p$ implies that $n - r$ is even. $\blacksquare$

A pair $(K, L)$ is called a mod $p$ **homology $n$-disk** if $H_i(K, L;\ \mathbf{Z}_p) = 0$ for $i \neq n$ and $H_n(K, L;\ \mathbf{Z}_p) \approx \mathbf{Z}_p$. By a proof which is similar to that of 5.1, but easier, we have the following result.

**5.2. Theorem** *Let $G$ be a $p$-group ($p$ prime) and let $K$ be a finite-dimensional regular $G$-complex and $L$ an invariant subcomplex. If $(K, L)$ is a* mod $p$ *homology $n$-disk, then $(K^G, L^G)$ is a* mod $p$ *homology $r$-disk for some $0 \leq r \leq n$. If $p$ is odd, then $n - r$ is even.* ▌

For later reference we shall show how to give a canonical isomorphism $H_n(K, L) \approx H_r(K^G, L^G)$ in the situation of 5.2. Similar considerations will hold for 5.1. We confine our attention to the case in which $G$ is cyclic of order $p$.

First note that the composition

$$H_n(K, L) \xrightarrow{\varrho_*} H_n{}^{\varrho}(K, L) \xrightarrow{i_*} H_n(K, L)$$

is just operation by $\varrho = \tau$ or $\sigma$. But operation by $G$ on $H_n(K, L)$ is trivial since $\mathbf{Z}_p$ has no automorphisms of order $p$. Thus $\varrho$ clearly induces the zero map $H_n(K, L) \to H_n(K, L)$, that is $i_*\varrho_* = 0$.

If $r = n$, then $H_n(K^G, L^G) \to H_n(K, L)$ must be an isomorphism, since $H^{\varrho}_{n+1}(K, L) = 0$ by 4.1. If $r < n$, then the exactness of

$$0 \longrightarrow H_n{}^{\varrho}(K, L) \xrightarrow{i_*} H_n(K, L) \xrightarrow{\bar{\varrho}_*} H_n{}^{\bar{\varrho}}(K, L)$$

shows that $i_*$ is injective and thus $\varrho_* = 0$ (since $i_*\varrho_* = 0$). Similarly $\bar{\varrho}_* = 0$ in degree $n$. Thus

$$i_*\colon\ H_n{}^{\varrho}(K, L) \xrightarrow{\approx} H_n(K, L).$$

The remainder of the Smith sequences show that

$$\begin{aligned}
\delta_*&\colon\ H_n{}^{\varrho}(K, L) \xrightarrow{\approx} H^{\bar{\varrho}}_{n-1}(K, L),\\
\delta_*&\colon\ H^{\bar{\varrho}}_{n-1}(K, L) \xrightarrow{\approx} H^{\varrho}_{n-2}(K, L),\\
&\ \ \vdots\\
\delta_*&\colon\ H^{\eta}_{r+1}(K, L) \xrightarrow{\approx} H_r(K^G, L^G),
\end{aligned}$$

where $\eta = \bar{\varrho}$ if $n - r$ is even and $\eta = \varrho$ if $n - r$ is odd (and necessarily $p = 2$ in the latter case). Thus the composition

$$\textbf{(5.3)}\quad H_n(K, L) \xleftarrow{\approx} H_n{}^{\varrho}(K, L) \xrightarrow{\approx} H^{\bar{\varrho}}_{n-1}(K, L) \xrightarrow{\approx} \cdots \xrightarrow{\approx} H^{\eta}_{r+1}(K, L) \xrightarrow{\approx} H_r(K^G, L^G)$$

is the desired explicit isomorphism. On the face of it, this isomorphism would seem to depend on the choice of $\varrho = \sigma$ or $\tau$. But it is not hard to prove that these give the same isomorphism, by using (3.6) and (3.7).

Note that the case $L = \emptyset$ and $n = 0$ of 5.2 shows that $K^G$ is mod $p$ acyclic (and *nonempty*) when $K$ is mod $p$ acyclic and finite-dimensional, and $G$ is a $p$-group. For $K^* = K/G$ there is a much stronger result.

**5.4. Theorem** *Let $G$ be any finite group and let $K$ be a finite-dimensional regular $G$-complex. Suppose that $K$ is acyclic over the integers. Then $K/G$ is also acyclic over the integers.*

*Proof* First, we shall show that, for each prime $p$, $K^*$ is mod $p$ acyclic. Also we consider, for the present, the case of a $p$-group $G$. Since, for $G'$ normal in $G$, we have $K/G \approx (K/G')/(G/G')$, an inductive argument reduces this case to the case in which $G$ is cyclic of order $p$. In this case 4.1, with $\varrho = \sigma$ and for each $n \geq 0$, shows that $H_n(K^*, K^G; \mathbf{Z}_p) = H_n{}^\sigma(K; \mathbf{Z}_p) = 0$ (for all $n$). Since $K^G$ is mod $p$ acyclic, this shows that $K^*$ is also mod $p$ acyclic.

Now for the case of general finite $G$, let $P$ be a Sylow $p$-group in $G$. Then $K/P$ is mod $p$ acyclic by the above remarks. But by (2.6) there are homomorphisms

$$H(K/G;\ \mathbf{Z}_p) \to H(K/P;\ \mathbf{Z}_p) \to H(K/G;\ \mathbf{Z}_p)$$

whose composition is multiplication by $|G|/|P|$. Since $|G|/|P|$ is prime to $p$ and $H(K/P;\ \mathbf{Z}_p) = 0$ in positive degrees, this shows that $K/G$ is mod $p$ acyclic (for all $p$ now).

The exact sequence (induced by $0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}_p \to 0$)

$$H_n(K^*;\ \mathbf{Z}) \xrightarrow{p} H_n(K^*; \mathbf{Z}) \to H_n(K^*; \mathbf{Z}_p) = 0$$

shows that $H_n(K^*; \mathbf{Z})$ is divisible. By (2.2) the composition

$$H_n(K^*; \mathbf{Z}) \to H_n(K; \mathbf{Z}) \to H_n(K^*; \mathbf{Z})$$

(through 0) is multiplication by $|G|$ and hence $H_n(K^*; \mathbf{Z}) = 0$ for $n > 0$. ∎

*Remark* Theorems 5.1 and 5.2 are essentially due to Smith [1]. The proofs given are due to Floyd [3], but the discussion below 5.2 is close to Smith's method. Theorem 5.4 is due to Floyd [10].

The following theorem is somewhat similar to 5.4, but substitutes an assumption on fixed point sets for the assumption of finite dimensionality, and has a strengthened conclusion. Note the case of *free* actions.

**5.5. Theorem** *Let $K$ be a regular $G$-complex, $G$ finite, and let $L$ be an invariant subcomplex. Assume that for a given prime $p$, $H_i(K^P, L^P; \mathbf{Z}_p) = 0$ for all $i \leq n$ and all $p$-subgroups $P$ of $G$ (including $P = \{e\}$). Then $H_i(K^*, L^*; \mathbf{Z}_p) = 0$ for $i \leq n$. Moreover, if this holds for all primes $p$ and also $H_i(K, L; \mathbf{Z}) = 0$ for $i \leq n$, then $H_i(K^*, L^*; \mathbf{Z}) = 0$ for $i \leq n$.*

*Proof* As in the proof of 5.4 we first consider the case in which $G$ is cyclic of prime order $p$, and prove only the first part. In this case let $k \leq n$ and suppose we have shown that $H_i^{\varrho}(K, L) = 0$ for $i < k$ and both $\varrho = \sigma$ and $\varrho = \tau$. The Smith sequence

$$0 = H_k(K, L) \to H_k^{\varrho}(K, L) \to H_{k-1}^{\bar{\varrho}}(K, L) \oplus H_{k-1}(K^G, L^G) = 0$$

shows that $H_k^{\varrho}(K, L) = 0$. Thus we conclude that

$$H_i(K^*, K^G \cup L^*) \approx H_i^{\sigma}(K, L) = 0 \qquad \text{for} \quad i \leq n.$$

Since $H_i(K^G \cup L^*, L^*) \approx H_i(K^G, L^G) = 0$ for $i \leq n$, by excision, the sequence

$$H_i(K^G \cup L^*, L^*) \to H_i(K^*, L^*) \to H_i(K^*, K^G \cup L^*)$$

shows that $H_i(K^*, L^*; \mathbf{Z}_p) = 0$ for $i \leq n$.

Now the usual inductive argument shows that this conclusion holds for all $p$-groups $G$. The proof for the general case now proceeds exactly as in the proof of 4.5 and will not be repeated. ▮

## 6. $G$-COVERINGS AND ČECH THEORY

In the next section we shall show how to generalize the results of the previous sections, by the use of Čech homology and cohomology, to actions on a large class of topological spaces. In the present section we shall give the necessary background concerning Čech theory and coverings of $G$-spaces. We shall assume that the reader is reasonably familiar with the construction of Čech theory as given, for example, in Eilenberg and Steenrod [1]. However, with one exception, we shall use only self-indexed open coverings in order to simplify notation and to avoid all logical technicalities. We consider only paracompact Hausdorff spaces.

First, we shall address ourselves to finding suitable types of open coverings of a paracompact $G$-space $X$, and $G$ is taken to be *finite* throughout this section.

If $\mathscr{U}$ is an open covering of $X$ and if $g$ is in $G$, then

$$g\mathscr{U} = \{g(U) \mid U \in \mathscr{U}\}$$

is also an open covering. If $g\mathscr{U} = \mathscr{U}$ for all $g$, we say that $\mathscr{U}$ is **invariant.** If $\mathscr{U}$ and $\mathscr{V}$ are coverings, then

$$\mathscr{U} \cap \mathscr{V} = \{U \cap V \mid U \in \mathscr{U},\ V \in \mathscr{V}\}$$

is a covering which refines both $\mathscr{U}$ and $\mathscr{V}$. Clearly

$$\bigcap_{g \in G} g\mathscr{U}$$

is an invariant cover refining $\mathscr{U}$. Moreover, this is locally finite if $\mathscr{U}$ is. Thus, for $X$ paracompact, the locally finite invariant coverings are cofinal in the set of all coverings of $X$.

We shall say that $X$ is **finitistic** if it satisfies the "Swan condition" that every covering has a finite-dimensional refinement. (The dimension of a covering is the dimension of its nerve, which is one less than the maximum number of members of the covering which intersect nontrivially.) Clearly, if $\dim \mathscr{U} \leq n$, then $\dim [\bigcap g\mathscr{U}] \leq (n+1)^{|G|} - 1$. In particular, if $X$ is *finitistic*, then the *finite-dimensional invariant coverings* are cofinal in the set of all coverings. Note that there are two main classes of finitistic spaces, the *compact* spaces and the *finite-dimensional* spaces.

Now let $\mathscr{U}$ be a locally finite invariant covering of $X$ and let

$$f = \{f_U \mid U \in \mathscr{U}\}$$

be a partition of unity subordinate to $\mathscr{U}$. Then $f$ is called a **$G$-partition of unity** if $f_{gU}(gx) = f_U(x)$ for all $g$, $x$, and $U$.

If $f = \{f_U\}$ is any partition of unity subordinate to the invariant covering $\mathscr{U}$, we define a $G$-partition $f'$ by putting

$$f_U'(x) = \frac{1}{|G|} \sum_g f_{gU}(gx).$$

Then

$$\sum_U f_U'(x) = \frac{1}{|G|} \sum_g \sum_U f_{gU}(gx) = \frac{1}{|G|} \sum_g 1 = 1,$$

and

$$f_{hU}'(hx) = \frac{1}{|G|} \sum_g f_{ghU}(ghx) = \frac{1}{|G|} \sum_{g'} f_{g'U}(g'x) = f_U'(x)$$

as claimed.

Now let $\mathbf{K}(\mathscr{U})$ be the nerve of the invariant covering $\mathscr{U}$ of $X$ (the simplicial complex with vertices the members $U \neq \varnothing$ of $\mathscr{U}$ and simplices $(U_0, \ldots, U_n)$ where $U_0 \cap \cdots \cap U_n \neq \varnothing$). Then $\mathbf{K}(\mathscr{U})$ is canonically a $G$-complex. Let $f = \{f_U\}$ be a $G$-partition of unity subordinate to $\mathscr{U}$ and let $\bar{f}$: $X \to |\mathbf{K}(\mathscr{U})|$ be the associated map with

$$\bar{f}(x) = \sum_U f_U(x)U.$$

Then $\bar{f}$ is *equivariant*, since

$$\begin{aligned}\bar{f}(gx) = \sum_U f_U(gx)U &= \sum_U f_{gU}(gx)gU \\ &= \sum_U f_U(x)gU = g\sum_U f_U(x)U = g\bar{f}(x).\end{aligned}$$

For any map $k$: $X \to |L|$ to a polyhedron, let $k^{-1}L$ denote the covering of $X$ by inverse images of open vertex stars of $|L|$. Suppose that $L$ is a $G$-complex and that $k$ is equivariant. Then $k^{-1}L$ is an invariant covering. Moreover, if $L$ satisfies condition (A′) of Section 1, then so does the nerve $\mathbf{K}(k^{-1}L)$. Similarly, if $L$ is a regular $G$-complex then so is $\mathbf{K}(k^{-1}L)$. [For these facts, it is convenient to interpret conditions (A′) and (B) of Section 1, for the nerve of a covering, directly in terms of the covering; see below.]

Returning to the equivariant map $\bar{f}$: $X \to |\mathbf{K}(\mathscr{U})|$, note that $\bar{f}^{-1}\mathbf{K}(\mathscr{U})$ is a refinement of $\mathscr{U}$ (in fact, it is the covering by the open sets $U' = \{x \mid f_U(x) \neq 0\} \subset U$). Let $L$ be the *second* barycentric subdivision of $\mathbf{K}(\mathscr{U})$ and regard the polyhedra $|L| = |\mathbf{K}(\mathscr{U})|$ as equal. Then the covering $\bar{f}^{-1}L$ is an invariant cover which is a refinement of $\mathscr{U}$ and its nerve $\mathbf{K}(\bar{f}^{-1}L)$ is canonically a regular $G$-complex. Also this covering is finite-dimensional when $\mathscr{U}$ is, since $L$ is finite-dimensional when $\mathbf{K}(\mathscr{U})$ is.

We shall call an invariant covering $\mathscr{U}$ a $G$-**covering** (sometimes called a **primitive covering**) if its nerve satisfies condition (A′) of Section 1; that is, if it satisfies the following condition:

*For $U \in \mathscr{U}$ and $g \in G$, $U \cap gU \neq \varnothing$ implies that $U = gU$.*

We shall call a $G$-covering $\mathscr{U}$ **regular** if its nerve is a regular $G$-complex, that is, if it satisfies the following condition for each subgroup $H$ of $G$:

*If $U_0, \ldots, U_n$ are members of $\mathscr{U}$ and $h_0, \ldots, h_n$ are in $H$ and if $U_0 \cap \cdots \cap U_n \neq \varnothing \neq h_0U_0 \cap \cdots \cap h_nU_n$, then there is an element $h$ of $H$ with $hU_i = h_iU_i$ for all $i$.*

Thus we have proved the following result.

**6.1. Theorem** *Let $X$ be a paracompact $G$-space, $G$ finite. Then the locally finite, regular $G$-coverings of $X$ are cofinal in the set of all coverings of $X$. If $X$ is also finitistic, then the finite-dimensional, regular $G$-coverings of $X$ are cofinal.* ▌

If $\mathscr{U}$ is a $G$-covering of $X$ let $\mathscr{U}/G$ be the set of orbits of $G$ on $\mathscr{U}$ (thus $U \in \mathscr{U}$ determines an element $U'$ of $\mathscr{U}/G$ and $U' = V'$ iff $U = gV$ for some $g$). Also $\mathscr{U}_G$ will denote the covering of $X/G$ by the sets $U^* = G(U)/G$ *but indexed by* $\mathscr{U}/G$ (thus if $G(U) = G(V)$ but $U \neq gV$ for any $g$, then $U^*$ and $V^*$ are regarded as different elements of the covering). Since an indexed covering and the associated self-indexed covering are refinements of one another, this will not effect homology or cohomology. Clearly, every covering $\mathscr{V}$ of $X/G$ is produced in this way since $(\pi^{-1}\mathscr{V})_G = \mathscr{V}$. If $\mathscr{U}$ is a regular $G$-covering of $X$ and if $U_0, \ldots, U_n$ are members of $\mathscr{U}$ with $U_0^* \cap \cdots \cap U_n^* \neq \varnothing$, then $g_0U_0 \cap \cdots \cap g_nU_n \neq \varnothing$ for some $g_i$ in $G$ and regularity of $\mathscr{U}$ implies that the *orbit* of the simplex $(g_0U_0, \ldots, g_nU_n)$ of $\mathbf{K}(\mathscr{U})$ is uniquely determined by the simplex $(U_0^*, \ldots, U_n^*)$ of $\mathbf{K}(\mathscr{U}_G)$. Thus we have the following fact.

**6.2. Proposition** *If $\mathscr{U}$ is a regular $G$-covering of $X$, then the assignment $\{gU \mid g \in G\} \mapsto U^* = G(U)/G$ gives an isomorphism of the simplicial complexes*

$$\mathbf{K}(\mathscr{U})/G \xrightarrow{\approx} \mathbf{K}(\mathscr{U}_G). \quad ▌$$

We need one further remark concerning $G$-coverings. Suppose that $\mathscr{U}$ and $\mathscr{V}$ are $G$-coverings and that $\mathscr{V}$ is a refinement of $\mathscr{U}$. Then we claim that there is a refinement projection $p: \mathscr{V} \to \mathscr{U}$ that is *equivariant*; that is, $V \subset p(V)$ and $p(gV) = gp(V)$. To see this, simply choose a representative out of each orbit of $G$ on $\mathscr{V}$ and define $p$ arbitrarily on these representatives [so that $V \subset p(V)$]. If $gV = g'V$, where $V$ is one of these representatives, then $[g^{-1}g'p(V)] \cap p(V) \supset [g^{-1}g'V] \cap V = V \neq \varnothing$ so that $gp(V) = g'p(V)$. Thus we can extend the definition by putting $p(gV) = gp(V)$.

Thus $G$-coverings, ordered by existence of *equivariant* refinement maps, form a *directed set.* This is *not true* of invariant coverings, as is easily seen. For this reason, we regard $G$-coverings as the basic notion, rather than invariant coverings.

Of course, a refinement projection $p: \mathscr{V} \to \mathscr{U}$ defines a simplicial map $\bar{p}: \mathbf{K}(\mathscr{V}) \to \mathbf{K}(\mathscr{U})$ which is equivariant when $p$ is equivariant. Also recall that two refinement projections $p_0, p_1: \mathscr{V} \to \mathscr{U}$ define contiguous simplicial maps $\bar{p}_0$ and $\bar{p}_1$ (see Eilenberg and Steenrod [1, p. 235]).

Let us now recall the definition of Čech homology and cohomology. For convenience, we shall restrict our attention to pairs $(X, A)$, where $X$ is paracompact and $A$ is *closed.* If $\mathscr{U}$ is a covering of $X$, then $\mathbf{K}(\mathscr{U} \mid A)$ will denote the subcomplex of $\mathbf{K}(\mathscr{U})$ consisting of those simplices $(U_0, \ldots, U_n)$ such that $U_0 \cap \cdots \cap U_n \cap A \neq \varnothing$, and $\bar{\mathbf{K}}(\mathscr{U} \mid A)$ will denote the corresponding *full* subcomplex consisting of those simplices $(U_0, \ldots, U_n)$ such that $U_i \cap A \neq \varnothing$ and $U_0 \cap \cdots \cap U_n \neq \varnothing$ in $X$.

If $\mathscr{V}$ is a refinement of $\mathscr{U}$ and $p: \mathscr{V} \to \mathscr{U}$ is any refinement projection, then the induced simplicial map $\bar{p}: \mathbf{K}(\mathscr{V}) \to \mathbf{K}(\mathscr{U})$ carries $\mathbf{K}(\mathscr{V} \mid A)$ into $\mathbf{K}(\mathscr{U} \mid A)$ and $\bar{\mathbf{K}}(\mathscr{V} \mid A)$ into $\bar{\mathbf{K}}(\mathscr{U} \mid A)$. Moreover, for any two such refinement projections $p, q: \mathscr{V} \to \mathscr{U}$, $\bar{p}$ and $\bar{q}$ are contiguous on $\mathbf{K}(\mathscr{V} \mid A) \to \mathbf{K}(\mathscr{U} \mid A)$ and on $\bar{\mathbf{K}}(\mathscr{V} \mid A) \to \bar{\mathbf{K}}(\mathscr{U} \mid A)$. This implies that the induced homology maps (arbitrary coefficients)

$$H(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid A)) \to H(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)),$$
$$H^*(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) \to H^*(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid A))$$

are independent of the choice of the refinement projection. Similarly, this holds with $\bar{\mathbf{K}}$ in place of $\mathbf{K}$. The definition of Čech homology and cohomology may be taken to be (since $A$ is closed)

$$\check{H}(X, A) = \varprojlim H(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)),$$
$$\check{H}^*(X, A) = \varinjlim H^*(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)),$$

where the limits are taken over all coverings $\mathscr{U}$ (or, of course, over any cofinal system of coverings). Now we claim that using $\bar{\mathbf{K}}$ rather than $\mathbf{K}$ produces the same groups. This is usually shown by producing cofinal coverings with $\mathbf{K} = \bar{\mathbf{K}}$ and in Eilenberg and Steenrod [1] it is done only for compact spaces. (However, it is not hard to generalize the proof to the paracompact case, as is well known.) However, this will not quite suffice for our purposes, and it is also more convenient to deduce this desired result from the following lemma which produces a refinement "suitable" for all subspaces $A \subset X$ simultaneously and "equivariantly."

Recall that a **star refinement** $\mathscr{V}$ of a covering $\mathscr{U}$ is a refinement possessing a refinement projection $p: \mathscr{V} \to \mathscr{U}$ such that $V' \cap V \neq \varnothing$ implies that $V' \subset p(V)$ (see Dugundji [1, p. 167]). We shall call such a refinement projection $p$ a **star projection.**

**6.3. Lemma** *If $\mathscr{U}$ is a covering of $X$ and if $\mathscr{V}$ is a star refinement of $\mathscr{U}$ with star projection $p: \mathscr{V} \to \mathscr{U}$, then for any subset $A$ of $X$, the image under*

*$\bar{p}$ of $\bar{\mathbf{K}}(\mathscr{V}\mid A)$ in $\bar{\mathbf{K}}(\mathscr{U}\mid A)$ is contained in $\mathbf{K}(\mathscr{U}\mid A)$. Moreover, if $X$ is a paracompact G-space and $\mathscr{U}$ is a G-covering, then there exists a G-covering $\mathscr{V}$ which is a star refinement of $\mathscr{U}$ and an equivariant star projection $p: \mathscr{V} \to \mathscr{U}$.*

***Proof*** If $(V_0, \ldots, V_n)$ is a simplex of $\bar{\mathbf{K}}(\mathscr{V}\mid A)$ (i.e., $V_i \cap A \neq \emptyset$ and $V_0 \cap \cdots \cap V_n \neq \emptyset$), then $V_0 \subset p(V_i)$ for all $i$. Thus

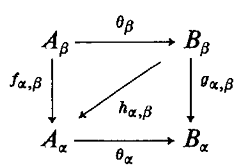
$$p(V_0) \cap \cdots \cap p(V_n) \supset V_0 \supset V_0 \cap A \neq \emptyset$$

so that $(p(V_0), \ldots, p(V_n))$ is a simplex of $\mathbf{K}(\mathscr{U}\mid A)$.

For the last statement, we may as well assume that $\mathscr{U}$ is locally finite. Let $f$ be a $G$-partition of unity subordinate to $\mathscr{U}$ and $\bar{f}: X \to |\mathbf{K}(\mathscr{U})|$ the associated equivariant map. Then if $L$ is the second barycentric subdivision of $\mathbf{K}(\mathscr{U})$, let $\mathscr{V} = \bar{f}^{-1}L$. Then $\mathscr{V}$ is a $G$-covering and it is well known to be a star refinement of $\mathscr{U}$ (see Dugundji [1, pp. 167–173]). The existence of a star projection which is equivariant follows from the discussion following 6.2; that is, choose $p$ arbitrarily on representatives $V$ for the orbits of $G$ on $\mathscr{V}$ such that $V' \subset p(V)$ for every $V'$ intersecting $V$ nontrivially, and then extend $p$ by equivariance: $p(gV) = gp(V)$. ▌

We also need the following algebraic lemma.

**6.4. Lemma** *Let $D$ be a directed set and let $\{A_\alpha, f_{\alpha,\beta}\}$ and $\{B_\alpha, g_{\alpha,\beta}\}$ be inverse systems of abelian groups based on $D$ (so that $f_{\alpha,\beta}: A_\beta \to A_\alpha$ and $g_{\alpha,\beta}: B_\beta \to B_\alpha$ for $\beta > \alpha$). Let $\{\theta_\alpha: A_\alpha \to B_\alpha\}$ be a homomorphism of directed systems* (i.e., $g_{\alpha,\beta}\theta_\beta = \theta_\alpha f_{\alpha,\beta}$). *Assume that for each index $\alpha$ there is an index $\beta > \alpha$ and a homomorphism $h_{\alpha,\beta}: B_\beta \to A_\alpha$ such that the diagram*




*commutes. Then the induced map*

$$\theta: \varprojlim A_\alpha \to \varprojlim B_\alpha$$

*is an isomorphism.*

*Similarly, the dual statement for direct limits of direct systems (reverse all the above arrows) holds.*

*Proof* Let $K_\alpha = \ker \theta_\alpha$ and $C_\alpha = \operatorname{coker} \theta_\alpha$ so that

$$0 \to K_\alpha \to A_\alpha \to B_\alpha \to C_\alpha \to 0$$

is exact for each $\alpha$. If $\beta > \alpha$ is such that $h_{\alpha,\beta}$ exists, then it is clear that the induced maps $K_\beta \to K_\alpha$ and $C_\beta \to C_\alpha$ are zero, giving the diagram

$$\begin{array}{ccccccccc} 0 \to & K_\beta & \to & A_\beta & \to & B_\beta & \to & C_\beta & \to 0 \\ & {\scriptstyle 0}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & \\ 0 \to & K_\alpha & \to & A_\alpha & \to & B_\alpha & \to & C_\alpha & \to 0. \end{array}$$

Let us write $\beta \gg \alpha$ when $K_\beta \to K_\alpha$ and $C_\beta \to C_\alpha$ are both zero. (In particular, we have that $\lim K_\beta = 0 = \lim C_\beta$. In the direct limit case this implies the result immediately since the direct limit is an exact functor. This is also the case when the inverse limit functor is exact; e.g., when $A_\alpha$ and $B_\alpha$ are finite-dimensional vector spaces over a field. This would suffice for our purposes, but we shall prove the general case for inverse limits, regardless. The proof is taken from Bredon [16, p. 12].)

If $a = \{a_\alpha\} \in \lim_{\leftarrow} A_\alpha$ (that is, $f_{\alpha,\beta} a_\beta = a_\alpha$ for all $\beta > \alpha$), and if $\theta a = 0$, then $a_\beta \in K_\beta$ and $a_\alpha = f_{\alpha,\beta} a_\beta = 0$ for $\beta \gg \alpha$ so that $a = 0$ and $\theta$ is a monomorphism. Now suppose that $b = \{b_\alpha\} \in \lim_{\leftarrow} B_\alpha$. If $\beta \gg \alpha$, then $b_\alpha = g_{\alpha,\beta} b_\beta = \theta_\alpha(a_\alpha')$ for some $a_\alpha' \in A_\alpha$. Choose such an $a_\alpha'$ for each $\alpha$. If $\beta \gg \alpha$, put $a_\alpha = f_{\alpha,\beta} a_\beta'$. This is well defined since if $\gamma > \beta \gg \alpha$, then

$$\theta_\beta(f_{\beta,\gamma}(a_\gamma') - a_\beta') = g_{\beta,\gamma}\theta_\gamma(a_\gamma') - \theta_\beta a_\beta' = g_{\beta,\gamma} b_\gamma - b_\beta = 0,$$

which implies that

$$0 = f_{\alpha,\beta}(f_{\beta,\gamma}(a_\gamma') - a_\beta') = f_{\alpha,\gamma}(a_\gamma') - f_{\alpha,\beta}(a_\beta'),$$

by definition of $\beta \gg \alpha$. Also, for $\beta \gg \alpha$,

$$\theta_\alpha(a_\alpha) = \theta_\alpha f_{\alpha,\beta}(a_\beta') = g_{\alpha,\beta}\theta_\beta(a_\beta') = g_{\alpha,\beta} b_\beta = b_\alpha$$

and, for any $\gamma > \alpha$ (and taking $\beta \gg \gamma$), we have

$$f_{\alpha,\gamma}(a_\gamma) = f_{\alpha,\gamma} f_{\gamma,\beta}(a_\beta') = f_{\alpha,\beta}(a_\beta') = a_\alpha.$$

Thus $a = \{a_\alpha\}$ defines an element of $\lim_{\leftarrow} A_\alpha$ with $\theta(a) = b$. ∎

Note that it follows from 6.3 and 6.4 that if $X$ is paracompact and if $A \supset B$ are closed subsets of $X$, then, for coverings of $X$, the inclusions of pairs

$$(\mathbf{K}(\mathscr{U} \mid A),\ \mathbf{K}(\mathscr{U} \mid B)) \to (\bar{\mathbf{K}}(\mathscr{U} \mid A),\ \bar{\mathbf{K}}(\mathscr{U} \mid B))$$

induce isomorphisms in the limit on the homology or cohomology of these simplicial pairs

$$\check{H}(A, B) \approx \lim_{\leftarrow} H(\bar{\mathbf{K}}(\mathscr{U} \mid A),\ \bar{\mathbf{K}}(\mathscr{U} \mid B)),$$
$$\check{H}^*(A, B) \approx \lim_{\rightarrow} H^*(\bar{\mathbf{K}}(\mathscr{U} \mid A),\ \bar{\mathbf{K}}(\mathscr{U} \mid B)).$$

(Here we use the obvious fact that $\check{H}(A, B)$ is canonically isomorphic to $\lim_{\leftarrow} H(\mathbf{K}(\mathscr{U} \mid A),\ \mathbf{K}(\mathscr{U} \mid B))$, and similarly for cohomology, *since* $A$ and $B$ are *closed*; see Eilenberg and Steenrod [1, p. 249].)

The importance of 6.3 and 6.4, for us, lies in the following fact and its consequences 6.6 and 6.7.

**6.5. Proposition** *Let $\mathscr{U}$ be a covering of the G-space $X$ and let $A \subset X$ be a closed invariant subspace. Then there exists a G-covering $\mathscr{V}$ refining $\mathscr{U}$ such that*

$$\mathbf{K}(\mathscr{V} \mid A^G) \subset \mathbf{K}(\mathscr{V} \mid A)^G \subset \bar{\mathbf{K}}(\mathscr{V} \mid A^G)$$

*and*

$$\mathbf{K}(\mathscr{V} \mid X^G) \subset \mathbf{K}(\mathscr{V})^G = \bar{\mathbf{K}}(\mathscr{V} \mid X^G).$$

*Proof* We may take $\mathscr{U}$ to be a $G$-covering. The first inclusion means that if $V \cap A^G \neq \emptyset$ (where $V$ is in $\mathscr{V}$), then $gV = V$ for all $g$, and this always holds for a $G$-covering. The second inclusion means that if $V \cap A \neq \emptyset$ and $gV = V$ for all $g$, then $V \cap A^G \neq \emptyset$. The equality means that $V \cap X^G \neq \emptyset$ iff $gV = V$ for all $g$. If $x \in X - X^G$ let $V_x$ be a small (contained in a member of $\mathscr{U}$) neighborhood of $x$ which is invariant under the action of $G_x$ and is such that $gV_x \cap V_x = \emptyset$ for $g \notin G_x$. [Note then that $gV_x \cap g'V_x \neq \emptyset$ iff $gV_x = g'V_x$ and $g(x) = g'(x)$.] If $x \in X^G$, let $V_x$ be a small invariant neighborhood of $x$, and if $x \in X^G - A$, require that $V_x \cap A = \emptyset$. Then the $V_x$ and their translates by $G$ clearly provide the desired $G$-covering $\mathscr{V}$. ▌

*Remark* Clearly we also have that $\bar{\mathbf{K}}(\mathscr{V} \mid A)^G = \bar{\mathbf{K}}(\mathscr{V} \mid A^G)$ in the situation of 6.5.

From 6.3–6.5 we obtain (see the method of proof of 6.7) the following theorem.

**6.6. Theorem** *If X is a paracompact G-space and if $A \subset X$ is a closed invariant subspace, then there are canonical isomorphisms*

$$\check{H}(X^G, A^G) \approx \varprojlim H(\mathbf{K}(\mathscr{U})^G, \mathbf{K}(\mathscr{U} \mid A)^G),$$
$$\check{H}^*(X^G, A^G) \approx \varinjlim H^*(\mathbf{K}(\mathscr{U})^G, \mathbf{K}(\mathscr{U} \mid A)^G),$$

*where $\mathscr{U}$ ranges over the G-coverings of X.* ▌

**6.7. Theorem** *Let X be a paracompact G-space and $A \subset X$ a closed invariant subspace. Then there are natural isomorphisms*

$$\check{H}(X^*, A^* \cup X^G) \approx \varprojlim H\left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U})^G}{G}\right),$$
$$\check{H}^*(X^*, A^* \cup X^G) \approx \varinjlim H^*\left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U})^G}{G}\right),$$

*where $\mathscr{U}$ ranges over the G-coverings of X.*

***Proof*** For $\mathscr{U}$ regular, the isomorphism $\mathbf{K}(\mathscr{U})/G \approx \mathbf{K}(\mathscr{U}_G)$ of 6.2 clearly induces

$$\frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U} \mid X^G)}{G} = \frac{\mathbf{K}(\mathscr{U} \mid (A \cup X^G))}{G} \approx \mathbf{K}(\mathscr{U}_G \mid (A^* \cup X^G)).$$

Thus the desired homology (and cohomology) is canonically isomorphic to that based on the pairs

$$\left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U} \mid X^G)}{G}\right).$$

By 6.5 there is a cofinal system of $G$-coverings $\mathscr{U}$ such that

$$\left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U} \mid X^G)}{G}\right) \subset \left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U})^G}{G}\right)$$
$$\subset \left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\bar{\mathbf{K}}(\mathscr{U} \mid A) \cup \bar{\mathbf{K}}(\mathscr{U} \mid X^G)}{G}\right).$$

By 6.3 there is a refinement $\mathscr{V}$ of $\mathscr{U}$ and an equivariant refinement projection which takes the right-hand pair (and hence the middle pair) for $\mathscr{V}$ into the left-hand pair for $\mathscr{U}$. Thus 6.4 implies that these inclusions induce isomorphisms on the limits of their homology and cohomology. ▌

*Remarks* Of course, we may take the limits in 6.6 and 6.7 over any cofinal system of $G$-coverings and thus can restrict the coverings to be regular (and finite-dimensional when $X$ is finitistic) even though we have not claimed 6.5 for regular $\mathscr{V}$. It is probably true that one can find a cofinal set of *regular* coverings $\mathscr{V}$ such that the inclusions of 6.5 are *equalities*. However, this is not necessary and would be difficult. In the case of cyclic groups of prime order, coverings with properties similar to this were constructed by Smith [1] and called "special coverings."

Note that if $f\colon X \to Y$ is an equivariant map of $G$-spaces and if $\mathscr{U}$ is a $G$-covering of $Y$, then $f^{-1}\mathscr{U} = \{f^{-1}(U) \mid U \in \mathscr{U}\}$ is a $G$-covering of $X$, and it is regular if $\mathscr{U}$ is regular. This will imply the naturality of constructions based on passage to limits over $G$-coverings or over regular $G$-coverings.

## 7. FINITE GROUP ACTIONS ON GENERAL SPACES

We shall now use the developments in Section 6 to extend the results of the previous sections to actions of finite groups on a wide class of topological spaces.

Let $X$ be a paracompact $G$-space, $G$ finite, and $A \subset X$ a closed invariant subspace. Let $\mathscr{U}$ be a regular $G$-covering of $X$. Then we have the transfer homomorphism

$$\mu_*\colon H(\mathbf{K}(\mathscr{U})/G,\ \mathbf{K}(\mathscr{U} \mid A)/G) \to H(\mathbf{K}(\mathscr{U}),\ \mathbf{K}(\mathscr{U} \mid A))$$

satisfying (2.2). If $\mathscr{V}$ is a regular $G$-covering refining $\mathscr{U}$, then there is an *equivariant* refinement projection $p\colon \mathscr{V} \to \mathscr{U}$ and we have the induced commutative diagram

$$\begin{array}{ccc} H(\mathbf{K}(\mathscr{V})/G, \mathbf{K}(\mathscr{V} \mid A)/G) & \xrightarrow{\mu_*} & H(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid A)) \\ \downarrow & & \downarrow \\ H(\mathbf{K}(\mathscr{U})/G, \mathbf{K}(\mathscr{U} \mid A)/G) & \xrightarrow{\mu_*} & H(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) \end{array}$$

by naturality of the transfer, and the vertical maps are independent of the choice of $p$ since contiguous equivariant simplicial maps $\mathbf{K}(\mathscr{V}) \to \mathbf{K}(\mathscr{U})$ clearly induce contiguous maps on $\mathbf{K}(\mathscr{V})/G \to \mathbf{K}(\mathscr{U})/G$.

Thus passage to the limit gives a transfer

$$\mu_*\colon \check{H}(X^*, A^*) \to \check{H}(X, A)$$

(where $X^* = X/G$ and $A^* = A/G$), which satisfies the analog of (2.2)

$$\textbf{(7.1)} \qquad \begin{aligned} \pi_*\mu_* &= |G|: \check{H}(X^*, A^*) \to \check{H}(X^*, A^*), \\ \mu_*\pi_* &= \sigma_* = \textstyle\sum g_*: \check{H}(X, A) \to \check{H}(X, A). \end{aligned}$$

Similarly, one can define the analog of the generalized transfer (2.5) and also the duals in cohomology. Clearly we have the following theorem.

**7.2. Theorem** *If X is a paracompact G-space, G finite, and A is a closed invariant subspace, then Theorem* 2.4 *holds for Čech homology with* (*K*, *L*) *replaced by* (*X*, *A*). *Also the dual statement in Čech cohomology holds.* ▌

Now we consider the generalization of the Smith sequences. Thus $G$ will now denote a *cyclic group of prime order p* and we work only with *coefficients in* $\mathbf{Z}_p$ *unless otherwise specified.*

Let $\mathscr{U}$ be a $G$-covering of $X$ and consider the special groups

$$H^\varrho(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)),$$
$$H_\varrho{}^*(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)),$$

where $\varrho = \tau^i$, $1 \le i \le p - 1$. These groups are natural with respect to equivariant maps as noted in Section 3. Since any $G$-covering $\mathscr{V}$ refining $\mathscr{U}$ possesses an *equivariant* refinement projection, we have the induced maps

$$H^\varrho(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid A)) \to H^\varrho(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)),$$
$$H_\varrho{}^*(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) \to H_\varrho{}^*(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid A)).$$

Any two equivariant refinement projections induce contiguous equivariant simplicial maps and the induced chain maps are *equivariantly* chain homotopic and, as noted at the end of Section 3, this implies that the above homology maps are independent of the choice of the equivariant refinement projection. Consequently, we can define

$$\textbf{(7.3)} \qquad \begin{aligned} \check{H}^\varrho(X, A) &= \varprojlim H^\varrho(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)), \\ \check{H}_\varrho{}^*(X, A) &= \varinjlim H_\varrho{}^*(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)), \end{aligned}$$

where $\mathscr{U}$ ranges over the $G$-coverings of $X$.

Since the Smith triangles 3.3 and (3.9) are natural (and by 6.6) we obtain the induced triangles

**(7.4)**
$$\begin{array}{ccc} & \check{H}(X, A) & \\ {\scriptstyle \varrho_*}\swarrow & & \nwarrow{\scriptstyle i_*} \\ \check{H}^{\varrho}(X, A) & \xrightarrow[\text{degree}-1]{\delta_*} & \check{H}^{\bar\varrho}(X, A) \oplus \check{H}(X^G, A^G) \end{array}$$

**(7.5)**
$$\begin{array}{ccc} & \check{H}^*(X, A) & \\ {\scriptstyle \varrho^*}\nearrow & & \searrow{\scriptstyle i^*} \\ \check{H}_{\varrho}{}^*(X, A) & \xleftarrow[\text{degree}+1]{\delta^*} & \check{H}_{\bar\varrho}{}^*(X, A) \oplus \check{H}^*(X^G, A^G). \end{array}$$

Since the direct limit functor is exact, (7.5) is *always exact* for $X$ paracompact and $A$ closed. However, restrictions must be placed on $X$ to insure that (7.4) is exact. Since the inverse limit of an exact sequence of *finite-dimensional* vector spaces over a field, $\mathbf{Z}_p$ here, is exact (see Eilenberg and Steenrod [1, p. 226]), the triangle (7.4) will be exact if $X$ is *compact* and $A$ is closed (since the *finite* $G$-coverings will then be cofinal).

Clearly we also have the generalization of the triangle (3.8) and its analog in cohomology. These will not be displayed.

Note that the isomorphism

$$H^{\sigma}(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) \approx H\left(\frac{\mathbf{K}(\mathscr{U})}{G}, \frac{\mathbf{K}(\mathscr{U} \mid A) \cup \mathbf{K}(\mathscr{U})^G}{G}\right)$$

of (3.4) is natural and similarly for the dual (3.10). Thus, by 6.7, we have

**(7.6)**
$$\begin{aligned} \check{H}^{\sigma}(X, A) &\approx \check{H}(X^*, A^* \cup X^G), \\ \check{H}_{\sigma}{}^*(X, A) &\approx \check{H}^*(X^*, A^* \cup X^G). \end{aligned}$$

Moreover the homology (and cohomology) diagrams induced by (3.5) are natural, as is the excision isomorphism $H(K^G \cup L^*, L^*) \approx H(K^G, L^G)$. Consequently (3.5), 6.6, and 6.7 provide the commutative diagrams

**(7.7)**
$$\begin{array}{ccccccccc} \cdots \to & \check{H}^{\sigma}_{n+1}(X, A) & \to & \check{H}_n{}^{\tau}(X, A) \oplus \check{H}_n(X^G, A^G) & \to & \check{H}_n(X, A) & \to & \check{H}_n{}^{\sigma}(X, A) & \to \cdots \\ & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle 0+1} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \approx} & \\ \cdots \to & \check{H}_{n+1}(X^*, A^* \cup X^G) & \to & \check{H}_n(X^G, A^G) & \to & \check{H}_n(X^*, A^*) & \to & \check{H}_n(X^*, A^* \cup X^G) & \to \cdots \end{array}$$

and

**(7.8)**
$$\begin{array}{ccccccccc} \cdots \to & \check{H}^n(X^*, A^* \cup X^G) & \to & \check{H}^n(X^*, A^*) & \to & \check{H}^n(X^G, A^G) & \to & \check{H}^{n+1}(X^*, A^* \cup X^G) & \to \cdots \\ & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle \approx} & \\ \cdots \to & \check{H}_{\sigma}{}^n(X, A) & \to & \check{H}^n(X, A) & \to & \check{H}_{\tau}{}^n(X, A) \oplus \check{H}^n(X^G, A^G) & \to & \check{H}_{\sigma}^{n+1}(X, A) & \to \cdots \end{array}$$

Similarly, we have the homology and cohomology diagrams induced by (3.6) and (3.7), which will not be displayed.

Now we turn to the question of generalizing the results of Sections 4 and 5. Basically, these results carry over in *cohomology* to all finitistic spaces $X$ and closed invariant subspaces $A$ in place of $(K, L)$, and, with two exceptions, in *homology* to all *compact* $(X, A)$. To avoid any possible misinterpretation, however, we shall restate all these results. For the main part, the proofs follow exactly the lines of their analogs in Sections 4 and 5 and we shall only comment on the few extra justifications necessary.

**7.9. Theorem** *Let $X$ be a finitistic $G$-space, $G$ cyclic of prime order $p$, and let $A$ be a closed invariant subspace. Then, with coefficients in $\mathbf{Z}_p$,*

$$\operatorname{rk} \check{H}_\varrho^n(X, A) + \sum_{i \geq n} \operatorname{rk} \check{H}^i(X^G, A^G) \leq \sum_{i \geq n} \operatorname{rk} \check{H}^i(X, A).$$

*The corresponding statement in homology holds when $X$ is compact.*

*Proof* The proof proceeds as in that of 4.1. However, we must justify the downwards induction from infinity. Thus, assume that $\check{H}^i(X, A) = 0$ for $i \geq N$. We claim then that $\check{H}_\varrho^i(X, A) = 0 = \check{H}^i(X^G, A^G)$ for $i \geq N$. Suppose not. Then there exists a *finite-dimensional* $G$-covering $\mathscr{U}$ of $X$ so fine that, with $K = \mathbf{K}(\mathscr{U})$ and $L = \mathbf{K}(\mathscr{U} \mid A)$,

$$H_\varrho^i(K, L) \oplus H^i(K^G, L^G) \to \check{H}_\varrho^i(X, A) \oplus \check{H}^i(X^G, A^G)$$

is nonzero for some such $i \geq N$.

Consider the commutative diagram

$$\begin{array}{ccccc} H_\varrho^i(K, L) \oplus H^i(K^G, L^G) & \xrightarrow{\delta^*} & H_{\bar{\varrho}}^{i+1}(K, L) & \to & H_\varrho^r(K, L) = 0 \\ \downarrow & & \downarrow & & \downarrow \\ \check{H}_\varrho^i(X, A) \oplus \check{H}^i(X^G, A^G) & \xrightarrow{\delta^*} & \check{H}_{\bar{\varrho}}^{i+1}(X, A) & \to & \check{H}_\varrho^r(X, A) \end{array}$$

where the horizontal maps are compositions of the maps $\delta^*: H_\varrho^j \to H_{\bar{\varrho}}^{j+1}$ and those with $\varrho$, $\bar{\varrho}$ interchanged (alternating), and where $r > \dim K$. Since $\check{H}^j(X, A) = 0$ for $j \geq N$ the bottom row of this diagram consists of *monomorphisms*, and this contradicts the nontriviality of the left-hand vertical map. The dual proof in homology will be left to the reader. This clearly justifies using the proof of 4.1, and, of course, is a special case of the statement of 7.9 itself. ∎

**7.10. Theorem** *Let $X$ be a finitistic $G$-space, $G$ cyclic of prime order $p$, and $A$ a closed invariant subspace. If* $\operatorname{rk} \check{H}^*(X, A; \mathbf{Z}_p) < \infty$ *and the euler characteristics are defined in terms of* mod $p$ *Čech cohomology, we have*

$$\chi(X, A) + (p-1)\chi(X^G, A^G) = p\chi(X^*, A^*).$$

*This also holds for homology when $X$ is compact.* ∎

**7.11. Theorem** *If $G$ is a $p$-group ($p$ prime) and if $X$ is a finitistic $G$-space which is a* mod $p$ *Čech cohomology $n$-sphere, then $X^G$ is a* mod $p$ *Čech cohomology $r$-sphere for some $-1 \le r \le n$. Similarly, if $A \subset X$ is closed and invariant and if $(X, A)$ is a* mod $p$ *Čech cohomology $n$-disk, then $(X^G, A^G)$ is a* mod $p$ *Čech cohomology $r$-disk for some $0 \le r \le n$. In both cases, $n - r$ is even when $p$ is odd. The analogous statements in Čech homology hold when $X$ is compact.* ∎

We remark that the discussion following 5.2 also applies to give explicit isomorphisms in the situation of 7.11. The proof of 5.4 gives the next theorem.

**7.12. Theorem** *Let $G$ be any finite group and let $X$ be a finitistic $G$-space which is acyclic with respect to integral Čech cohomology. Then $X/G$ is also acyclic in integral Čech cohomology.* ∎

**Example** That 7.12 does not hold in homology, even for compact spaces, is shown by the following example. Consider $\mathbf{S}^2$ as the (unreduced) suspension of $\mathbf{S}^1$, the unit circle in the complex plane. Let $f\colon \mathbf{S}^2 \to \mathbf{S}^2$ be the suspension of the map $z \mapsto z^3$ of $\mathbf{S}^1 \to \mathbf{S}^1$. Then $f$ is of degree 3 and clearly commutes with the antipodal map $g\colon \mathbf{S}^2 \to \mathbf{S}^2$. Thus, regarding $\mathbf{S}^2$ as a $\mathbf{Z}_2$-space via the antipodal map $g$, we may form the $\mathbf{Z}_2$-space $\Sigma$ as the inverse limit of

$$\cdots \xrightarrow{f} \mathbf{S}^2 \xrightarrow{f} \mathbf{S}^2 \xrightarrow{f} \mathbf{S}^2.$$

Then $\Sigma/\mathbf{Z}_2$ is the inverse limit of

$$\cdots \xrightarrow{f^*} \mathbf{P}^2 \xrightarrow{f^*} \mathbf{P}^2 \xrightarrow{f^*} \mathbf{P}^2.$$

Now $\check{H}_2(\Sigma; \mathbf{Z}) = 0$ since it is the inverse limit of

$$\cdots \xrightarrow{3} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{3} \mathbf{Z}$$

and hence $\Sigma$ is *acyclic over* $\mathbf{Z}$. However, $f^*$: $\mathbf{P}^2 \to \mathbf{P}^2$ induces an isomorphism in homology [since $3 \equiv 1 \pmod 2$] and thus $\check{H}_1(\Sigma/\mathbf{Z}_2;\ \mathbf{Z}) \approx \mathbf{Z}_2$. [Note, of course, that $\Sigma$ is *not* acyclic over $\mathbf{Z}_2$, however, since $\check{H}_2(\Sigma;\ \mathbf{Z}_2) \approx \mathbf{Z}_2$.] It can be shown that 7.12 would hold for homology, with $X$ *compact*, if one also assumes that $X$ is homologically acyclic over $\mathbf{Z}_p$ for all primes $p$. This follows from the fact that a compact space having trivial mod $p$ Čech homology for all $p$ also has trivial integral Čech homology (the same statement in cohomology is false). However, this contention is not easy to prove and, since we have no use for it, we shall not give the proof. Similar remarks apply to the next theorem which, for cohomology, follows from the proof of 5.5.

**7.13. Theorem** *Let $X$ be a paracompact $G$-space, $G$ finite, and let $A$ be a closed invariant subspace. Assume that, for a given prime $p$, $\check{H}^i(X^P, A^P;\ \mathbf{Z}_p) = 0$ for all $i \leq n$ and all $p$-subgroups $P$ of $G$. Then $\check{H}^i(X^*, A^*;\ \mathbf{Z}_p) = 0$ for all $i \leq n$. If this holds for all primes $p$ and also $\check{H}^i(X, A;\ \mathbf{Z}) = 0$ for all $i \leq n$, then $\check{H}^i(X^*, A^*;\ \mathbf{Z}) = 0$ for $i \leq n$.* ▌

The main case of interest, for this theorem and the following corollary, is that of *free* actions. In that case it will be used in the next section to study those groups which can act freely on spheres.

**7.14. Corollary** *Let $X$ and $Y$ be paracompact $G$-spaces, $G$ finite, and let $f$: $X \to Y$ be an equivariant map. For a given prime $p$, suppose that*

$$f^*:\ \check{H}^i(Y^P;\ \mathbf{Z}_p) \to \check{H}^i(X^P;\ \mathbf{Z}_p)$$

*is an isomorphism for $i < n$ and a monomorphism for $i = n$, for all $p$-subgroups $P$ of $G$. Then*

$$(f/G)^*:\ \check{H}^i(Y^*;\ \mathbf{Z}_p) \to \check{H}^i(X^*;\ \mathbf{Z}_p)$$

*is also an isomorphism for $i < n$ and a monomorphism for $i = n$. If this holds for all primes $p$ and if $f^*$: $\check{H}^i(Y;\ \mathbf{Z}) \to \check{H}^i(X;\ \mathbf{Z})$ is also an isomorphism for $i < n$ and a monomorphism for $i = n$, then the same is true of $(f/G)^*$ with integral coefficients.*

***Proof*** This follows easily from 7.13 applied to the mapping cylinder $M_f$ of $f$: $X \to Y$ modulo the top $X$, and using the exact sequence

$$\cdots \to \check{H}^{i-1}(X) \to \check{H}^i(M_f, X) \to \check{H}^i(Y) \xrightarrow{f^*} \check{H}^i(X) \to \cdots$$

(and similarly for $X^* \to Y^*$ and $X^P \to Y^P$) which is obtained from the exact sequence of the pair $(M_f, X)$ by replacing $M_f$ by its deformation retract $Y$. ∎

**7.15. Corollary** *Let $G$ be a finite group acting freely on the connected and compact spaces $X$ and $Y$. Suppose that $\check{H}^i(X; \mathbf{Z}) = 0$ and $\check{H}^i(Y; \mathbf{Z}) = 0$ for all $0 < i < n$. Then there is a canonical isomorphism*

$$\check{H}^i(X/G; \mathbf{Z}) \approx \check{H}^i(Y/G; \mathbf{Z})$$

*for $i < n$. Similarly this result holds with $\mathbf{Z}$ replaced by $\mathbf{Z}_p$ throughout, for any prime $p$.*

*Proof* Consider the equivariant projections $X \leftarrow X \times Y \to Y$, where $G$ acts diagonally on $X \times Y$. These induce isomorphisms in cohomology in degrees less than $n$ and monomorphisms in degree $n$. By 7.14 the same is true of the induced maps

$$X/G \leftarrow X \times_G Y \to Y/G$$

on the orbit spaces. ∎

*Remark* Since $X$ and $Y$ are compact, the Universal Coefficient Theorem implies that the conclusion of 7.15 holds with arbitrary coefficients. Also note that such $G$-spaces $X$ with trivial cohomology below degree $n$, for arbitrary $n$, always exist; it suffices to take $X$ to be the join of $n + 1$ copies of $G$. Corollary 7.15 shows that $\check{H}^i(X/G; \mathbf{Z})$ depends only on $G$ (for $n > i$). This group is called the "$i$th cohomology group $H^i(G; \mathbf{Z})$ of $G$." One can define it in an algebraic way, of course, and for general $G$-modules as coefficients (see Cartan and Eilenberg [1] and MacLane [1]).

*Remark* One may extend somewhat the results concerning *homology* by using compactly supported Čech homology. This is defined to be

$$\check{H}^c(X, A) = \varinjlim \check{H}(B, C)$$

where $(B, C)$ ranges over the *compact* subspaces of $(X, A)$. This gives exact Smith sequences for all pairs $(X, A)$ of $G$-spaces. Of course, for most of the results, one would have to justify the induction from infinity used in the proofs of most of the theorems. This would require an assumption, in the nature of finite dimensionality of $X$, sufficiently strong to draw the conclusion that $\check{H}_i^c(X^*, A^* \cup X^G) = 0$ for sufficiently large $i$. We shall not develop the theory in this direction since the cohomological results are at least as strong as anything obtainable with homology, and are considerably simpler.

***Remark*** The examples of Chapter I, Section 7 show that the use of mod $p$ coefficients in 7.9 and 7.11 is essential. Also the examples of Chapter I, Section 8 show that results similar to 7.9 and 7.11 are not obtainable for actions of finite groups of composite order, in general. Also the assumption that the spaces involved are finitistic is essential. To see this, note that the union $\mathbf{S}^\infty$ of the spheres $\mathbf{S}^0 \subset \mathbf{S}^1 \subset \mathbf{S}^2 \subset \cdots$ (as a CW-complex) carries a *free* involution (the antipodal map). Also $\mathbf{S}^\infty$ is contractible since its homotopy groups are obviously trivial. If $K$ is any polyhedron, then the join $\mathbf{S}^\infty \star K$ is also contractible and carries an involution with fixed set $K$ and orbit space $\mathbf{P}^\infty \star K$. Also $(\mathbf{S}^\infty \star K) \times \mathbf{S}^n$ is a homology $n$-sphere and carries an involution with fixed set $K \times \mathbf{S}^n$. Thus 7.9–7.12 fail miserably without the finitistic assumption.

## 8. GROUPS ACTING FREELY ON SPHERES

In this section we discuss the problem of determining which compact Lie groups $G$ (including finite groups) can act freely on some sphere $\mathbf{S}^n$.

If $n$ is even, the problem is almost trivial since the Lefschetz Fixed Point Theorem then implies that each element $g \neq e$ of $G$ must reverse orientation. Since the composition of two orientation reversing homeomorphisms preserves orientation, $G$ must be $\mathbf{Z}_2$ or trivial.

When $n$ is odd the problem is considerably more difficult and, in fact, a complete solution is not known. First we prove a well-known result which is originally due to Smith [7], but which can be found in various other places, for example, see Cartan and Eilenberg [1, p. 358].

**8.1. Theorem** *The group $\mathbf{Z}_p \oplus \mathbf{Z}_p$ cannot act freely on a finitistic* mod $p$ *cohomology n-sphere $\Sigma^n$.*

***Proof*** Let $N > n$ be an odd integer and consider the usual free action of $\mathbf{Z}_p$ on $\mathbf{S}^N$ (the generator of which is given by the diagonal complex matrix $\mathrm{diag}(e^{2\pi i/p}, \ldots, e^{2\pi i/p})$ acting on $\mathbf{C} \times \cdots \times \mathbf{C} = \mathbf{C}^k \supset \mathbf{S}^N$, where $N = 2k - 1$). Then $G = \mathbf{Z}_p \oplus \mathbf{Z}_p$ acts freely, in the obvious way, on $\mathbf{S}^N \times \mathbf{S}^N$ with orbit space $\mathbf{L}^N \times \mathbf{L}^N$, where $\mathbf{L}^N = \mathbf{S}^N/\mathbf{Z}_p$. Let $G$ act "diagonally" on the product $\Sigma^n \times (\mathbf{S}^N \times \mathbf{S}^N)$ and consider the equivariant projections

$$\Sigma^n \xleftarrow{f} \Sigma^n \times \mathbf{S}^N \times \mathbf{S}^N \xrightarrow{h} \mathbf{S}^N \times \mathbf{S}^N.$$

In cohomology, $f^*$ is an isomorphism in degrees less than $N$, $h^*$ is an isomorphism in degrees less than $n$ and a monomorphism in degree $n$. From 7.14 applied to both $f$ and $h$, it follows that $\check{H}^n(\Sigma^n/G; \mathbf{Z}_p)$ contains a subgroup isomorphic to $\check{H}^n(\mathbf{L}^N \times \mathbf{L}^N; \mathbf{Z}_p)$. Recall the fact that

$$\check{H}^i(\mathbf{L}^N; \mathbf{Z}_p) \approx \begin{cases} \mathbf{Z}_p & \text{for } 0 \le i \le N, \\ 0 & \text{otherwise.} \end{cases}$$

(This follows, for example, from the Smith sequence for the action of $\mathbf{Z}_p$ on $\mathbf{S}^N$, and can be proved in many other ways.) From the Künneth formula (Spanier [1, p. 360])

$$\check{H}^n(\mathbf{L}^N \times \mathbf{L}^N; \mathbf{Z}_p) \approx \mathbf{Z}_p \oplus \mathbf{Z}_p \oplus \cdots \oplus \mathbf{Z}_p \qquad (n+1 \text{ times}).$$

Thus $\check{H}^n(\Sigma^n/G; \mathbf{Z}_p)$ must contain a copy of $\mathbf{Z}_p \oplus \mathbf{Z}_p$.

(If $\Sigma^n = \mathbf{S}^n$, then $\Sigma^n/G$ would be a topological manifold and this would contradict the well-known fact that $\check{H}^n(M^n; \mathbf{Z}_p) \approx \mathbf{Z}_p$ or $0$ for an $n$-manifold $M^n$. In the general case we shall obtain a contradiction using Smith theory.)

Let $\mathbf{Z}_p \approx K \subset G = \mathbf{Z}_p \oplus \mathbf{Z}_p$ so that $G/K \approx \mathbf{Z}_p$. Then $K$ acts freely on $\Sigma^n$ and $G/K$ acts freely in $\Sigma^n/K$ with orbit space $(\Sigma^n/K)/(G/K) \approx \Sigma^n/G$. Applying 7.9 to both of these actions, and using the fact that $\check{H}_\sigma{}^*$ is the cohomology of the orbit space in the free case, we see that (with $\mathbf{Z}_p$ coefficients)

$$\operatorname{rk} \check{H}^n(\Sigma^n/G) \le \operatorname{rk} \check{H}^n(\Sigma^n/K) \le \operatorname{rk} \check{H}^n(\Sigma^n) = 1$$

which contradicts the previous discussion. ∎

See Exercise 10 for a stronger result, also due to Smith [7], as well as an indication of an alternative method of proof.

Since an abelian group is cyclic iff it does not contain a subgroup of the form $\mathbf{Z}_p \oplus \mathbf{Z}_p$ for some prime $p$, we have the following theorem.

**8.2. Theorem** *If $G$ is a finite group which can act freely on a finitistic integral cohomology sphere, then every abelian subgroup of $G$ is cyclic.* ∎

Groups with each abelian subgroup cyclic are precisely the groups with periodic cohomology (see Cartan and Eilenberg [1, p. 262]). Swan [1] has shown that each such group acts freely on some compact polyhedral integral homology sphere. However, not all such groups can act freely on an actual sphere, since, in this case, there is the following additional restriction due to Milnor [1]. Note that it rules out actions by some groups, such as the dyhedral groups or order $2r$, $r$ odd, which satisfy the conclusion of 8.2.

**8.3. Theorem** *Let $G$ be a group acting freely on $\mathbf{S}^n$. Then each element of order two in $G$ is in the center of $G$.*

***Proof*** Let $g$ have order two in $G$ and let $f$ be any other element of $G$. If $fg \neq gf$, then, since $G$ acts freely, $fg(x) \neq gf(x)$ for all $x$. Since $f$ is a homeomorphism, it has degree $\pm 1$. Thus 8.3 will follow from the next theorem.

**8.4. Theorem** *Let $g$ be a homeomorphism of period 2 on $\mathbf{S}^n$ without fixed points and let $f$ be any map from $\mathbf{S}^n$ to itself. If $gf(x) \neq fg(x)$ for all $x$, then $f$ has even degree.*

***Proof*** Let $G = \{e, g\} \approx \mathbf{Z}_2$. Put $M = \mathbf{S}^n \times \mathbf{S}^n - \Delta$, where $\Delta = \{(x, x) \mid x \in \mathbf{S}^n\}$ is the diagonal. Define an action of $G$ on $M$ by putting

$$g(x, y) = (gy, gx).$$

Note that

$$M^G = \{(x, gx) \mid x \in \mathbf{S}^n\} \subset M.$$

We claim that $M^G$ is a deformation retract of $M$. To see this, let $(x, y) \in M$ so that $x, y \in \mathbf{S}^n$ and $x \neq y$. For $0 \leq t \leq 1$ define $F(t, x, y)$ to be the point on $\mathbf{S}^n$ at which the line joining $x$ and $ty + (1 - t)g(x)$ intersects $\mathbf{S}^n - \{x\}$; see Figure III-1. Then the map $\mathbf{I} \times M \to M$ defined by $(t, (x, y)) \mapsto (x, F(t, x, y))$ is clearly a strong deformation retraction of $M$ onto $M^G$.

Thus the inclusion $M^G \to M$ induces an isomorphism in cohomology. Using $\mathbf{Z}_2$ coefficients everywhere, we conclude from the inequalities 7.9 and from (7.6) for $A = \varnothing$, that

$$\check{H}^i(M/G, M^G) \approx \check{H}_\sigma{}^i(M) = 0 \qquad \text{for all } i.$$

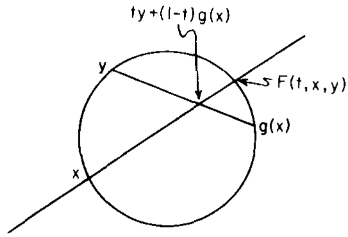



Figure III-1

Thus the inclusion induces an isomorphism

$$\check{H}^i(M/G) \xrightarrow{\approx} \check{H}^i(M^G)$$

for all $i$. (Alternatively, this follows directly from 7.14 applied to the inclusion $M^G \to M$.) The diagram

$$\begin{array}{ccc} \check{H}^*(M/G) & \xrightarrow{\pi_M^*} & \check{H}^*(M) \\ & \approx\searrow \quad \swarrow\approx & \\ & \check{H}^*(M^G) & \end{array}$$

shows that $\pi_M{}^*$ is an isomorphism.

Since $G$ acts freely on $\mathbf{S}^n$, the diagram (mod 2 coefficients)

$$\begin{array}{ccccccc} 0 \to & \check{H}^n(\mathbf{S}^n/G) & \to & \check{H}^n(\mathbf{S}^n/G) & \to & 0 & \to \\ & \downarrow\approx & & \downarrow\pi^* & & \downarrow & \\ \to & \check{H}_\sigma{}^n(\mathbf{S}^n) & \to & \check{H}^n(\mathbf{S}^n) & \to & \check{H}_\tau{}^n(\mathbf{S}^n) & \to 0 \end{array}$$

of (7.8) shows that $\pi^*$ is trivial (since $\tau = \sigma$, whence $\check{H}^n(\mathbf{S}^n) \to \check{H}_\tau{}^n(\mathbf{S}^n)$ is an isomorphism and the other map on the bottom is trivial; also see the discussion following 5.2 whose dual holds in cohomology).

Now define an *equivariant* map

$$\varphi\colon \mathbf{S}^n \to M$$

by $\varphi(x) = (f(x), gfg(x))$. The commutative diagram (mod 2 coefficients)

$$\begin{array}{ccc} \check{H}^n(M/G) & \xrightarrow{(\varphi/G)^*} & \check{H}^n(\mathbf{S}^n/G) \\ \pi_M^*\downarrow\approx & & \downarrow\pi^*=0 \\ \check{H}^n(M) & \xrightarrow{\varphi^*} & \check{H}^n(\mathbf{S}^n) \end{array}$$

shows that $\varphi^* = 0$. If $p\colon M \to \mathbf{S}^n$ is the projection $p(x, y) = x$, then

$f = p\varphi$ so that

$$f^* = \varphi^* p^* = 0: \check{H}^n(\mathbf{S}^n; \mathbf{Z}_2) \to \check{H}^n(\mathbf{S}^n; \mathbf{Z}_2)$$

which means that $f$ has even degree. ▮

*Remark* Theorem 8.4, and hence 8.3, also hold if $\mathbf{S}^n$ is replaced by any *n-manifold* $\Sigma^n$ having the mod 2 homology of $\mathbf{S}^n$. The proof of this requires the use of somewhat deeper results in algebraic topology than we wish to assume at this point, so that we have replaced a more general algebraic argument with the geometric argument concerning Figure III-1. However we shall briefly outline the proof of the more general result. The details may be found in Milnor [1]. Clearly it suffices to prove that the inclusion $M^G \subset M$ induces an isomorphism in mod 2 cohomology. This can be done by showing that the projection $p: M \to \Sigma^n$ is a fiber bundle map with fiber $\Sigma^n - \{\text{point}\}$. (This part is an easy geometric argument.) Clearly $M^G$ is a cross section for $p$. Now the Lefschetz and Poincaré dualities imply that the fiber $\Sigma^n - \{\text{point}\}$ is mod 2 acyclic, and the Vietoris Mapping Theorem implies that $p$ induces an isomorphism in mod 2 cohomology whose inverse comes from the inclusion $M^G \to M$. [Milnor's proof uses singular homology. We have used Čech cohomology instead, merely to be able to use the results of Section 7 directly. In singular homology, the proof that $\pi_M$ induces an isomorphism requires more justification.] We emphasize that any proof of 8.3 must make some use of the "geometry" of the situation, since results of Swan [1], mentioned above, show that it does not hold for actions on polyhedral homology spheres (nonmanifolds).

*Remark* The *orthogonal* free actions of finite groups on spheres have been completely classified by Wolf [1], completing earlier work of Vincent. There is a very simple necessary condition for a group to possess a free orthogonal action on some sphere, namely that every subgroup of order $pq$ be cyclic (where $p$ and $q$ are primes, possibly equal). (This condition is also sufficient when the group is solvable.) It was a long standing conjecture that a group which can act freely on a sphere can also act freely and orthogonally on a sphere. Recently T. Petrie [1,2] and R. Lee [4] have independently disproved this by showing that nonabelian groups of order $pq$ ($p$ and $q$ distinct odd primes) can act freely on suitable spheres. The construction of such examples uses the nonsimply connected surgery theory of C. T. C. Wall [3] and is beyond our scope.

We now turn to the same question for compact Lie groups of positive dimension. In this case there is the following complete and simple answer.

Denote the groups of complex numbers and quaternions (respectively) of norm 1 by $\mathbf{S}^1$ and $\mathbf{S}^3$. We regard $\mathbf{S}^1 \subset \mathbf{S}^3$ and let $N(\mathbf{S}^1)$ be its normalizer, which is just the group generated by $\mathbf{S}^1 = \{a + bi \mid a^2 + b^2 = 1\}$ and $j$, and which has two components $\mathbf{S}^1$ and $j\mathbf{S}^1$.

**8.5. Theorem** *If $G$ is a compact Lie group of positive dimension which can act freely on a sphere, then $G$ is isomorphic to a subgroup of $\mathbf{S}^3$. Specifically, there are precisely three possibilities: $G \approx \mathbf{S}^1$, $G \approx N(\mathbf{S}^1)$, and $G \approx \mathbf{S}^3$.*

*Proof* The group $G$ cannot contain a 2-dimensional torus since it cannot contain $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus $G$ has rank 1. We shall use the fact that there are precisely three connected Lie groups of rank 1, namely $\mathbf{S}^1$, $\mathbf{S}^3$, and $\mathbf{SO}(3)$ (which is doubly covered by $\mathbf{S}^3$). Thus the identity component $G_0$ of $G$ must be one of these three groups. Since $\mathbf{SO}(3)$ contains $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ as a subgroup, this case is ruled out. Suppose first that $G_0 = \mathbf{S}^1$. Then consider conjugation on $\mathbf{S}^1$ by an element $g \in G - G_0$. There are only two possibilities, $gtg^{-1} = t$ or $gtg^{-1} = t^{-1}$. Suppose the first possibility holds, that is, that $g$ commutes with $\mathbf{S}^1$. Let $n$ be the least integer for which $g^n$ is in $\mathbf{S}^1$, and let $h$ be an element of $\mathbf{S}^1$ with $h^n = g^n$. Then, for $k = gh^{-1}$, $g\mathbf{S}^1 = k\mathbf{S}^1$ and $k^n = 1$. Then $k$ and some element of $\mathbf{S}^1$ of order $n$ generate a subgroup of $G$ isomorphic to $\mathbf{Z}_n \oplus \mathbf{Z}_n$, contrary to 8.1. It follows that every element $g \in G - G_0$ satisfies $gtg^{-1} = t^{-1}$ for all $t \in G_0 = \mathbf{S}^1$. This clearly implies that $G/G_0$ has order 1 or 2. Suppose $G \neq G_0 = \mathbf{S}^1$ and let $g \in G - G_0$. Then $gtg^{-1} = t^{-1}$ for $t$ in $G_0$, and $g^2 \in G_0$. Now $g^2 = g(g^2)g^{-1} = (g^2)^{-1}$ since $g^2 \in G_0$, and hence $g^2 = \pm 1$. If $g^2 = 1$, then $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \approx \{1, -1, g, (-1)g\} \subset G$ contrary to 8.1. Thus $g^2 = -1$. Then taking $g$ to $j \in \mathbf{S}^3$ gives an isomorphism $G \approx N(\mathbf{S}^1)$.

Now suppose that $G_0 = \mathbf{S}^3$. It is well known that every automorphism $\varphi$ of $\mathbf{S}^3$ is inner. [This fact follows from the classification theory of compact connected Lie groups, but we shall *indicate* a direct proof. The automorphism $\varphi$ permutes the 1-parameter subgroups of $\mathbf{S}^3$ and, since all these are circles, they have a natural metric which must be preserved by $\varphi$. It this way one sees that $\varphi$ is orthogonal on the tangent space to $\mathbf{S}^3$ at the origin. However, the inner automorphisms constitute the action by $\mathbf{S}^3/\{1, -1\} \approx \mathbf{SO}(3)$ on this tangent space. Thus, if an outer automorphism exists, then there is one which operates by $-I \in \mathbf{O}(3)$ on the tangent space at the origin. This would imply that $g \mapsto g^{-1}$ is an automorphism, and it is not.] Thus, for any $g \in G - G_0$, there is an $h \in G_0$ such that $gh^{-1}$ commutes with $G_0$. However, this will yield a subgroup isomorphic to $\mathbf{Z}_n \oplus \mathbf{Z}_n$ exactly as in the proof of the case $G_0 = \mathbf{S}^1$. Thus $G = G_0$ when $G_0 = \mathbf{S}^3$. ∎

## 9. NEWMAN'S THEOREM

In this section we shall prove a theorem of Newman [1] in a version due to Smith [4]. This states that a compact Lie group (e.g., a finite group) cannot act on a manifold in such a way as to have "uniformly small" orbits.

Let $\pi\colon X \to Y$ be a map, let $B \subset Y$ be closed and put $A = \pi^{-1}B$. For a covering $\mathscr{V}$ of $Y$ we let $\pi^{-1}\mathscr{V}$ be the covering of $X$ by inverse images of members of $\mathscr{V}$. Assume that $\pi$ is onto.

**9.1. Lemma** *Let $\mathscr{U}$ be a covering of $X$ and suppose that the canonical homomorphism $H^n(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) \to \check{H}^n(X, A)$ is onto. If there exists a covering $\mathscr{V}$ of $Y$ such that $\pi^{-1}\mathscr{V}$ refines $\mathscr{U}$, then $\pi^*\colon \check{H}^n(Y, B) \to \check{H}^n(X, A)$ is onto.*

*Proof* Note that $\mathbf{K}(\pi^{-1}\mathscr{V}) \approx \mathbf{K}(\mathscr{V})$ in a canonical way and that $\pi^*\colon \check{H}^n(Y, B) \to \check{H}^n(X, A)$ is induced by passage to the direct limit of

$$H^n(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid B)) \xrightarrow{\approx} H^n(\mathbf{K}(\pi^{-1}\mathscr{V}), \mathbf{K}(\pi^{-1}\mathscr{V} \mid A)) \to \check{H}^n(X, A).$$

Since $\pi^{-1}\mathscr{V}$ refines $\mathscr{U}$, we have the commutative diagram

$$\begin{array}{ccccc} H^n(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) & \to & H^n(\mathbf{K}(\pi^{-1}\mathscr{V}), \mathbf{K}(\pi^{-1}\mathscr{V} \mid A)) & \to & \check{H}^n(X, A) \\ & & \uparrow{\scriptstyle \approx} & & \uparrow{\scriptstyle \pi^*} \\ & & H^n(\mathbf{K}(\mathscr{V}), \mathbf{K}(\mathscr{V} \mid B)) & \longrightarrow & \check{H}^n(Y, B) \end{array}$$

and this shows that $\pi^*$ must be onto when the composition along the top is onto. ∎

**9.2. Theorem** *Let $X$ be a finitistic space and $A \subset X$ a closed subspace. Let $p$ be a prime and suppose that $\check{H}^i(X, A; \mathbf{Z}_p) = 0$ for $i > n$ and that $\check{H}^n(X, A; \mathbf{Z}_p) \approx \mathbf{Z}_p$. Also assume that if $C \subset X$ is any proper closed subspace, then the restriction $\check{H}^n(X, A; \mathbf{Z}_p) \to \check{H}^n(C, C \cap A; \mathbf{Z}_p)$ is 0. Let $\mathscr{U}$ be any covering of $X$ such that*

$$H^n(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A); \mathbf{Z}_p) \to \check{H}^n(X, A; \mathbf{Z}_p)$$

*is onto. Then there does not exist an effective action of $\mathbf{Z}_p$ on $X$ leaving $A$ invariant and such that each orbit is contained in some member of $\mathscr{U}$.*

*Proof* Suppose such an action exists and let $G = \mathbf{Z}_p$. Then each orbit is contained in a saturated open set contained in some member of $\mathscr{U}$. The projections of these open sets in the orbit space $X^*$ give a covering $\mathscr{V}$ of $X^*$ with $\pi^{-1}\mathscr{V}$ refining $\mathscr{U}$. By 9.1, this implies that $\pi^*\colon \check{H}^n(X^*, A^*) \to \check{H}^n(X, A)$ is onto (coefficients in $\mathbf{Z}_p$).

Theorem 7.9 implies that $\check{H}_\varrho{}^i(X, A) = 0 = \check{H}^i(X^G, A^G)$ for $i > n$, and the Smith sequence

$$\check{H}^n(X, A) \to \check{H}_\varrho^n(X, A) \oplus \check{H}^n(X^G, A^G) \to \check{H}_\varrho^{n+1}(X, A) = 0$$

shows that $\check{H}^n(X^G, A^G) = 0$, (by the assumption on restrictions). Diagram (7.8) has the form

$$\begin{array}{ccccc}
\check{H}^n(X^*, A^* \cup X^G) & \to & \check{H}^n(X^*, A^*) & \to & 0 \\
\downarrow \approx & & \downarrow \pi^* & & \downarrow \\
\check{H}_\sigma{}^n(X, A) & \xrightarrow{\sigma^*} & \check{H}^n(X, A) & \longrightarrow & \check{H}_\tau{}^n(X, A) \to 0
\end{array}$$

and implies that $\check{H}_\tau{}^n(X, A) = 0$ and that $\sigma^*$ is onto. Then the Smith sequence

$$0 = \check{H}_\tau{}^n(X, A) \to \check{H}^n(X, A) \xrightarrow{i^*} \check{H}_\sigma{}^n(X, A) \to 0$$

shows that $i^*$ is an isomorphism. The composition $\sigma^* i^*\colon \check{H}^n(X, A) \to \check{H}^n(X, A)$ is just $\sum g^*$. However, $g^* = 1$, since $\check{H}^n(X, A) \approx \mathbf{Z}_p$ has no automorphisms of order $p$, and hence $\sigma^* i^* = p = 0$ contradicting the previous deduction that $\sigma^* i^*$ is onto. (Note that this argument also occurs following 5.2.) ∎

**9.3. Theorem** *Let $X$ be a finitistic space and $A \subset X$ a closed subspace. Suppose that $\check{H}^i(X, A; \mathbf{Z}) = 0$ for $i > n$ and that $\check{H}^n(X, A; \mathbf{Z}) \approx \mathbf{Z}$. Also assume that if $C$ is any proper closed subspace of $X$, then $\check{H}^n(X, A; \mathbf{Z}) \to \check{H}^n(C, C \cap A; \mathbf{Z})$ is trivial. Let $\mathscr{U}$ be any covering of $X$ such that*

$$H^n(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A); \mathbf{Z}) \to \check{H}^n(X, A; \mathbf{Z})$$

*is onto. Then there does not exist an effective action of any compact Lie group on $X$ leaving $A$ invariant and such that each orbit is contained in some member of $\mathscr{U}$.*

*Proof* It is easily seen that the conditions are inherited for $\mathbf{Z}_p$ coefficients by using the coefficient sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0$. Since any compact

Lie group contains a cyclic group of prime order $p$ for some $p$, the theorem follows from 9.2. ▮

**9.4. Theorem** *Let $M$ be a connected topological $n$-manifold. Then there is a finite open covering $\mathscr{U}$ of the 1-point compactification of $M$ such that there is no effective action of a compact Lie group on $M$ with each orbit contained in some member of $\mathscr{U}$.*

***Proof*** First suppose that $M$ is orientable and let $X$ be the 1-point compactification of $M$ with $A = \{\infty\}$. Then $\check{H}^n(X, A) \approx \mathbf{Z}$ and $\check{H}^i(X, A) = 0$ for $i > n$. Also, if $V \subset M$ is an open $n$-disk, then the inclusion $(X, A) \to (X, X - V)$ induces an isomorphism

$$\check{H}^n(\mathbf{D}^n, \partial\mathbf{D}^n) \approx \check{H}^n(X, X - V) \xrightarrow{\approx} \check{H}^n(X, A).$$

(These facts are well known and shall be assumed. They follow from Lefschetz duality, for example; see Spanier [1, p. 297]. They can also be proved by a fairly elementary direct argument.) For $C \subset X$ a proper closed subset, we can find an open $n$-disk $V \subset X - C$, and the diagram

$$\begin{array}{ccc} & \check{H}^n(X, X - V) & \xrightarrow{\approx} & \check{H}^n(X, A) \\ & \downarrow & & \downarrow \\ 0 = \check{H}^n(C, C) = & \check{H}^n(C, C \cap (X - V)) & \to & \check{H}^n(C, C \cap A) \end{array}$$

shows that the conditions of 9.3 are satisfied. This finishes the orientable case.

In order to treat the nonorientable case we first note that there is a simple explicit covering $\mathscr{U}$ of an orientable manifold which satisfies the requirement of 9.3, where $(X, A)$ is as above. To construct this, let $D$ be an $n$-disk in $M$, let $\Delta^{n+1}$ be the standard $(n + 1)$-simplex with vertices $v_0, \ldots, v_{n+1}$, and let $f\colon X \to \partial\Delta^{n+1}$ be a map which is a homeomorphism of int $D$ onto the complement of $v_0$ and which takes $X -$ int $D$ onto $v_0$. Let $U_i = f^{-1}(\text{star } v_i)$. Then the covering $\mathscr{U} = \{U_i \mid i = 0, \ldots, n + 1\}$ clearly satisfies the requirement that

$$H^n(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid A)) \xrightarrow{\approx} \check{H}^n(X, A).$$

Now let $M$ be nonorientable. Let $L$ be the polyhedron obtained from $\partial\Delta^{n+1}$ by adding an edge $(w, v_0)$; see Figure III-2. Let $D \subset$ int $D'$ be two $n$-disks in $M$ and let $h\colon M \to L$ be a map taking int $D$ homeomorphically onto $\partial\Delta^{n+1} - \{v_0\}$, taking $\partial D$ to $v_0$, taking $(\text{int } D') - D$ into the open edge $(w, v_0)$, and taking $M - \text{int } D'$ to $w$. (See Figure III-2, where the $u_i$ are points mapping to $v_i$.)

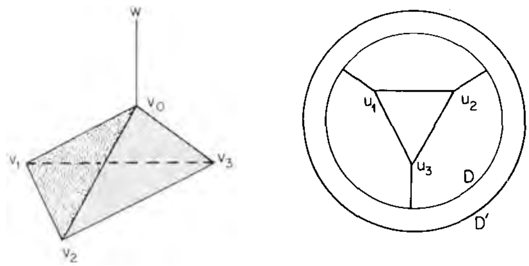



FIGURE III-2

Let $\mathscr{U}$ be the covering by inverse images of open stars of $L$. We claim that no action of a compact Lie group on $M$ can have all orbits contained in members of $\mathscr{U}$. Suppose there is such an action by $G$. Then clearly any transform of $D$ must intersect $D$ and $G(D) \subset \text{int } D'$. By compactness, there is a connected open neighborhood $V$ of $D$ with $G(V) \subset \text{int } D'$. Since all transforms of $V$ intersect $D$, $G(V)$ is connected. Thus we may as well take $V = G(V)$. However, $V$ is an orientable $n$-manifold and $\mathscr{U} \cap V$ is a covering of $V$ of the type constructed above for orientable manifolds. It follows that $G$ acts trivially on $V$. This contradicts the next result, which is another theorem of Newman [1].

**9.5. Theorem** *If $G$ is a compact Lie group acting effectively on a connected topological $n$-manifold $M$, then $M^G$ is nowhere dense.*

*Proof* Assume that $D$ is an $n$-disk contained in $M^G$. By lifting the action to the orientable double covering, using I.9.4, we can assume that $M$ is orientable. As before, map $M$ to $\partial\Delta^{n+1}$ taking $M - D$ to $v_0$. The induced covering of the 1-point compactification $X$ of $M$ then contradicts 9.3 with $A = \{\infty\}$, as noted in the proof of the orientable case of 9.4. ∎

*Remark* Theorem 9.5 is a special case of a much stronger theorem of Smith which states that, for $G$ cyclic of prime order $p$, $M^G$ is a generalized manifold over $\mathbf{Z}_p$. This is called the "Local Smith Theorem," and proofs can be found in Borel [5], Bredon [3, 13], and Smith [2]. However, the theory of generalized manifolds will not be treated in this book. One can give a fairly direct proof of 9.5, using the Smith sequences, which is independent of 9.3. For a recent treatment of this material along the lines of Newman's original proof, see Dress [1]. The present method of Smith is, however, more generally applicable.

**9.6. Corollary** *Let $M$ be a connected manifold and suppose we are given a metric on $M$. Then there is an $\varepsilon > 0$ such that, for every action of a compact Lie group $G$ on $M$, there exists an orbit of diameter larger than $\varepsilon$.*

*Proof* Let $\{U_0, \ldots, U_n\}$ cover $M \cup \{\infty\}$ as in 9.4 with $\infty \in U_0$. Then $M - U_0$ is compact and contained in $U_1 \cup \cdots \cup U_n$. Let $\varepsilon > 0$ be such that $B_{2\varepsilon}(x) \subset U_i$ for some $1 \leq i \leq n$, for each $x \in M - U_0$, where $B_{2\varepsilon}(x)$ is the closed ball of radius $2\varepsilon$ about $x$. Suppose that $G$ acts on $M$ and diam $G(y) \leq \varepsilon$ for all $y \in M$. Then $G(y) \subset B_\varepsilon(y)$ for all $y$. If $x \in B_\varepsilon(y) - U_0$, then $y \in B_\varepsilon(x)$ and $G(y) \subset B_\varepsilon(y) \subset B_{2\varepsilon}(x) \subset U_i$ for some $i$. If $B_\varepsilon(y) - U_0 = \varnothing$, then $G(y) \subset B_\varepsilon(y) \subset U_0$. This contradicts the choice of the covering $\{U_i\}$. ∎

**9.7. Corollary** *Let $\mathbf{R}^n$ have the usual euclidean metric. Then an action of a compact Lie group $G$ on $\mathbf{R}^n$ cannot have orbits of uniformly bounded diameter.*

*Proof* Let $\varepsilon > 0$ be as in 9.6 for $M = \mathbf{R}^n$ and let $N$ be a bound for the diameters of the orbits of a given action $\Theta$ of $G$ on $\mathbf{R}^n$. Define a new action $\Theta'$ by putting

$$\theta_g'(x) = \frac{\varepsilon}{N}\,\theta_g\left(\frac{N}{\varepsilon}\,x\right).$$

Then

$$\| x - \theta_g'(x) \| = \frac{\varepsilon}{N}\left\| \left(\frac{N}{\varepsilon}\,x\right) - \theta_g\left(\frac{N}{\varepsilon}\,x\right)\right\| \leq \frac{\varepsilon}{N}\,N = \varepsilon$$

contrary to 9.6. ∎

## 10. TORAL ACTIONS

In this section we shall prove some results analogous to 7.10 and 7.11 for actions of a toral group. Our main purpose in giving these results here is for their application in the next two chapters. We first give some immediate consequences for toral actions of the results of the preceding sections. These suffice for most applications, but we also discuss a direct approach which gives stronger results. We will denote by $\mathbf{T}$ the circle group and by $\mathbf{T}^k$ the $k$-dimensional torus; that is, the product of $k$ copies of $\mathbf{T}$.

**10.1. Lemma** *Let $p$ be a prime. Then the set of elements of $\mathbf{T}^k$ of order a power of $p$ is dense in $\mathbf{T}^k$.*

*Proof* It suffices to show that the union of the $p$-subgroups of $\mathbf{T}^k$ is dense, and this follows from the same statement for the circle group ($k = 1$) where it is obvious. ▌

**10.2. Theorem** *Let $G = \mathbf{T}^k$ act on a finitistic integral cohomology $n$-sphere $X$. Assume that $\check{H}^*(X^G; \mathbf{Z})$ has finite type and that there are only a finite number of orbit types. Then $X^G$ is an integral cohomology $r$-sphere for some $-1 \leq r \leq n$ and $n - r$ is even.*

*Proof* Let $p$ be a prime and let $H_1, \ldots, H_m$ be the isotropy types. Since the elements of $G$ of order a power of $p$ are dense, there exists such an element $g$ not in any of the $H_i$. Thus $X^G = X^g$ and 7.11 implies that $X^g$ is a mod $p$ cohomology $r_p$-sphere for some $-1 \leq r_p \leq n$, and $n - r_p$ is even if $p$ is odd.

Assume that $X^G \neq \varnothing$ and consider cohomology modulo some given point in $X^G$, which we shall omit from the notation. We claim that $\check{H}^i(X^G; \mathbf{Z})$ has no $p$-torsion. If it does, then, since it is finitely generated, it has a summand of the form $\mathbf{Z}_{p^j}$ for some $j \geq 1$. Then the exact sequence

$$\cdots \to \check{H}^{i-1}(X^G; \mathbf{Z}_p) \to \check{H}^i(X^G; \mathbf{Z}) \xrightarrow{p} \check{H}^i(X^G; \mathbf{Z}) \to \check{H}^i(X^G; \mathbf{Z}_p) \to \cdots$$

shows that $\check{H}^{i-1}(X^G; \mathbf{Z}_p)$ and $\check{H}^i(X^G; \mathbf{Z}_p)$ are both nonzero, and this contradicts the assertion that $X^G$ is a mod $p$ sphere.

Since this holds for all $p$ we conclude that each $\check{H}^i(X^G; \mathbf{Z})$ is free abelian and (again by the above sequence) it must vanish for $i \neq r_p$ for each $p$ and has rank 1 for $i = r_p$. Thus $r = r_p$ is independent of $p$, lies between $-1$ and $n$, and $n - r$ is even. ▌

A similar proof gives the next theorem.

**10.3. Theorem** *Let $G = \mathbf{T}^k$ act on a finitistic integral cohomology $n$-disk $(X, A)$, where $A$ is closed in $X$. Assume that $\check{H}^*(X^G, A^G; \mathbf{Z})$ has finite type and that there are only finitely many orbit types. Then $(X^G, A^G)$ is an integral cohomology $r$-disk for some $0 \leq r \leq n$, and $n - r$ is even.* ▌

The above results are sufficient for most of our purposes, but they are not sufficiently strong to prove, for example, that the fixed point set of a toral action on euclidean space is acyclic over the integers (even if we *assume* finiteness of number of orbit types and that the action is differentiable). Thus we shall now develop a tool for studying toral actions *directly*. For this, we shall have to assume slightly more background knowledge, namely that of the Thom–Gysin isomorphism for orientable disk bundles.

Let the circle group $\mathbf{T}$ act on the paracompact space $X$ and, as usual, regard $X^{\mathbf{T}}$ as also being a subspace of the orbit space $X^* = X/\mathbf{T}$. Let $M_\pi$ denote the mapping cylinder of the orbit map $\pi\colon X \to X^*$. Let $F$ denote $X^{\mathbf{T}} \times \mathbf{I}$ embedded in $M_\pi$ as the mapping cylinder of the restriction of $\pi$ to $X^{\mathbf{T}}$, and put $X' = X \cup F$ which has $X$ as a deformation retract. Consider the triple $(M_\pi, X', F)$. The cohomology sequence of this triple, together with the three exact sequences of the pairs contained in this triple, form the commutative "braid diagram of the triple" shown in Figure III-3.

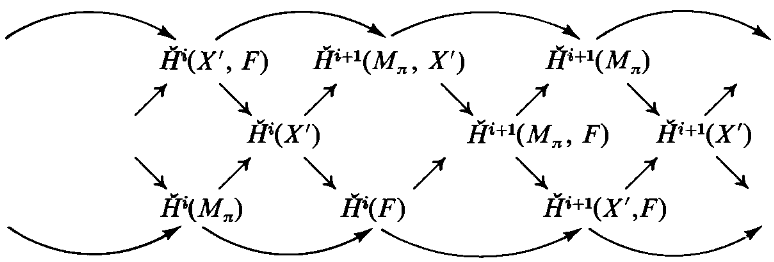



FIGURE III-3

Now we may replace the pair $(X', F)$ by $(X, X^{\mathbf{T}})$ and the pair $(M_\pi, F)$ by $(X^*, X^{\mathbf{T}})$. Also we may replace the absolute groups of $M_\pi$, $X'$, and $F$ by those of $X^*$, $X$, and $X^{\mathbf{T}}$, respectively. For the pair $(M_\pi, X')$ we use the following lemma.

**10.4. Lemma** *If* $\mathbf{T}$ *acts semifreely* (i.e., *freely outside* $X^{\mathbf{T}}$), *then* $\check{H}^i(M_\pi, X') \approx \check{H}^{i-2}(X^*, X^{\mathbf{T}})$ *for arbitrary coefficients. For a general* $\mathbf{T}$ *action with finitely many orbit types, this isomorphism holds with rational coefficients.*

***Proof*** If $N^*$ ranges over the closed neighborhoods of $X^{\mathbf{T}}$ in $X^*$ and if $N$ is the part of $M_\pi$ over $N^*$, then $\check{H}^*(X^*, X^{\mathbf{T}}) \approx \lim_{\to} \check{H}^*(X^*, N^*)$ and $\check{H}^*(M_\pi, X') \approx \lim_{\to} \check{H}^*(M_\pi, N \cup X')$. (This well-known fact is almost an immediate consequence of the definition of Čech cohomology and of the relationship between double limits and iterated limits. Proofs of closely related facts will be found in Bredon [13, p. 52] and in Spanier [1, p. 316].) Now we may excise a neighborhood of $X^{\mathbf{T}}$ in $N^*$ and of $F$ in $N$. Also note that by II.5.8, $M_\pi - F \to X^* - X^{\mathbf{T}}$ is a 2-disk bundle map, in the semifree case, with $X' - F \to X^* - X^{\mathbf{T}}$ the bounding 1-sphere bundle. Orientability of this bundle follows from the fact that it arises from a **T**-action. Thus the claimed isomorphism follows from the relative Thom-Gysin isomorphism for 2-disk bundles. (We must assume this. It can be found in Spanier [1, p. 259] for the absolute case in singular theory and it is not hard to copy the proof in Čech theory. A sufficiently general version can also be found in Bredon [13, p. 150].)

For the case of finitely many orbit types, note that **T** contains a cyclic subgroup $H$ containing all isotropy groups other than **T** itself. Since **T** acts trivially in cohomology, 7.2 shows that $\check{H}^i(M_\pi, X') \approx \check{H}^i(M_\pi/H, X'/H)$ for rational coefficients. Now $M_\pi/H$ is the mapping cylinder of $X/H \to X/\mathbf{T} = X^*$ and, since $\mathbf{T}/H$ acts semifreely on $X/H$, this case follows from the semifree case. ▌

*Remark* It can be shown that the second part of 10.4 holds without the restriction of finiteness of number of orbit types.

Now, using these isomorphisms, the above braid diagram may be replaced by that in Figure III-4 in either of the cases of 10.4.

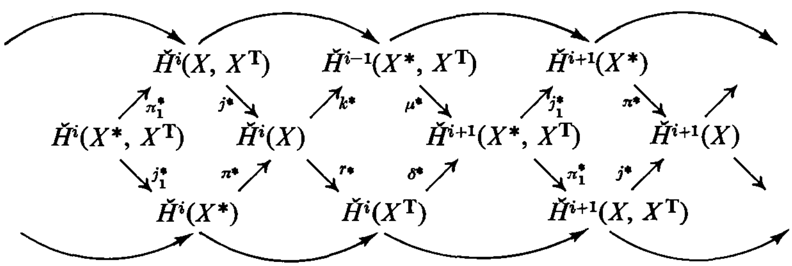



FIGURE III-4

From this we can extract out the following exact sequence (for **T** semifree and arbitrary coefficients, or for finitely many orbit types and rational coefficients).

**(10.5)** $$\cdots \xrightarrow{\mu^*-\delta^*} \check{H}^i(X^*, X^{\mathbf{T}}) \xrightarrow{q^*} \check{H}^i(X) \xrightarrow{(k^*, r^*)} \check{H}^{i-1}(X^*, X^{\mathbf{T}}) \oplus \check{H}^i(X^{\mathbf{T}}) \xrightarrow{\mu^*-\delta^*} \check{H}^{i+1}(X^*, X^{\mathbf{T}}) \xrightarrow{q^*} \cdots,$$

where $q^* = j^*\pi_1^* = \pi^* j_1^*$ in Figure III-4. This sequence is called the **Smith-Gysin sequence.** It was first derived in Bredon [8] from the Fary spectral sequence; also see Bredon [13, p. 169]. The exactness is easily proved by a diagram chase in Figure III-4. (Also see Wall [1] for generalities about braid diagrams.) We remark that the homomorphism $\mu^*$ is the cup product with an element, the euler class, of $\check{H}^2(X^* - X^{\mathbf{T}})$, which follows from the same fact for the classical Gysin sequence.

For the sake of simplicity of notation, we have discussed only the absolute case. However, it is clear that for $A \subset X$ closed and invariant, the discussion generalizes to give the following Smith–Gysin sequence

**(10.6)**
$$\begin{aligned} \cdots &\to \check{H}^i(X^*, A^* \cup X^{\mathbf{T}}) \to \check{H}^i(X, A) \\ &\to \check{H}^{i-1}(X^*, A^* \cup X^{\mathbf{T}}) \oplus \check{H}^i(X^{\mathbf{T}}, A^{\mathbf{T}}) \\ &\to \check{H}^{i+1}(X^*, A^* \cup X^{\mathbf{T}}) \to \cdots. \end{aligned}$$

Using this sequence exactly as the Smith sequences were used, one may prove theorems for circle group actions which are analogous to the theorems of Section 7. To justify induction from infinity, we need to impose a condition which insures that $\check{H}^i(X^*, A^* \cup X^{\mathbf{T}}) = 0$ for sufficiently large $i$. Finite dimensionality of $X^*$ would clearly suffice for this, and certainly this is enough for most applications. However, for completeness, we shall show that it suffices that $X$ and $X^*$ be finitistic. (We do not know whether or not $X^*$ is finitistic when $X$ is, but this does hold in the two main cases of compact spaces and finite-dimensional, separable, metric spaces; see Chapter II, Exercise 3.) Thus we shall prove the following proposition.

**10.7. Proposition** *Let the circle group* $\mathbf{T}$ *act with only finitely many orbit types on* $X$ *with* $X$ *and* $X^*$ *both finitistic. Let* $A \subset X$ *be closed and invariant and assume that* $\check{H}^i(X, A) = 0$ *for* $i > n$. *Then also* $\check{H}^i(X^{\mathbf{T}}, A^{\mathbf{T}}) = 0$ *for* $i > n$ *and* $\check{H}^i(X^*, A^* \cup X^{\mathbf{T}}) = 0$ *for* $i \geq n$. *This holds for either rational or integral coefficients.*

***Proof*** First note that for any finite subgroup $K$ of $\mathbf{T}$ we have $\check{H}^i(X/K, A/K) = 0$ for $i > n$. For rational coefficients, this follows from 7.2. For integral coefficients, note that, by an induction, we may assume that $K$ is cyclic of prime order $p$. Then $\check{H}^i(X/K, A/K; \mathbf{Z}_p) = 0$ for $i > n$ by 7.9 and the exact sequence of the triple $(X/K, A/K \cup X^K/K, A/K)$. This implies that $\check{H}^i(X/K, A/K; \mathbf{Z})$ is divisible by $p$ for $i > n$, and the transfer shows it to be zero. (See the proof of 5.4 and also Exercise 9.) This reduces the proposition to the semifree case.

Suppose that

$$\check{H}^{i-1}(X^*, A^* \cup X^{\mathrm{T}}) \oplus \check{H}^i(X^{\mathrm{T}}, A^{\mathrm{T}}) \neq 0$$

for some $i > n$. Then the composition (from the Smith–Gysin sequence)

$$\check{H}^{i-1}(X^*, A^* \cup X^{\mathrm{T}}) \oplus \check{H}^i(X^{\mathrm{T}}, A^{\mathrm{T}}) \xrightarrow[\approx]{\mu^*-\delta^*} \check{H}^{i+1}(X^*, A^* \cup X^{\mathrm{T}})$$
$$\xrightarrow{\mu^*} \check{H}^{i+3}(X^*, A^* \cup X^{\mathrm{T}}) \xrightarrow{\mu^*} \cdots$$

consists of monomorphisms. It suffices to show that a high power of $\mu^*$ kills any given element $\beta_0$ of $\check{H}^{i+1}(X^*, A^* \cup X^{\mathrm{T}})$. Now $\beta_0$ is in the image of $\check{H}^{i+1}(X^*, A^* \cup N^*)$ for some sufficiently small *closed* neighborhood $N^*$ of $X^{\mathrm{T}}$ and, by excision, corresponds to an element $\beta$ of $\check{H}^{i+1}(X^* - U^*, (A^* \cup N^*) - U^*)$ for some open neighborhood $U^*$ of $X^{\mathrm{T}}$ in $N^*$. (In Čech theory, $U^*$ can be taken to be the interior of $N^*$, but this is of no consequence here.) We shall use the fact that $\mu^*$ is the cup product by some element $\alpha_0$ of $\check{H}^2(X^* - X^{\mathrm{T}})$. Hence $\mu^*(\beta_0)$ is the image in $\check{H}^{i+3}(X^*, A^* \cup X^{\mathrm{T}})$ of $\alpha\beta$ where $\alpha$ is the restriction of $\alpha_0$ to $\check{H}^2(X^* - U^*)$. Since a closed subspace of a finitistic space is clearly finitistic, it suffices to prove the following lemma.

**10.8. Lemma** *Let $Y$ be finitistic and $B \subset Y$ be closed. Let $\alpha \in \check{H}^i(Y)$ and $\beta \in \check{H}^j(Y, B)$ where $i > 0$. Then $\alpha^k\beta = 0$ for sufficiently large $k$.*

*Proof* There is a finite-dimensional covering $\mathscr{U}$ of $Y$ such that $\alpha$ and $\beta$ are images of elements $\alpha'$ and $\beta'$ of $H^i(\mathbf{K}(\mathscr{U}))$ and $H^j(\mathbf{K}(\mathscr{U}), \mathbf{K}(\mathscr{U} \mid B))$, respectively. Then $(\alpha')^k\beta' = 0$ when $ki + j > \dim \mathscr{U}$ and this implies that $\alpha^k\beta = 0$. ▮

Using 10.6, 10.7, and the argument of 4.1, the proof of the following theorem is clear.

**10.9. Theorem** *Let the circle group $\mathbf{T}$ act with only finitely many orbit types on $X$ with both $X$ and $X^* = X/\mathbf{T}$ finitistic. Let $A \subset X$ be a closed invariant subspace. Then, for coefficients in the rationals $\mathbf{Q}$,*

$$\operatorname{rk} \check{H}^{n-1}(X^*, A^* \cup X^{\mathrm{T}}) + \sum_{i=0}^{\infty} \operatorname{rk} \check{H}^{n+2i}(X^{\mathrm{T}}, A^{\mathrm{T}}) \leq \sum_{i=0}^{\infty} \operatorname{rk} \check{H}^{n+2i}(X, A).$$

*If $\check{H}^*(X, A)$ is finitely generated over $\mathbf{Q}$, then so are $\check{H}^*(X^{\mathrm{T}}, A^{\mathrm{T}})$ and $\check{H}^*(X^*, A^*)$, and we also have that $\chi(X, A) = \chi(X^{\mathrm{T}}, A^{\mathrm{T}})$.* ▮

***Remark*** Theorem 10.9 can be proved without the condition on finiteness of number of orbit types, and the same is true of all our results involving only rational coefficients (10.10–10.12).

For convenience we will call a $\mathbf{T}^k$-space $X$ $\mathbf{T}^k$-finitistic if the orbit spaces of all subtori are finitistic.

**10.10. Corollary** *If $X$ is $\mathbf{T}^k$-finitistic, then* 10.2 *and* 10.3 *hold with rational coefficients throughout.* ▮

**10.11. Corollary** *Let the circle group $\mathbf{T}$ act on the finitistic space $X$ with $X^*$ finitistic and with only finitely many orbit types. Let $A$ be an invariant closed subspace. Suppose that $\check{H}^j(X, A; \mathbf{Q}) = 0$ for all odd $j$ (respectively, even $j$). Then $\check{H}^j(X^{\mathbf{T}}, A^{\mathbf{T}}; \mathbf{Q}) = 0$ for all odd $j$ (respectively, even $j$), and*

$$\sum \mathrm{rk}\, \check{H}^i(X^{\mathbf{T}}, A^{\mathbf{T}}; \mathbf{Q}) = \sum \mathrm{rk}\, \check{H}^i(X, A; \mathbf{Q}).$$

***Proof*** The first part is immediate from 10.9. For the second part, we note that $\check{H}^j(X^*, A^* \cup X^{\mathbf{T}}) = 0$ for $j$ even, by 10.9, and thus the Smith–Gysin sequence consists of the short exact sequences

$$0 \to \check{H}^{2i}(X, A) \to \check{H}^{2i-1}(X^*, A^* \cup X^{\mathbf{T}}) \oplus \check{H}^{2i}(X^{\mathbf{T}}, A^{\mathbf{T}}) \to \check{H}^{2i+1}(X^*, A^* \cup X^{\mathbf{T}}) \to 0.$$

Thus

$$\mathrm{rk}\, \check{H}^{2i}(X^{\mathbf{T}}, A^{\mathbf{T}}) + \mathrm{rk}\, \check{H}^{2i-1}(X^*, A^* \cup X^{\mathbf{T}}) = \mathrm{rk}\, \check{H}^{2i}(X, A) + \mathrm{rk}\, \check{H}^{2i+1}(X^*, A^* \cup X^{\mathbf{T}})$$

and the desired equality results upon adding and canceling. ▮

**10.12. Theorem** *Let $\Sigma$ be a $\mathbf{T}^k$-finitistic $\mathbf{T}^k$-space which is a rational cohomology $n$-sphere, and assume that there are only a finite number of orbit types and no stationary points. If $H \subset \mathbf{T}^k$ is a subtorus of dimension $k - 1$, let $r(H)$ denote that integer, between $-1$ and $n$, for which $\Sigma^H$ is a rational cohomology $r(H)$-sphere. Then, with $H$ ranging over all subtori of dimension $k - 1$, we have*

$$n + 1 = \sum_H (r(H) + 1).$$

*Proof* Since the circle subgroups are dense in $\mathbf{T}^k$, there is one, say $S$, with no stationary points. Then the Smith–Gysin sequence shows that

$$\check{H}^i(\Sigma^H/S;\ \mathbf{Q}) \approx \begin{cases} \mathbf{Q} & \text{for } i = 0, 2, 4, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that a point $x$ in $\Sigma$ is fixed by a subtorus of codimension 1 iff its orbit has dimension 1 [and hence coincides with $S(x)$]. This holds iff the image of $x$ in $\Sigma/S$ is fixed by the action of $\mathbf{T}^k/S$. Also, the fixed sets $\Sigma^H$ are mutually disjoint. Thus $(\Sigma/S)^{\mathbf{T}^k/S}$ consists of disjoint sets, each of the form $\Sigma^H/S$. As above

$$\check{H}^i(\Sigma^H/S;\ \mathbf{Q}) \approx \begin{cases} \mathbf{Q} & \text{for } i = 0, 2, \ldots, r(H)-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum \operatorname{rk} \check{H}^i(\Sigma/S;\ \mathbf{Q}) = \tfrac{1}{2}(n+1)$$

and

$$\sum_i \operatorname{rk} \check{H}^i((\Sigma/S)^{\mathbf{T}^k/S}; \mathbf{Q}) = \sum_H \sum_i \operatorname{rk} \check{H}^i(\Sigma^H/S; \mathbf{Q}) = \tfrac{1}{2} \sum_H (r(H)+1).$$

But these are equal by 10.11 and an obvious induction. ∎

*Remark* Theorem 10.12 is due to Borel [5]. It is easily generalized to the case in which there are stationary points, as done by Borel, but the case given is the only one of interest to us here. Borel also proved an analogous result for actions of $\mathbf{Z}_p \oplus \mathbf{Z}_p \oplus \cdots \oplus \mathbf{Z}_p$.

**10.13. Theorem** *Let the circle group* $\mathbf{T}$ *act with only finitely many orbit types on* $X$ *with* $X$ *and* $X^*$ *finitistic, Let* $A \subset X$ *be closed and invariant. If* $\check{H}^*(X, A; \mathbf{Z})$ *is finitely generated, then so are* $\check{H}^*(X^{\mathbf{T}}, A^{\mathbf{T}}; \mathbf{Z})$ *and* $\check{H}^*(X^*, A^*; \mathbf{Z})$.

*Proof* If $\mathbf{Z}_p \subset \mathbf{T}$, we first show that $\check{H}^*(X/\mathbf{Z}_p, A/\mathbf{Z}_p; \mathbf{Z})$ is finitely generated. (Note that this is a special case of Exercise 1.) The composition $\check{H}^*(X/\mathbf{Z}_p, A/\mathbf{Z}_p;\ \mathbf{Z}) \to \check{H}^*(X, A; \mathbf{Z}) \to \check{H}^*(X/\mathbf{Z}_p, A/\mathbf{Z}_p;\ \mathbf{Z})$ with the transfer is multiplication by $p$ and its image is finitely generated since it factors through a finitely generated group. Thus it suffices to show that the kernel of multiplication by $p$ on this group is finitely generated. But the exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0$ shows this to be the case, since $\check{H}^*(X/\mathbf{Z}_p, A/\mathbf{Z}_p; \mathbf{Z}_p)$ is finitely generated by 7.9. By induction we see that $\check{H}^*(X/\mathbf{Z}_n, A/\mathbf{Z}_n; \mathbf{Z})$ is finitely generated for each cyclic subgroup $\mathbf{Z}_n$ of $\mathbf{T}$. Thus the problem reduces to the semifree case. But this case is an easy consequence of the Smith–Gysin sequence (10.6) with integral coefficients by an induction from infinity, and by the exact sequence of the triple $(X^*, A^* \cup X^{\mathbf{T}}, A^*)$. (Compare the proof of 10.7.) ∎

**10.14. Corollary** *If $X$ is $\mathbf{T}^k$-finitistic, then the conditions of finite type in* 10.2 *and* 10.3 *may be removed.* ∎

*Example* That finiteness of number of orbit types is necessary in 10.2 and 10.3 is shown by the following example for the case $n = 0$ and $A = \varnothing$ of 10.3. Let $\mathbf{T}$ act on $\mathbf{S}^1$ $(= \mathbf{T})$ by $(z, z') \mapsto z^n z'$ of $\mathbf{T} \times \mathbf{S}^1 \to \mathbf{S}^1$. The circle $\mathbf{S}^1$ with this action will be denoted by $\mathbf{S}^1(n)$. The map $z \mapsto z^2$ gives an equivariant map $f\colon \mathbf{S}^1(n) \to \mathbf{S}^1(2n)$. Let $\mathbf{P}^2$ be the projective plane with trivial $\mathbf{T}$ action and let $X$ be the union of the mapping cylinders of the maps of joins

$$\mathbf{P}^2 \star \mathbf{S}^1(1) \xrightarrow{1\star f} \mathbf{P}^2 \star \mathbf{S}^1(2) \xrightarrow{1\star f} \mathbf{P}^2 \star \mathbf{S}^1(4) \xrightarrow{1\star f} \cdots.$$

Then $X$ is simply connected and acyclic over $\mathbf{Z}$ (since $1 \star f$ induces the trivial map in homology), and hence $X$ is *contractible*. But $X^{\mathbf{T}} \approx \mathbf{P}^2 \times [0, \infty)$ has $\check{H}^2(X^{\mathbf{T}}; \mathbf{Z}) \approx \mathbf{Z}_2$.

Let $x$ be a fixed point of this $\mathbf{T}$-space $X$ and let $X_i$ be copies of $X$ with $x_i$ corresponding to $x$. Attach each $X_i$ to the real line by identifying $x_i$ with $i$. Then the resulting space $Y$ is still contractible, but $\check{H}^2(Y^{\mathbf{T}}; \mathbf{Z})$ is the direct product of infinitely many copies of $\mathbf{Z}_2$ and hence is not finitely generated. This shows that the condition of finiteness of the number of orbit types is necessary for 10.13. (It can be shown, however, that this cannot happen when $X$ is compact.)

*Remark* Theorems of the type considered in this section were first proved in Conner [3] and in Floyd [9]. Also see Borel [5].

## EXERCISES FOR CHAPTER III

**1.** If $K$ is a finite-dimensional regular $G$-complex, $G$ finite, and if $H(K; \mathbf{Z})$ is finitely generated, show that $H(K/G; \mathbf{Z})$ is also finitely generated. Generalize this in cohomology for finitistic $G$-spaces.

**2.** Show how to derive 5.1 from 5.2 and similarly for their generalizations to finitistic $G$-spaces (see 7.11).

**3.** Let $K$ be a finite-dimensional regular $G$-complex, where $G$ is cyclic of prime order $p$. Suppose that $K$ is a mod $p$ homology $n$-sphere and that $K^G$ is a mod $p$ homology $r$-sphere. Show that

$$H_i(K/G; \mathbf{Z}_p) \approx \begin{cases} \mathbf{Z}_p & \text{for } i = 0, \\ \mathbf{Z}_p & \text{for } r + 2 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

If $K$ is an *integral* homology $n$-sphere, then (with no further assumptions on $K^G$) show that, if $n - r$ is even, then

$$H_i(K/G; \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{for } i = 0 \text{ and } i = n, \\ \mathbf{Z}_p & \text{for } i = r + 2,\ r + 4, \ldots, n - 2, \\ 0 & \text{otherwise} \end{cases}$$

and, if $n - r$ is odd (and necessarily $p = 2$), then

$$H_i(K/G; \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{for } i = 0, \\ \mathbf{Z}_p & \text{for } i = r + 2,\ r + 4, \ldots, n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $p = 2$, show that $n - r$ is even iff $G$ acts trivially on $H_n(K; \mathbf{Z})$. Also restate the problem in cohomology and generalize it to finitistic spaces.

**4.** Show that the sequence 3.1 is exact for *arbitrary* coefficients when $L \supset K^G$ and $\varrho = \sigma$ or $\tau$, $\bar{\varrho}$ being the other. Also prove this for arbitrary cyclic $G$ acting *freely* on the set of simplices of $K$ not in $L$.

**5.** Let $K$ be a $G$-complex. Consider the condition that whenever $(u, v)$ and $(gu, v)$ are edges of $K$ then either $gu = u$ or $gv = v$. Show that this condition implies regularity, but not conversely. Also show that this condition always holds for the action on the second barycentric subdivision of $K$.

**6.** Let $\mathscr{U}$ be a locally finite regular $G$-covering of a $G$-space $X$. Show that there exists a minimal invariant subcover. If $\mathscr{U}$ is minimal, show that $\mathbf{K}(\mathscr{U})/G \approx \mathbf{K}(\mathscr{U}_G)$ when $\mathscr{U}_G$ is *self-indexed.*

**7.** Generalize 7.14 to maps $f$: $(X, A) \to (Y, B)$ of closed pairs of paracompact $G$-spaces. [*Hint*: If $f'$: $A \to B$ is the restriction of $f$: $X \to Y$, then the inclusion $(X, A) \to (M_{f'} \cup X, M_{f'})$ and the retraction $(M_f, M_{f'}) \to (Y, B)$ both induce isomorphisms in cohomology. Consider the exact sequence of the triple $(M_f, M_{f'} \cup X, M_{f'})$.]

**8.** Let $G$ be cyclic of prime order $p$ and let $E$ be a connected, compact, *free* $G$-space such that $\check{H}^i(E) = 0$ for $0 < i < N$. Let $B = E/G$. The equivariant projection $(X \times E, X^G \times E) \to (X, X^G)$ induces the map $(X \times_G E, X^G \times B) \to (X^*, X^G)$ of orbit spaces (where $X$ is a $G$-space). If $X$ is paracompact, show that the induced map

$$\check{H}^i(X^*, X^G) \to \check{H}^i(X \times_G E, X^G \times B)$$

is an isomorphism for $i < N$ and a monomorphism for $i = N$ (coefficients in $\mathbf{Z}_p$ or in $\mathbf{Z}$). If $X$ is finitistic and $\check{H}^i(X; \mathbf{Z}_p) = 0$ for $i > n$, then conclude that the restriction

$$\check{H}^i(X \times_G E; \mathbf{Z}_p) \to \check{H}^i(X^G \times B; \mathbf{Z}_p)$$

is an isomorphism for $n < i < N - 1$ and an epimorphism for $i = n$ (when $n < N$). (*Hint*: Use Exercise 7. We remark that this fact is the basis of a powerful method for studying transformations of prime period, which we shall study later. It requires, however, a knowledge of the theory of spectral sequences.)

**9.** Let $G$ be a finite group acting on a finitistic space $X$, and let $A \subset X$ be closed and invariant. Suppose that $\check{H}^i(X, A; \mathbf{Z}) = 0$ for $i > n$. Show that $\check{H}^i(X/G, A/G; \mathbf{Z}) = 0$ for $i > n$.

**10.** Show that $\mathbf{Z}_p \oplus \mathbf{Z}_p$ cannot act semifreely on a finitistic mod $p$ cohomology $n$-sphere $X$ with fixed set $F$ a mod $p$ cohomology $r$-sphere with $-1 \leq r < n$. (*Hint*: By suspending twice, one may assume that $r \geq 1$. Let $G = G_1 \times G_2$, $G_i$ cyclic of order $p$. Note that $F \approx (X^{G_1})^{G_2}$. Consider the orbit map $X \to X/G_1$ as a $G_2$-equivariant map and look at the map of Smith sequences (of $G_2$ actions) induced by this map.)

**11.** If $G$ is a finite group which can act freely on a compact integral cohomology $n$-sphere $\Sigma^n$, show that $G$ has periodic cohomology of period $n + 1$; that is, $H^i(G; \mathbf{Z}) \approx H^{i+n+1}(G; \mathbf{Z})$ for all $i > 0$. (*Hint*: By taking joins of copies of $\Sigma^n$, $G$ also acts freely on a cohomology $N$-sphere $\Sigma^N$ for $N$ arbitrarily large. Consider the twisted product $\Sigma^n \times_G \Sigma^N$ and the projections to $\Sigma^n/G$ and $\Sigma^N/G$, which are fiber bundle maps with fibers $\Sigma^N$ and $\Sigma^n$, respectively, and group $G$. Use the Gysin sequences of these maps.)

**12.** Suppose that the group $G = \mathbf{S}^3$ of unit quaternions acts *semifreely* on the paracompact space $X$. Derive the following exact Smith–Gysin sequence (arbitrary coefficients)

$$\cdots \to \check{H}^i(X^*, X^G) \to \check{H}^i(X) \to \check{H}^{i-3}(X^*, X^G) \oplus \check{H}^i(X^G) \to \check{H}^{i+1}(X^*, X^G) \to .$$

Also generalize this to pairs $(X, A)$, where $A$ is closed and invariant.

**13.** Let $X$ be a finitistic space such that $\check{H}^*(X; \mathbf{Z})$ is finitely generated. Suppose that $G = \mathbf{Z}_p$, $p$ prime, acts on $X$ with trivial action on $\check{H}^*(X; \mathbf{Z})$. Then show that $\chi(X) = \chi(X^G)$. (*Hint*: Apply 7.2 and Exercise 1.)

# CHAPTER IV

## LOCALLY SMOOTH ACTIONS ON MANIFOLDS

In this chapter we begin to study, systematically, actions of compact Lie groups on manifolds. With a few exceptions, the results considered in this chapter are of a rather general nature. Our study is carried out in the context of locally smooth actions, which should be regarded as the analog of the category TOP of topological manifolds. (We do not prove many deep results about this category of actions, in the sense of results unattainable for arbitrary actions of compact Lie groups on manifolds, but it is hoped that the study of the locally smooth category may prove to be fruitful in the future.)

Locally smooth actions are defined in Section 1 and it is shown that such actions on $\mathbf{R}^n$ have only finitely many orbit types. Orientability of fixed point sets and related questions are considered in Section 2. The concepts of principal, exceptional, and singular orbits are studied in Sections 3 and 4. In Section 5 a theorem is proved which allows the reduction of certain questions to the case in which the principal isotropy group is finite. Section 6 is devoted to a result of Borel concerning actions on a sphere with one orbit type. Some general facts about the set of nonprincipal orbits are considered in Section 7. In Section 8 we prove a theorem of Montgomery, Samelson, and Yang which states that a locally smooth action on $\mathbf{R}^n$ is orthogonal if it has an orbit of codimension 1 or 2. A theorem of Conner and Montgomery, showing that actions on tori are particularly simple, is proved in Section 9. In Section 10 we prove a result of Mann which states that an action on an orientable manifold with finitely generated homology has only finitely many orbit types.

We shall assume all manifolds to be paracompact, whether or not this is explicitly stated.

### 1. LOCALLY SMOOTH ACTIONS

Let $M$ be a $G$-space, $G$ compact Lie. Let $P$ be an orbit of type $G/H$ and let $V$ be a euclidean space on which $H$ operates *orthogonally.* Then a **linear**

**tube** about $P$ in $M$ is a tube ($G$-equivariant embedding onto an open neighborhood of $P$) of the form

$$\varphi\colon G \times_H V \to M.$$

If $x = \varphi[e, v]$ is in $P$, then $v$ is stationary under $H$ and, since the translation of $V$ moving $v$ to 0 is equivariant, it is no loss of generality to assume that $v = 0$.

Let $S$ be a slice at $x$ in $M$. Then $S$ is called a **linear slice** if the canonically associated tube

$$G \times_{G_x} S \to M$$

$([g, s] \mapsto g(s))$ is equivalent to a linear tube; that is, if the $G_x$-space $S$ is equivalent to an orthogonal $G_x$-space.

Note that II.5.2 says that there is a linear tube about each orbit of an orthogonal action on $\mathbf{R}^n$. (In a later chapter we shall show that this is true for any differentiable action.)

We will say that the $G$-space $M$ is **locally smooth** if there exists a linear tube about each orbit.

Since $G \times_H V$ is a $V$-bundle over $G/H$, it is a manifold. Thus $M$ must be a topological manifold if it is locally smooth. The category of locally smooth actions is the analog, in the theory of $G$-spaces, of the category TOP of topological manifolds.

If the $G$-space $M$ is locally smooth and if $x$ is a fixed point, then a neighborhood of $x$ in $M$ is equivalent to an orthogonal action. This shows that $M^G$ is a topological submanifold of $M$ in this situation.

If one takes the open cones over the examples of Chapter I, Section 7, then one obtains actions on euclidean space whose fixed sets are not locally euclidean at the origin. Thus not all actions on manifolds are locally smooth.

**1.1. Proposition** *If $M$ is a locally smooth $G$-space and if $K \subset G$ is a closed subgroup, then $M$ is locally smooth as a $K$-space.*

*Proof* Let $P$ be a $G$-orbit of type $G/H$ in $M$. Considering $G$ as an orbit of the $(G \times G)$-action (by left and right translation) recall that $G$ can be embedded as an orbit in an orthogonal $(G \times G)$-action on some $\mathbf{R}^n$; see Chapter II, Section 5. Restrict this action to one of $K \times H$, and consider the orbit $KH$ of $e$ in $G \subset \mathbf{R}^n$. (Note that the isotropy group at $e$ is $\{(l, l^{-1}) \mid l \in K \cap H\} \approx K \cap H$.) By II.5.2 an $\varepsilon$-ball $V_\varepsilon$ in the normal space to $KH$ in $\mathbf{R}^n$ at $e$ is a slice for this $K \times H$ action on $\mathbf{R}^n$. Then it is clear that

$W = V_\varepsilon \cap G$ is a slice for the action of $K \times H$ on $G$ at $e$. Now $W$ is equivalent (as a $(K \times H)$-space) to the intersection of $V_\varepsilon$ with the tangent space of $G$, for $\varepsilon$ small, and hence is a linear slice at $e$ in $G$ for the action of $K \times H$.

Put $L = K \cap H$. Since $K$ acts from the left and $H$ from the right in this action of $K \times H$ on $G$, $L = K \cap H$ acts on $W$ by $(l, w) \mapsto lwl^{-1}$.

Letting $L$ act by translation on the right of $K$ and on the left of $H$ [and hence on $W \times H$ by $\big(l, (w, h)\big) \mapsto (lwl^{-1}, lh)$], the above remarks show that the map

$$\varphi \colon K \times_L (W \times H) \to G$$

defined by $\varphi[k, (w, h)] = kwh$, is a tube about $KH$ in $G$ for the action of $K \times H$.

Now let

$$\psi \colon G \times_H U \to M$$

be a linear tube about the given orbit $P$ with $x = \psi[e, 0]$. (Recall that $H = G_x$.) Then the composition

$$\big(K \times_L (W \times H)\big) \times_H U \xrightarrow{\varphi \times_H U} G \times_H U \xrightarrow{\psi} M$$

is a $K$-equivariant embedding onto an open set, by II.2.1. But

$$\begin{aligned}\big(K \times_L (W \times H)\big) \times_H U &\approx K \times_L \big((W \times H) \times_H U\big) \\ &\approx K \times_L \big(W \times (H \times_H U)\big) \approx K \times_L (W \times U)\end{aligned}$$

and the induced map

$$\theta \colon K \times_L (W \times U) \to M$$

is just $\theta[k, (w, u)] = \psi[\varphi[k, (w, e)], u] = \psi[kw, u]$. Thus $\theta$ is a $K$-equivariant homeomorphism onto an open neighborhood of the $K$-orbit of $\theta[e, (e, 0)] = \psi[e, 0] = x$. Since $W$ and $U$ are orthogonal $L$-spaces with $W \times U$ having the diagonal action, and since $L = K \cap H = K_x$, $\theta$ is a linear tube about $K(x)$ in $M$. ▌

*Remark* This proposition is, of course, a trivial corollary of the fact, which we shall prove in Chapter VI, that a differentiable action is locally smooth. The reader with good background in differential geometry may well prefer that approach.

Since an orthogonal $G$-action has only finitely many orbit types (see Chapter II, Exercise 2) and since the isotropy types of $G$ on $G \times_H V$ come from those of $H$ on $V$ (see II.(3.4)) the following fact is clear.

**1.2. Proposition** *Let $M$ be a locally smooth $G$-space and let $C$ be a compact subset of $M$. Then the orbit types of orbits touching $C$ are finite in number.* ∎

*Remark* This result is true for arbitrary actions on manifolds (and on generalized manifolds) by compact Lie groups, but the proof is considerably more difficult. The general case is due to Floyd [8], Mann [1], and Mostow [2]. For an exposition of this result see Borel [5, Chapters VI, VII]. The proof of the following result is adapted from Floyd [11] and contains a few aspects of the proof of this general result.

**1.3. Theorem** *Let $M$ be a paracompact topological manifold with acyclic* mod $p$ *cohomology for all primes $p$* (e.g., $M = \mathbf{R}^n$). *Then any locally smooth action of a toral group $T$ on $M$ has only a finite number of orbit types.*

*Proof* Suppose that $x_1, x_2, \ldots$ is a sequence of points in $M$ with distinct isotropy groups. Since the collection of closed subsets of $T$ form a compact metric space with the Hausdorff metric

$$d(A, B) = \max d(a, B) + \max d(A, b),$$

we may assume, by passing to a subsequence, that the $T_{x_i}$ converge to $G \subset T$ (necessarily a subgroup). By II.5.6 we see that $T_{x_i} \subset G$ for $i$ large, since $T$ is abelian. We may as well assume this for all $i$. Suppose that we could show that $F(T_{x_i}, M) = F(T_{x_j}, M)$ for $i$, $j$ sufficiently large. Then $T_{x_i}$ fixes $x_j$, so that $T_{x_i} \subset T_{x_j}$, and similarly $T_{x_j} \subset T_{x_i}$, for $i$, $j$ large, contradicting our assumption. Thus we have reduced the proof to the following lemma.

**1.4. Lemma** *Let $M$ be as in* 1.3. *Let $G$ be a compact abelian Lie group acting locally smoothly on $M$ and let $\{G_i \mid i = 1, 2, \ldots\}$ be a sequence of subgroups of $G$ converging to $G$. Then $F(G_i, M) = F(G_j, M)$ for all sufficiently large $i$ and $j$.*

*Proof* Let us first note the following fact. If $S$ is a toral group operating on $M$ and if $K_1 \subset K_2 \subset \ldots$ is a sequence of $p$-groups ($p$ a fixed prime) converging to $S$, then $F(K_1, M) \supset F(K_2, M) \supset \cdots$ is a sequence of mod $p$ acyclic (and hence connected) manifolds with intersection $F(S, M)$. However, dim $F(K_i, M)$ is eventually constant and hence the $F(K_i, M)$ are eventually constant by the invariance of domain. Thus $F(S, M)$ is a mod $p$ acyclic manifold (for all $p$).

Also note that $F(G, M) = \bigcap F(G_i, M)$.

Now we shall prove the lemma by induction on $\dim G$, and we may assume that $\dim G > 0$. By passing to a subsequence, we may assume that the $F(G_i, M)$ are *all distinct*. Since the $G_i$ converge to $G$, it is clear that ord $G_i$ converges to infinity (where ord $G_i = \infty$ when $\dim G_i > 0$). (To see this, intersect $G_i$ with the identity component of $G$ and look at the projections on some circle factor of this.) Thus we may choose subgroups $H_i$ of $G_i$ of *prime power order* (for varying primes) such that ord $H_i$ converges to infinity, (since the largest prime power dividing $n$ converges to infinity as $n$ goes to infinity). By passing to a subsequence, we see that we may as well assume that the $H_j$ converge to a closed subgroup $S$ of $G$. Then $\dim S > 0$, since ord $H_i$ goes to infinity. By intersecting the $H_i$ with the identity component of $S$, we see that we may assume that $S$ is connected, and hence is a torus.

Since $G$ acts locally smoothly, and hence has only finitely many orbit types near any given fixed point $x$, it follows that $F(H_i, M)$ coincides with $F(S, M)$ near $x$ for $i$ large. But these are both connected manifolds and hence $F(H_i, M) = F(S, M)$ for $i$ large. Then, for $i$ large,

$$F(G_i, M) = F(G_i, F(H_i, M)) = F(G_i, F(S, M)) = F((G_iS)/S, F(S, M)).$$

Since $\dim G/S < \dim G$, the inductive assumption implies that the fixed sets on the right side of this equality are all equal for sufficiently large $i$, which contradicts our assumption that the $F(G_i, M)$ are all distinct. ∎

*Remark* In Section 10 we shall prove a result of Mann [2] which shows that the conclusion of 1.3 holds for all orientable manifolds having finitely generated homology.

By 1.3 and III.10.14 we have

**1.5. Corollary** *If $M$ is a paracompact manifold which is acyclic over the integers* (e.g., $M = \mathbf{R}^n$) *and if the toral group $T$ acts locally smoothly on $M$, then $M^T$ is also acyclic over the integers.* ∎

If $\varphi$: $G \times_H V \to M$ is a linear tube with $\varphi[e, 0] = x$, then the representation of $G_x = H$ on $V$ is called a **slice representation** at $x$. We do not call it *the* slice representation at $x$, since it is by no means clear that it is uniquely determined by $x$ (up to linear equivalence). It is, in fact, unknown whether or not this is the case. (We remark, however, that for *differentiable* actions, where $\varphi$ is required to be a diffeomorphism, the slice representation *is* uniquely determined by $x$.) In the case of a fixed point $x$, this question reduces to the question of whether two topologically equivalent representations of a compact Lie group are necessarily linearly equivalent, and this is unknown in general.

We conclude this section by remarking on the definition of local smoothness for manifolds $M$ with boundary. For this we require, for orbits $P$ lying in $\partial M$, tubes of the form

$$\varphi\colon G \times_H V^+ \to M,$$

where $\varphi[e, 0] \in P$ and $V^+$ denotes the half space $x_1 \geq 0$ in $V = \mathbf{R}^n$, where $x_1$ denotes the first coordinate, and where $H$ acts orthogonally on $V^+$. (Of course, for $V^+$ to be invariant, $H$ must leave the $x_1$-axis stationary.)

## 2. FIXED POINT SETS OF MAPS OF PRIME PERIOD

In this section we shall prove some results concerning fixed point sets of locally smooth actions on $M$ of a cyclic group $G$ having prime order $p$. Our first result shows that if $M$ is orientable and $p$ is odd, then $M^G$ is also orientable. That this does not hold for $p = 2$ is shown by the involution $(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : x_2 : x_3)$ on real projective 3-space $\mathbf{P}^3$ whose fixed set is the disjoint union of a point and $\mathbf{P}^2$.

**2.1. Theorem** *If $G$ is a finite group of odd order or is a torus and if $G$ acts locally smoothly on an orientable n-manifold $M$, then each component of $M^G$ is orientable.*

***Proof*** The toral case reduces to the case of a $p$-group since each component of $M^G$ is also a component of the fixed set of some subgroup of order a power of $p$ (see the proof of 1.4). Also an obvious induction (using the fact that a group of odd order is solvable) reduces the finite group case to that of cyclic $G$ of odd prime order $p$, and we now assume this.

Let $x$ be a fixed point and assume that its component in $M^G$ has dimension $r$. Let $U$ be an open $n$-cell about $x$ on which $G$ acts orthogonally (by local smoothness). Let $M_+$ be the one-point compactification of $M$ with ideal point $\infty$. Since $(M_+, M_+ - U)$ is a cohomology $n$-cell, by excision, the remarks following III.5.2 (translated to cohomology) leading to the isomorphism III.(5.3), show that there is a commutative diagram (where coefficients are in $\mathbf{Z}_p$ and $B = M_+ - U$)

$$\begin{array}{ccccccccc}
\check{H}^r(M_+^G, B^G) & \xrightarrow{\approx} & \check{H}_\sigma^{r+1}(M_+, B) & \xrightarrow{\approx} & \cdots & \xrightarrow{\approx} & \check{H}_\tau^n(M_+, B) & \xleftarrow{\approx} & \check{H}^n(M_+, B) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow\approx \\
\check{H}^r(M_+^G, \infty) & \xrightarrow{\delta^*} & \check{H}_\sigma^{r+1}(M_+, \infty) & \xrightarrow{\delta^*} & \cdots & \xrightarrow{\delta^*} & \check{H}_\tau^n(M_+, \infty) & \xleftarrow{i^*} & \check{H}^n(M_+, \infty).
\end{array}$$

(The right-hand vertical map is an isomorphism since $M$ is orientable.) Now $\check{H}^n(M_+, \infty) \approx \mathbf{Z}_p$ and the same argument with Smith sequences given under III.5.2 (and in the proof of III.9.2) shows that the homomorphism $i^*\colon \check{H}^n(M_+, \infty) \to \check{H}_\tau^n(M_+, \infty)$, on the bottom row, is an *isomorphism.* Thus the diagram shows that $\check{H}^r(M_+^G, (M_+ - U)^G) \to \check{H}^r(M_+^G, \infty)$ is a monomorphism.

If $F$ is the component of $M^G$ containing $x$ and $F_+ = F \cup \{\infty\}$, then the map $j_F\colon M_+^G \to F_+$ taking all other components of $M^G$ to $\infty$, is a retraction. This gives the diagram

$$\begin{array}{ccc}
\mathbf{Z}_p \approx \check{H}^r(F_+, F_+ - U) & \xrightarrow{\approx} & \check{H}^r(M_+^G, (M_+ - U)^G) \\
\downarrow & & \downarrow \\
\check{H}^r(F_+, \infty) & \xrightarrow{j_F^*} & \check{H}^r(M_+^G, \infty)
\end{array}$$

which shows that $\check{H}^r(F_+, \infty) \neq 0$, whence $F$ is orientable (since $p$ is odd). ∎

Note that this discussion also holds with $p = 2$ (but does not show $F$ to be orientable, of course). Let us look at this proof a bit more closely in the case for which $M$ is *compact* (for convenience only). Let $\theta_r$ denote the composition

$$\check{H}^r(M^G) \xrightarrow{\delta^*} \check{H}_\sigma^{r+1}(M) \xrightarrow{\delta^*} \cdots \xrightarrow{\delta^*} \check{H}_\varrho^n(M) \underset{\approx}{\xrightarrow{i^{*-1}}} \check{H}^n(M)$$

($\varrho$ may be $\sigma$ when $p = 2$). Then we have shown that

$$\theta_r j_F^*\colon \check{H}^r(F) \to \check{H}^n(M)$$

is nontrivial, and hence is an isomorphism, when $F$ is a component of $M^G$ and $M$ is orientable (or $p = 2$). Now

$$\check{H}^r(M^G) \approx \prod_F \check{H}^r(F)$$

over all components $F$ of $M^G$, with canonical inclusions $j_F^*$ (and projections, the restriction maps $\check{H}^r(M^G) \to \check{H}^r(F)$ induced by the inclusions $F \subset M^G$).

Let $0 \neq \lambda \in \check{H}^n(M)$ be an "orientation" over $\mathbf{Z}_p$ and let $\lambda_F' \in \check{H}^r(F)$ be the induced orientation of $F$, where $F$ is a component of $M^G$ of dimension $r$. That is, $\lambda_F'$ is chosen such that

$$\theta_r j_F^*(\lambda_F') = \lambda.$$

Put $\lambda_F = j_F^*(\lambda_F') \in \check{H}^r(M^G)$. Suppose that $F_1, \ldots, F_k$ are the components of $M^G$ of dimension $r$ (where $r$ is now fixed) and consider the element

$$\alpha = \sum_{i=1}^{k} a_i \lambda_{F_i}; \qquad a_i \in \mathbf{Z}_p$$

of $\check{H}^r(M^G)$. We have $\theta_r(\alpha) = (\sum a_i)\lambda$. Now $\delta^*\colon \check{H}^r(M^G) \to \check{H}_\sigma^{r+1}(M)$ kills $\alpha$ only if $\theta_r$ does, and thus only if $\sum a_i = 0$ in $\mathbf{Z}_p$. The diagram III.(7.8) has the form

$$\begin{array}{ccccc} \check{H}^r(M^*) & \longrightarrow & \check{H}^r(M^G) & \longrightarrow & \check{H}^{r+1}(M^*, M^G) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle \approx} \\ \check{H}^r(M) & \longrightarrow & \check{H}_\tau^{\,r}(M) \oplus \check{H}^r(M^G) & \xrightarrow{\delta^*} & \check{H}_\sigma^{r+1}(M) \end{array}$$

This shows that $\alpha$ is the image of an element of $\check{H}^r(M^*)$ only if $\delta^*(\alpha) = 0$ and thus only if $\sum a_i = 0$. Therefore we have proved the following theorem.

**2.2. Theorem** *Let $M$ be a compact, orientable (over $\mathbf{Z}_p$) manifold on which the cyclic group $G$ of prime order $p$ acts locally smoothly. Then $M^G$ is homologous to* 0 (mod $p$) *in $M^* = M/G$ in the following sense: If $r$ is any integer and $F_1, \ldots, F_k$ are the components of $M^G$ of dimension $r$, then there are nonzero classes $0 \neq \lambda_i' \in \check{H}^r(F_i;\ \mathbf{Z}_p)$, corresponding to $\lambda_i \in \check{H}^r(M^G; \mathbf{Z}_p)$, such that if*

$$\sum a_i \lambda_i \in \operatorname{Im}\big(\check{H}^r(M^*; \mathbf{Z}_p) \to \check{H}^r(M^G; \mathbf{Z}_p)\big),$$

*then $\sum a_i = 0$ in $\mathbf{Z}_p$.* ∎

We remark that the corresponding statement in *homology* would state that there are classes $0 \neq \mu_i' \in \check{H}_r(F_i; \mathbf{Z}_p)$ and $\mu' \in \check{H}_r(M^G - \bigcup F_i; \mathbf{Z}_p)$ (possibly 0), with images $\mu_i$ and $\mu$ in $\check{H}_r(M^G)$, such that $\mu + \sum \mu_i$ goes to 0 in $\check{H}_r(M^*; \mathbf{Z}_p)$.

**2.3. Corollary** *If $M$ is a compact manifold, orientable over $\mathbf{Z}_p$, then there is no locally smooth action of a cyclic group $G$ of prime order $p$ having precisely one fixed point.* ∎

**2.4. Corollary** *If $M$ is a compact manifold and if $T$ is a locally smooth involution on $M$ having only isolated fixed points, then the number of fixed points is even.* ∎

**2.5. Corollary** *Let $M$ be a compact manifold, orientable over $\mathbf{Z}_p$, and let $r$ be the maximum dimension of the components of $M^G$. Suppose that $M^G$ has just one component of dimension $r$, say $F$. Then $F$ is not a retract of $M^*$.*

*Proof* If $F$ is a retract of $M^*$, then $\check{H}^r(M^*; \mathbf{Z}_p) \to \check{H}^r(F; \mathbf{Z}_p)$ is onto, since the retraction gives a right inverse to this map. However, $\check{H}^r(F; \mathbf{Z}_p) = \check{H}^r(M^G; \mathbf{Z}_p)$ since the other components of $M^G$ all have dimension less than $r$, and this contradicts 2.2. ∎

*Remarks* Theorem 2.1 is due to Smith [2]. Mostow noticed the fact that Smith's proof of 2.1 also essentially proved 2.3. Theorem 2.2 and its corollaries are from Bredon [3] and we asked there whether these results extend to actions of general $p$-groups. That this is not the case is shown by the map $(z_0: z_1: z_2) \mapsto (\bar{z}_0: -\bar{z}_2: \bar{z}_1)$ on $\mathbf{CP}^2$ which has period 4 and exactly one fixed point $(1:0:0)$. (This is from Conner and Floyd [8].) Also the maps $(z_0: z_1: z_2: z_3) \mapsto (z_0: z_2: z_3: z_1)$ and $(z_0: z_1: z_2: z_3) \mapsto (z_0: z_1: z_2: \omega z_3)$ on $\mathbf{CP}^3$, where $\omega$ is a primitive cube root of unity, generate a group of order $3^4$ which has just one stationary point $(1:0:0:0)$. However, for differentiable actions of *cyclic* groups of *odd* prime power order, it was conjectured in Conner and Floyd [8] that 2.3 generalizes. This was later proved to be the case in Atiyah and Bott [2] and in Conner and Floyd [9]. Whether the more general result 2.2 holds for actions of cyclic groups of odd prime power order, is still in question.

We conclude this section by proving a relatively simple global version of a result of Yang [2]. It will be generalized considerably in Chapter V. To motivate it we recall that the fixed point set of an involution on a sphere is a mod 2 cohomology sphere. Also, the examples of Chapter I, Section 7 show that the fixed point set may have odd torsion in its homology and may have codimension 2. The following result shows that this cannot happen in codimension 1.

**2.6. Theorem** *Let $G \approx \mathbf{Z}_2$ act locally smoothly on a compact integral cohomology n-sphere $\Sigma$ with $\Sigma^G$ of dimension $n - 1$. Then $\Sigma^G$ is an integral cohomology $(n - 1)$-sphere, and $\Sigma^*$ is acyclic over the integers and has a global cross section in $\Sigma$.*

*Proof* We know that $\Sigma^G$ has the mod 2 cohomology of $\mathbf{S}^{n-1}$. The exact sequence

$$0 \to \check{H}^{n-1}(\Sigma^G; \mathbf{Z}_2) \to \check{H}^n(\Sigma, \Sigma^G; \mathbf{Z}_2) \to \check{H}^n(\Sigma; \mathbf{Z}_2) \to 0$$

of the pair $(\Sigma, \Sigma^G)$ and Alexander duality $\check{H}^n(\Sigma, \Sigma^G; \mathbf{Z}_2) \approx H_0(\Sigma - \Sigma^G; \mathbf{Z}_2)$ show that $\Sigma - \Sigma^G$ has two components. Let $M_1$ and $M_2$ be the closures of these components, so that $M_1 \cup M_2 = \Sigma$ and $M_1 \cap M_2 = \Sigma^G$. Consideration of a neighborhood of a point in $\Sigma^G$ shows that $G$ permutes $M_1$ and $M_2$ and hence that $M_1 \approx M_2 \approx \Sigma^*$. The map $\Sigma \to \Sigma^* \approx M_1$ shows that $M_1$ is a retract of $\Sigma$. Thus $\check{H}^*(\Sigma^*; \mathbf{Z})$ is a direct summand of $\check{H}^*(\Sigma; \mathbf{Z})$ and it follows that $\Sigma^*$ is acyclic. Poincaré–Lefschetz duality $\tilde{\check{H}}^{i-1}(\Sigma^G; \mathbf{Z}) \approx \check{H}^i(\Sigma^*, \Sigma^G; \mathbf{Z}) \approx H_{n-i}(\Sigma^*; \mathbf{Z})$ implies that $\Sigma^G$ is an integral cohomology $(n - 1)$-sphere. ∎

## 3. PRINCIPAL ORBITS

In this section we shall prove the existence of a maximum orbit type for a locally smooth action on a connected manifold, and shall discuss its consequences. As with most of the results of this chapter, this result can be proved for actions which are not locally smooth (see Montgomery [6]) but it is considerably more difficult in that case. Throughout this section $G$ will be a compact Lie group acting locally smoothly on the $n$-manifold $M$ with $M^* = M/G$ *connected.*

**3.1. Theorem** *There exists a maximum orbit type $G/H$ for $G$ on $M$* (i.e., *$H$ is conjugate to a subgroup of each isotropy group*). *The union $M_{(H)}$ of the orbits of type $G/H$ is open and dense in $M$ and its image $M^*_{(H)}$ in $M^* = M/G$ is connected.*

*Proof* The proof will proceed by induction on the dimension $n$ of $M$. Using the inductive assumption, we shall first prove the result "locally" on $M^*$. Consider a linear tube

$$G \times_K V \to M.$$

Then $K$ acts orthogonally on $V$ and thus its action may be thought of as that on the open cone over the action of $K$ on the unit sphere $S$ in $V$ about the origin. Now $\dim S < \dim M$ (in fact $\dim V < \dim M$ unless $K = G$) and $K$ is locally smooth on $S$ by II.5.2. By the inductive assumption there exists a subgroup $H$ of $K$ with $S_{(H)}$ open and dense and $S^*_{(H)}$ connected. (There is one exception to this, when $S$ is the 0-sphere and $K$ acts trivially on $S$; but then all orbits of $K$ on $V$ are the same, namely trivial.) Then $V_{(H)}$ and $V^*_{(H)}$ are the open cones *less the vertex* on $S_{(H)}$ and $S^*_{(H)}$, respectively; or, when $H = K$, $V_{(H)} = V$ and $V^*_{(H)} = V^*$. Thus $V^*_{(H)}$ is connected, open, and dense in $V^*$, and all other orbits of $K$ in $V$ have strictly smaller type (by the inductive assumption or simply by the density of $V_{(H)}$ and by II.5.5). Suppose that $K/L$ is another orbit type occuring in $V$. Then, by conjugating, we may assume that $H \subset L \subset K$ with $H \neq L$. Then $L$ and $H$ differ either in dimension or number of components and thus cannot be conjugate in $G$. Since subgroups of $K$ which are conjugate in $K$ are, a fortiori, conjugate in $G$, it follows from this that

$$(G \times_K V)_{(H)} = G \times_K (V_{(H)}).$$

Also

$$(G \times_K V)^*_{(H)} \approx V^*_{(H)}$$

under the homeomorphism $(G \times_K V)^* \approx V^*$ of II.3.3, and this is connected, open, and dense in $(G \times_K V)^*$.

Thus, for each point $x$ in $M$, we have found a neighborhood $U_x{}^*$ of $x^*$ and a connected, open, dense subset $W_x{}^*$ of $U_x{}^*$ such that all orbits in $W_x{}^*$ have the same type and all other orbits in $U_x{}^*$ have strictly smaller type.

Now if $H$ is any subgroup of $G$, let $C_{(H)}$ be the *closure* of int $M^*_{(H)}$. Then $x^*$ is in $C_{(H)}$ iff $W_x{}^*$ consists of orbits of type $G/H$ and, in this case, $C_{(H)} \supset U_x{}^*$. Thus $C_{(H)}$ is both open and closed and it follows that $C_{(H)} = M^*$ for some $H$ (now fixed) and $C_{(K)} = \varnothing$ if $K$ is not conjugate to $H$. Then $M^*_{(H)}$ is open since $M^*_{(H)} \cap U_x{}^* = W_x{}^*$, and it is also dense. Also all other orbits have type strictly smaller than that of $G/H$. If $D$ is a component of $M^*_{(H)}$, then, since $W_x{}^*$ is connected for each $x$, we see that $\bar{D}$ is open (and closed) in $M^*$. Hence $M^*_{(H)} = D$ is connected. ∎

The maximum orbit type for orbits in $M$ guaranteed by 3.1 is called the **principal orbit type** and orbits of this type are called **principal orbits**. The corresponding isotropy groups are called **principal isotropy groups**. If $P$ is a principal orbit and $Q$ is any orbit, then there is an equivariant map

$P \to Q$. If $P \approx G/H$ and $Q \approx G/K$, then $H$ is conjugate to a subgroup of $K$ and, without loss of generality, we may assume that $K \supset H$. Then an equivariant map $P \to Q$ is a fiber bundle projection $G/H \to G/K$ with fiber $K/H$. If $\dim P > \dim Q$ (i.e., $\dim K/H > 0$), then $Q$ is called a **singular orbit**. If $\dim P = \dim Q$, but $P$ and $Q$ are not equivalent (i.e., $P \to Q$ is a nontrivial covering map; that is, $K/H$ is finite and nontrivial), then $Q$ is called an **exceptional orbit.**

Let us collect some elementary properties which will be used continually.

**3.2. Theorem** *Let $x \in M$ with $K = G_x$ and let $V$ be a linear slice at $x$. Let $v \in V$. Then*

(i) *$K(v)$ is principal, exceptional, or singular for $K$ on $V$ according as $G(v)$ is principal, exceptional or singular for $G$ on $M$.*

(ii) *If $H \subset K$ is a principal isotropy group for $K$ on $V$ (and hence principal for $G$ on $M$), then $G(x)$ is nonsingular* (i.e., *principal or exceptional*) *iff $K/H$ is finite. In this case $H$ is normal in $K$ and is precisely the noneffective part of $K$ on $V$.*

(iii) *$G(x)$ is principal iff $K = G_x$ acts trivially on $V$. In this case $G \times_K V = (G/K) \times V$.*

(iv) $(G \times_K V)_{(K)} = G \times_K V^K = (G/K) \times V^K$.

(v) *If $H \subset K$ is a principal isotropy group for $K$ on $V$, then $(G \times_K V)_{(H)} = G \times_K (V_{(H)})$.*

*Proof* Let $K(w)$ be principal in $V$ and suppose, as we may, that $H = K_w \subset K_v$. Since $G \times_K (V_{(H)})$ is open in $G \times_K V$, and consists of orbits of type $G/H$, we see that $H$ is a principal isotropy group for $G$ on $M$. Since $K(v)$ is principal, exceptional, or singular according as $K_v/K_w = G_v/G_w$ is trivial, finite and nontrivial, or infinite, we obtain (i). Part (v) is immediate from this. Part (iii) is clear, as is the first part of (ii). If $K/H$ is finite, then $K$-orbits in $V$ are finite and a slice at a point in $V$ is just a small open neighborhood of the point in $V$. Thus (in this case) $H$ leaves an open set stationary and hence $H$ acts trivially on $V$. From this it is clear that $H$ is just the noneffective part of $K$ on $V$ and, in particular, $H$ is normal in $K$. This proves (ii). If $G(v)$ is an orbit of type $G/K$, then $K \sim G_v \subset K$ and this implies that $G_v = K$ (that is, $v \in V^K$). This implies (iv). ▮

Note that if $M$ is a sphere, for example, then by 3.2(iii), a principal orbit embeds in the sphere $M$ with a *product neighborhood.* This is known to be a strong restriction on the possibilities for the principal orbit. (It implies that the principal orbit is stably parallelizable and hence that all of its characteristic classes are zero.)

We shall now proceed to analyze the orbit structure of a locally smooth action a little more closely.

**3.3. Theorem** *Let $K$ be a subgroup of $G$ and let $M_{(K)}$ denote the set of points on orbits of type $G/K$. Then $M_{(K)}$ is a topological manifold (locally closed in $M$), and its closure $\overline{M_{(K)}}$ consists of orbits of type less than or equal to* type $G/K$. *The orbit map $M_{(K)} \to M^*_{(K)}$ is a fiber bundle projection with fiber $G/K$ and structure group $N(K)/K$.*

*Proof* If $x \in \overline{M_{(K)}}$, then type $G(x) \leq$ type $G/K$ by II.5.5. If equality holds, then any orbit in $\overline{M_{(K)}}$ near $G(x)$ has type greater than or equal to type $G(x)$ and hence has type equal to type $G(x)$. Thus $M_{(K)}$ is open in $\overline{M_{(K)}}$ and hence is locally closed in $M$. Let $P$ be an orbit in $M_{(K)}$. Then $P$ has a linear tube of the form $G \times_K V$. Clearly, the subspace of this consisting of orbits of type $G/K$ is just $G \times_K V^K \approx (G/K) \times V^K$ which is a manifold since $V^K$ is a linear subspace of $V$. The last statement follows from II.5.8 (or directly from the discussion above). ▌

**3.4. Lemma** *For any $K \subset G$, $M_{(K)}$ consists of only finitely many components in the neighborhood of any orbit of $M$.*

*Proof* By induction on dim $M$ this is satisfied for the action on the unit sphere in a linear slice. Compactness shows that it holds globally on the sphere and then it is clearly satisfied on the associated linear tube. ▌

**3.5. Lemma** *Let $C \subset M$ be locally closed and invariant. Then* $\dim C = \max \dim(C \cap M_{(K)})$ *and* $\dim C^* = \max \dim(C^* \cap M^*_{(K)})$, *where $K$ ranges over the subgroups of $G$.*

*Proof* Since dimension has a local character, we may assume that there are only finitely many orbit types in $C$. Then we can prove the lemma by induction on the number of orbit types occurring in $C$. Let $L \subset G$ be such that $C \cap M_{(L)} \neq \varnothing$ and such that no larger orbit type occurs in $C$. Then $M_{(L)} \cap C$ is open and, by induction, $\dim(C - M_{(L)}) = \max\{\dim(C \cap M_{(K)}) \mid K \not\sim L\}$. But $\dim C = \max\{\dim(C - M_{(L)}), \dim(C \cap M_{(L)})\}$ by Hurewicz and Wallman [1, p. 32], which proves the assertion (the statement on $C^*$ following in the same way). ▌

**3.6. Theorem** *Let $K$ be any subgroup of $G$. If $D$ is a component of $M^*_{(K)}$, then* $\dim(\bar{D} - D) < \dim D$. *Thus*

$$\dim(\overline{M^*_{(K)}} - M^*_{(K)}) < \dim M^*_{(K)} .$$

*Proof* Note that the orbit types in $\bar{D} - D$ are strictly less than type $G/K$. By 3.5 it suffices to show that if $L$ contains $K$ properly, then $\dim(\bar{D} \cap M^*_{(L)}) < \dim D$. By taking a linear slice at a point of an orbit in $\bar{D} \cap M^*_{(L)}$ this is reduced to the case $L = G$ (and to an orthogonal action); that is, to showing that $\dim M^G < \dim D$ (where $D$ consists of orbits of nontrivial type). However, if $S$ is a sphere in the orthogonal complement of $M^G$ (in the orthogonal case), then $S^* \cap D \subset S^*_{(K)}$ is a manifold and $D$ is just the product of $M^G$ with $(S^* \cap D) \times \mathbf{R}^+$, whence $\dim D \geq \dim M^G + 1$. ∎

For any subgroup $K$ of $G$ we put

$$B_{(K)} = \{x \in \overline{M_{(K)}} \mid \dim G(x) < \dim G/K\};$$

$$E_{(K)} = \{x \in \overline{M_{(K)}} \mid \dim G(x) = \dim G/K \text{ but type } G(x) \neq \text{type } G/K\}.$$

Note that $B_{(K)}$ and $B_{(K)} \cup E_{(K)} = \overline{M_{(K)}} - M_{(K)}$ are both *closed sets*. If $H$ is a principal isotropy group, then $\overline{M_{(H)}} = M$ and thus $B_{(H)}$ is just the set of points on *singular* orbits, and $E_{(H)}$, those on *exceptional* orbits. We put $B = B_{(H)}$ and $E = E_{(H)}$ when $H$ is principal.

**3.7. Proposition** *For any subgroup $K$ of $G$ we have*

$$\dim B_{(K)} \leq \dim M_{(K)} - 2,$$
$$\dim E_{(K)} \leq \dim M_{(K)} - 1.$$

*Proof* By 3.5 it suffices to show that $\dim (B_{(K)} \cap M_{(L)}) \leq \dim M_{(K)} - 2$ (and similarly for $E_{(K)}$). However, $B_{(K)} \cap M_{(L)}$ is a bundle over $B^*_{(K)} \cap M^*_{(L)}$ with fiber $G/L$. By 3.6 we have

$$\begin{aligned} \dim B_{(K)} \cap M_{(L)} &= \dim(B^*_{(K)} \cap M^*_{(L)}) + \dim (G/L) \\ &\leq \dim M^*_{(K)} - 1 + \dim(G/L) \\ &\leq \dim M^*_{(K)} - 1 + \dim(G/K) - 1 \\ &= \dim M_{(K)} - 2. \end{aligned}$$

The other inequality follows in the same way. ∎

**3.8. Theorem** *If $d$ is the dimension of a principal orbit* (i.e., *the maximal dimension among all orbits*), *then* $\dim M^* = n - d$ *and* $\dim(B^* \cup E^*) \leq n - d - 1$. *Also the union of all nonprincipal orbits of dimension at most $t$ is a closed set of dimension at most $n - d + t - 1$. Thus* $\dim B \leq n - 2$ *and* $\dim(B \cup E) \leq n - 1$.

*Proof* If $H$ is the principal isotropy group, then $M_{(H)}$ is open and thus has dimension $n$. This is a $G/H$-bundle over $M^*_{(H)}$, so that $\dim M^*_{(H)} = n - d$. By 3.6, $\dim(B^* \cup E^*) \leq n - d - 1$ and thus $M^* = M^*_{(H)} \cup B^* \cup E^*$ has dimension $n - d$. The next to last statement follows from the inequality $\dim(B^* \cup E^*) \leq n - d - 1$ and from 3.5 (see the proof of 3.7). Of course, the last statement follows from this, or directly from 3.7. ▮

We shall need the following fact later in the section.

**3.9. Proposition** *Let $K$ be a subgroup of $G$ and suppose that $M_{(K)}$ has dimension $k$. Then*

$$\check{H}^k(G(M^K)_+, \infty; \mathbf{Z}_2) \neq 0.$$

*Proof* Let $C = G(M^K)$ which is just the union of all orbits of type less than or equal to type $G/K$. Let $N = N(K)$ and consider the twisted product $G \times_N M^K$. This is a manifold since it is a bundle over $G/N$ with fiber $M^K$. Let

$$\varphi\colon G \times_N M^K \to C$$

be the action map $\varphi[g, x] = g(x)$. Now $G \times_N (M^K \cap M_{(K)})$ is an open subset of $G \times_N M^K$ and $\varphi$ takes it *homeomorphically onto* $M_{(K)}$ by II.5.9. Let

$$V \subset G \times_N (M^K \cap M_{(K)})$$

be an open $k$-cell and let $W$ be the component of $G \times_N M^K$ containing $V$ (so that $W$ is a connected $k$-manifold). Let $V' = \varphi(V)$ be the *homeomorphic image* of $V$ in $M_{(K)} \subset C$. The map $\varphi$ (which is proper) and the inclusions give a commutative diagram (where $+$ denotes one-point compactification, as usual, and coefficients are in $\mathbf{Z}_2$)

$$\begin{array}{ccc}
\check{H}^k(C_+, C_+ - V') & \xrightarrow[\approx]{\varphi^*} & \check{H}^k(W_+, W_+ - V) \\
\downarrow & & \downarrow \approx \\
\check{H}^k(C_+, \infty) & \xrightarrow{\varphi^*} & \check{H}^k(W_+, \infty)
\end{array}$$

which shows that $\check{H}^k(C_+, \infty) \neq 0$ as claimed. ▮

*Remark* For later reference let us note that the proof of 3.9 can easily be sharpened to yield the following further information. Suppose that $A$ is a closed set containing $G(M^K)$ and that there exists a $k$-manifold $U$ which is an open subset of $A$ with $U \subset A \cap M_{(K)}$, where $k = \dim M_{(K)}$. Then $\check{H}^k(A_+, \infty; \mathbf{Z}_2) \neq 0$. Moreover, if $U$ is nonorientable, then $\check{H}^k(A_+, \infty; \mathbf{Z})$ contains 2-torsion.

Suppose that $G(x)$ is an exceptional orbit and let $S$ be a linear slice at $x$. If $H \subset G_x$ is a principal isotropy group for $G_x$ on $S$, then, by 3.2, $H$ is just the ineffective part of $G_x$ on $S$ and thus $G_x/H$ is a finite group acting effectively on $S$. If $S^{G_x}$ has codimension 1 in $S$, then $G(x)$ will be called a **special exceptional orbit**. In this case, $G_x/H$ has order 2 and acts by reflection across the hyperplane $S^{G_x}$ of $S$. Thus the set $SE$ of points on special exceptional orbits is open in $E$ and has dimension $d + \dim S - 1 = d + (n - d) - 1 = n - 1$. If $x$ is on an exceptional orbit which is *not* special exceptional, then, for a slice $S$ at $x$, $G(S)_{(G_x)} = G(S^{G_x})$ has dimension equal to $\dim G(x) + \dim S^{G_x} < d + (n - d - 1) = n - 1$. Thus it follows that

**(3.10)**
$$\dim(E - SE) \leq n - 2,$$
$$\dim(E^* - SE^*) \leq n - d - 2.$$

**3.11. Proposition** *If $M$ is orientable, then principal orbits are orientable. If $M$ is orientable and the principal orbits are connected (so that all orbits are connected), then the special exceptional orbits are nonorientable.*

*Proof* If $P = G/H$ is principal, then $P$ has a product neighborhood in $M$ which is orientable, and thus $P$ must be orientable. If $Q \approx G/K$, $K \supset H$, is special exceptional and $G/H$ is connected, then $Q$ has a neighborhood of the form

$$G \times_K V \approx (G/H) \times_{K/H} V$$

and $K/H \approx \mathbf{Z}_2$ reverses orientation on $V$. Since this neighborhood is orientable and since $G/H$ is connected, $K/H$ must reverse the orientation of $G/H$. Hence $G/K = (G/H)/(K/H)$ is nonorientable. ∎

The following result shows that $SE$ is empty in most cases of interest.

**3.12. Theorem** *If $H_1(M; \mathbf{Z}_2) = 0$ and if a principal orbit is connected (and hence all orbits are connected), then there are no special exceptional orbits* (i.e., $\dim E \leq n - 2$).

*Proof* Poincaré duality implies that $\check{H}^{n-1}(M_+, \infty; \mathbf{Z}_2) = 0$. Let $K$ be the isotropy group of a point on a special exceptional orbit. Then $M_{(K)}$ has dimension $n - 1$ and 3.9 shows that

$$\check{H}^{n-1}(D_+, \infty; \mathbf{Z}_2) \neq 0,$$

where $D = G(M^K)$. The exact sequence (mod 2)

$$0 = \check{H}^{n-1}(M_+, \infty) \to \check{H}^{n-1}(D_+, \infty) \to \check{H}^n(M_+, D_+) \to \check{H}^n(M_+, \infty) \to 0$$

of the triple $(M_+, D_+, \infty)$ then shows that

$$H_0(M - D;\ \mathbf{Z}_2) \approx \check{H}^n(M_+, D_+;\ \mathbf{Z}_2)$$

has rank at least 2. Thus $M - D$ is disconnected. Since the orbits are connected, this implies that $M^* - D^*$ is disconnected. But the set $U^* \subset M^* - D^*$ of principal orbits is connected by 3.1, and is dense in $M^*$ (hence in $M^* - D^*$), so that $M^* - D^*$ is connected; giving a contradiction. ∎

*Remark* The main results of this section were proved, in more generality, in Montgomery, Samelson, and Zippin [1], Montgomery, Samelson, and Yang [1], Yang [2], and Montgomery and Yang [2]. Also see Montgomery [6].

## 4. THE MANIFOLD PART OF $M^*$

In this section, as in the last, $G$ will be a compact Lie group acting locally smoothly on an $n$-manifold $M$ with $M^*$ connected, and $d$ denotes the maximum orbit dimension, so that $\dim M^* = n - d$. We shall first investigate the question of how closely $M^*$ resembles a manifold with boundary. Of course, $M^*$ need not be a manifold with boundary, the simplest example being the antipodal map in $\mathbf{R}^3$. However, we shall prove a useful elementary result stating that $M^*$ is a manifold with boundary outside of some closed set of codimension at least 3.

**4.1. Lemma** *If $n - d \leq 2$, then $M^*$ is a manifold with boundary.*

*Proof* Put $k = n - d$, the codimension of the principal orbit (i.e., the dimension of the orbit space). One can analyze the local structure of $M^*$ by induction on $k$ as follows. A linear tube in $M$ has the form $G \times_K V$

and $(G \times_K V)^* \approx V^*$, which is the open cone over $S^*$ where $S$ is the unit sphere in $V$. But $\dim M^* = \dim V^* = \dim S^* + 1$. If $k = 0$, then $M^*$ is discrete, and if $M$ is a sphere, then $M^*$ is one or two points (admitting here the disconnected case). Thus when $k = 1$, $M^*$ has the local structure of the open cone over one or two points, and hence is a 1-manifold with boundary. When $k = 1$ and $M$ is a sphere, then $M^*$ is a compact, connected 1-manifold and hence is an arc or a circle. If $k = 2$, then $M^*$ has the local structure of an open cone over an arc or a circle, and hence is a 2-manifold with boundary. ▮

**4.2. Lemma** *Let $K \subset G$ and let $D$ be a component of $M^*_{(K)}$ having dimension $n - d - 2$ at least. Then $M^*$ is an $(n - d)$-manifold (possibly with boundary) in the neighborhood of any point of $D$.*

*Proof* It suffices to consider a linear tube $G \times_K V$ about an orbit corresponding to a point of $D$. Since

$$(G \times_K V)_{(K)} = G \times_K V^K = (G/K) \times V^K$$

we see that $V^K$ has dimension $n - d - 2$ at least. Also $V^*$ has dimension $n - d$. Let $W$ be the normal plane to $V^K$ in $V$. Then $V^* \approx W^* \times V^K$ and

$$\dim W^* = \dim V^* - \dim V^K \leq (n - d) - (n - d - 2) = 2.$$

By 4.1, $W^*$ is a manifold with boundary and hence $(G \times_K V)^* \approx V^* = W^* \times V^K$ is also a manifold with boundary. ▮

**4.3. Theorem** *Let $C^* \subset M^*$ be the union of all components of dimension less than or equal to $n - d - 3$ of $M^*_{(K)}$ for all $K \subset G$. Then $C^*$ is a closed set,* $\dim C^* \leq n - d - 3$, *and $M^* - C^*$ is an $(n - d)$-manifold with boundary.*

*Proof* By 3.6, $C^*$ is the union of the *closures* of these components and, by 3.4 and local finiteness of number of orbit types, this union is locally finite. Thus $C^*$ is closed. By 3.5, $\dim C^* \leq n - d - 3$. The complement of $C^*$ consists of points on components of $M^*_{(K)}$ having dimension at least $n - d - 2$, so that the conclusion follows from 4.2. ▮

It is of interest to ask when $M^* - C^*$ is orientable. The following result shows that this is the case for most situations of interest.

**4.4. Theorem** *If* $H_1(M; \mathbf{Z}_2) = 0$ *and if all orbits are connected, then* (*with the notation of* 4.3) $H_1(M^* - C^*; \mathbf{Z}_2) = 0$ *and* $M^* - C^*$ *is orientable.*

***Proof*** For some base point in $M^* - C^*$, consider the diagram (where $C$ is the inverse image of $C^*$ in $M$)

$$\begin{array}{ccc} \pi_1(M - C) & \longrightarrow & H_1(M - C; \mathbf{Z}) \\ \downarrow{\scriptstyle \pi_\#} & & \downarrow{\scriptstyle \pi_*} \\ \pi_1(M^* - C^*) & \longrightarrow & H_1(M^* - C^*; \mathbf{Z}). \end{array}$$

By II.6.3, $\pi_\#$ is onto. The horizontal maps are onto by the Hurewicz Theorem. Thus $\pi_*$ is also onto. Since $H_1(\cdot\,; \mathbf{Z}) \to H_1(\cdot\,; \mathbf{Z}_2)$ is onto by the Universal Coefficient Theorem (or by the sequence induced by $0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}_2 \to 0$) the same argument shows that $H_1(M - C; \mathbf{Z}_2) \to H_1(M^* - C^*; \mathbf{Z}_2)$ is onto. Now dim $C \leq d + (n - d - 3) = n - 3$, by the proof of 3.7. Thus Poincaré duality and the exact sequence of the triple $(M_+, C_+, \infty)$ show that

$$H_1(M - C; \mathbf{Z}_2) \approx \check{H}^{n-1}(M_+, C_+; \mathbf{Z}_2) \approx \check{H}^{n-1}(M_+, \infty; \mathbf{Z}_2) \approx H_1(M; \mathbf{Z}_2) = 0.$$

Consequently, the homomorphic image $H_1(M^* - C^*; \mathbf{Z}_2)$ is also 0 as claimed. This implies the orientability of $M^* - C^*$ as follows: If $M^* - C^*$ were nonorientable, then $\pi_1(M^* - C^*)$ would have $\mathbf{Z}_2$ as a quotient group and thus its abelianization $H_1(M^* - C^*; \mathbf{Z})$ would have $\mathbf{Z}_2$ as a quotient group. Thus, multiplication by 2 on $H_1(M^* - C^*; \mathbf{Z})$ would not be onto, and the exact sequence

$$H_1(M^* - C^*; \mathbf{Z}) \xrightarrow{2} H_1(M^* - C^*; \mathbf{Z}) \to H_1(M^* - C^*; \mathbf{Z}_2)$$

would contradict the triviality of the group on the right. (Alternatively, Poincaré duality mod 2 can be used to show that $M^* - C^*$ is orientable.) ∎

**4.5. Corollary** *If* $H_1(M; \mathbf{Z}_2) = 0$ *and if all orbits are connected, then every orbit of* (*maximal*) *dimension d is orientable.*

***Proof*** Note that $M$ is orientable by the remarks at the end of the proof of 4.4. A principal orbit $G/H$ is orientable by 3.11. Let $K \supset H$ and let $Q \approx G/K$ be an orbit of dimension $d$. Then a linear tube about $Q$ has the form

$$G \times_K V \approx (G/H) \times_{K/H} V$$

by 3.2. Since this tube is orientable, the finite group $K/H$ must preserve the orientation of $(G/H) \times V$. If $Q = (G/H)/(K/H)$ is nonorientable, then $K/H$ reverses the orientation of $G/H$ and hence of $V$. By 3.12 the complement of $(G \times_K V)_{(H)} = G \times_K (V_{(H)})$ has codimension at least 2, and the same must then be true of $V - V_{(H)}$ in $V$. Thus $V_{(H)}$ is open, *connected*, and dense in $V$ and thus $K/H$ reverses its orientation. Thus $(G \times_K V)_{(H)} \approx V^*_{(H)}$ is nonorientable. However, this is an open subset of $M^* - C^*$ which is orientable by 4.4. ∎

*Remarks* More general versions of 4.1 and 4.3 can be found in Bredon [1, 5]. In the cases $M = \mathbf{R}^n$ or $\mathbf{S}^n$, 4.5 was proved in Montgomery [5, 6] by a somewhat different method, and 4.4 is similar to a theorem proved there.

**Examples** Let $G = \mathbf{SO}(3)$, let $H = \mathbf{SO}(2)$, and let $N = N(H)$. Then $G/H \approx \mathbf{S}^2$, $G/N \approx \mathbf{P}^2$ and $N/H \approx \mathbf{Z}_2$. Let $N$ act on $\mathbf{S}^2$ via $N \to N/H$ and the antipodal action of $N/H$ on $\mathbf{S}^2$. Let

$$M = G \times_N \mathbf{S}^2 = (G/H) \times_{\mathbf{Z}_2} \mathbf{S}^2 \approx \mathbf{S}^2 \times_{\mathbf{Z}_2} \mathbf{S}^2$$

(where $\mathbf{Z}_2$ acts antipodally on both factors of $\mathbf{S}^2 \times \mathbf{S}^2$). Then $M$ is orientable since $\mathbf{Z}_2$ preserves orientation on $\mathbf{S}^2 \times \mathbf{S}^2$, but $M^* \approx \mathbf{S}^2/N \approx \mathbf{P}^2$ is nonorientable. This shows that orientability of $M$ is not enough for 4.4.

Now let $N$ act on $\mathbf{S}^3$ via $N \to N/H$ and the action of $N/H$ on $\mathbf{S}^3$ by reflection across a 2-sphere. Put

$$M = G \times_N \mathbf{S}^3 = (G/H) \times_{\mathbf{Z}_2} \mathbf{S}^3 \approx \mathbf{S}^2 \times_{\mathbf{Z}_2} \mathbf{S}^3$$

which is orientable. Also $M^* \approx \mathbf{S}^3/N \approx \mathbf{D}^3$ is orientable. However, the orbits of $G$ on $M$ have the types $G/H \approx \mathbf{S}^2$ which is principal and $G/N \approx \mathbf{P}^2$ which is (special) exceptional. This shows that orientability of $M$ and $M^* - C^*$ is not enough for the conclusion of 4.5 *in the presence* of special exceptional orbits. If $SE = \varnothing$ however, the proof of 4.5 shows that orientability of $M$ and $M^* - C^*$ is sufficient for the conclusion of 4.5.

The next result shows that quite often $M^*$ is a manifold with boundary outside a set of codimension 4 rather than 3.

**4.6. Theorem** *Suppose that $H_1(M; \mathbf{Z}_2) = 0$ and that all orbits are connected. Let $C^*$ be the union of all components of dimension less than or equal to $n - d - 4$ of $M^*_{(K)}$ for all $K \subset G$. Then $C^*$ is a closed set,* $\dim C^* \leq n$

$-d-4$, *$M^*-C^*$ is an orientable $(n-d)$-manifold with boundary and* $H_1(M^*-C^*;\ \mathbf{Z}_2)=0$.

*Proof* Recall the proof of 4.3. We shall only show how to modify it to obtain the present improvement. Suppose first that we are given a locally smooth action on a sphere $S$ with orbit space $S^*$ of dimension 2. Then by 4.1, $S^*$ is a 2-manifold with boundary and, by II.6.5, $H_1(S^*;\mathbf{Q})=0$. This implies that $S^*$ is either $\mathbf{S}^2$, $\mathbf{D}^2$, or $\mathbf{P}^2$. Recalling the proof of 4.2 we see that if $M^*-C^*$ is not a manifold with boundary, then there is a point in it with a neighborhood homeomorphic to the product of the cone over $\mathbf{P}^2$ with a euclidean space. Removal of a set of codimension 3 in this cannot alter the fact that it is nonorientable and this would contradict 4.4. ▌

**4.7. Corollary** *Suppose that $d=n-3$, that $M$ is compact and simply connected and that all orbits are connected. Then $M^*$ is a simply connected 3-manifold with or without boundary.* ▌

## 5. REDUCTION TO FINITE PRINCIPAL ISOTROPY GROUPS

In this section we shall prove a theorem from Bredon [2, 5] which allows the reduction of certain questions about actions on spheres or euclidean space to the case in which the principal isotropy group is finite.

We assume throughout that $G$ is a compact Lie group acting locally smoothly on an $n$-manifold $M$ with $M^*$ connected, and $d$ denotes the maximum orbit dimension.

First let us note the following technical fact.

**5.1. Lemma** *If $K\subset G$, then $N(K)$ acts locally smoothly on $M^K$.*

*Proof* Since $N(K)$ acts locally smoothly on $M$ by 1.1, we may as well assume that $K$ is *normal* in $G$. If $x$ is fixed by $K$ and $L=G_x$ (so that $K\subset L$), then a linear tube about $G(x)$ has the form $G\times_L V$. Since $K$ is normal, we have $K[g,v]=[Kg,v]=[gK,v]=[g,Kv]$, and it follows that

$$(G\times_L V)^K=G\times_L(V^K)$$

which is a linear tube about $G(x)$ in $M^K$. ▌

The following theorem is our main result.

**5.2. Theorem** *Let $H$ be a principal isotropy group and let $T$ be a maximal torus of $H$. Let $M_0{}^T$ denote the union of those components of $M^T$ which intersect $M_{(H)}$ nontrivially. Then the natural map*

$$\varphi\colon\ M_0{}^T/N(T) \to M/G$$

*is a homeomorphism and takes the set of principal orbits for $N(T)$ on $M_0{}^T$ onto the set of principal orbits for $G$ on $M$.*

*Proof* Let $N = N(T)$. A linear tube about a principal orbit has the form $G \times_H V \approx (G/H) \times V$ and the fixed set of $T$ on this is

$$\frac{NH}{H} \times V \approx \frac{N}{N \cap H} \times V$$

by the remarks in Chapter I, Section 5. Thus $(N/(N \cap H)) \times V$ is a linear tube for $N$ on $M_0{}^T$. This shows that $M^T_{(H)}/N \approx M_{(H)}/G$ (which is connected, open, and dense in $M^*$). The closure of $M^T_{(H)}$ is contained in $M_0{}^T$ and $\varphi$ maps this to a closed set containing $M^*_{(H)}$ and hence *onto* $M^*$. Also $M^T_{(H)}$ is open in $M_0{}^T$ and consists of $N$-orbits of type $N/(N \cap H)$. Since, by definition, $M^T_{(H)}$ touches each component of $M_0{}^T$ and since $M^T_{(H)}/N$ is connected, it follows that $M_0{}^T/N$ is connected. Also, $N/(N \cap H)$ is the principal orbit type for $N$ on $M_0{}^T$. (In particular, each component of $M_0{}^T$ has dimension $m = n - d + \dim N/(N \cap H) = n - d + \dim N/T$.) If $K \supset H$ is an isotropy group of the *same rank* as $H$, so that $T$ is also a maximal torus of $K$, then $N$ is transitive on $(G/K)^T$. Hence $\varphi$ is one–one on the set of all such orbits.

Now let $D \subset M$ be the set of points $x$ for which $G_x$ has *larger rank* than does $H$. Then $D$ is contained in the singular set $B$ for $G$ on $M$. Also, if $x \in D \cap M_0{}^T$, then $G_x \cap N$ contains a maximal torus of $G_x$ containing $T$ (and hence has rank larger than that of $H$) so that $N(x)$ is *singular* for $N$ on $M_0{}^T$.

By II.5.7, $N$ has only finitely many orbits on each $(G/K)^T$, $K \supset H$. This means that $\varphi$ is *finite to one* and, as remarked above, $\varphi$ is *one–one on* $(M_0{}^T - D)/N \to M^* - D^*$.

Since $\varphi$ has a natural extension to the one-point compactifications, it is proper, and hence closed. Thus to show that $\varphi$ is a homeomorphism, it suffices to show that it is one–one. Let $x^*$ be in $D^*$ and let $\varphi^{-1}(x^*) = \{x_1, \ldots, x_k\}$, $k > 1$. Let $U_1, \ldots, U_k$ be pairwise disjoint neighborhoods of the $x_i$ in $M_0{}^T/N$. Since $\varphi$ is closed, there exists an open connected neighborhood $V^*$ of $x^*$ in $M^*$ such that $\varphi^{-1}(V^*) \subset U_1 \dot{\cup} \cdots \cup U_k$. Put $V_i = \varphi^{-1}(V^*) \cap U_i$. Let $\pi\colon M \to M^*$ be the orbit map and let

$V = \pi^{-1}(V^*)$. Since $D \cap M_0{}^T$ is in the singular set for $N$, we see that each $V_i - (D/N)$ is nonempty, so that $\bigcup_i(V_i - (D/N))$ is disconnected. Since $\varphi$ is proper and one–one on $M_0{}^T/N - D/N \to M^* - D^*$, it is a homeomorphism there. Thus $V^* - D^* \approx \varphi^{-1}(V^* - D^*) = \bigcup_i(V_i - (D/N))$ is *disconnected.* However, $V^*$ is connected and $D \cap V$ is in the singular set for the action of $G$ on $V$. But the set $V^*_{(H)}$ of principal orbits in $V$ is *connected* and dense in $V^*$, and hence dense in $V^* - D^*$. Thus $V^* - D^*$ is *connected*, a contradiction. Thus $\varphi$ is a homeomorphism.

It remains to show that principal orbits correspond under $\varphi$. The discussion has already shown that $Q \in M_0{}^T/N$ is principal *when* $\varphi(Q) \in M^*$ is principal. Suppose that $Q$ is principal for $N$ and consider $\varphi(Q) \in M^*$. Then $\varphi(Q) = G(x)$ for some point $x$ and we may take $x$ such that $H$ occurs as an isotropy group $H = G_y = K_y$ of $K = G_x$ on a linear slice $S$ at $x$. Then $T$ fixes the segment in $S$ between $x$ and $y$ so that $x \in M_0{}^T$ and hence $Q = N(x)$. Also $K/H$ is principal for $K$ on $S$ so that $K/H$ is orientable by 3.11. Since $Q$ is principal, $N \cap H = N \cap K$, and we must show that $K = H$. Now

$$(K/H)^T = \frac{(N \cap K)H}{H} \approx \frac{N \cap K}{N \cap H} = \text{a point.}$$

If $\dim H = \dim K$, then $(K/H)^T = K/H$, so that $H = K$. If $\dim K > \dim H$, then each component of $K/H$ has positive dimension. Since the elements of prime order in $T$ are dense, there exists one, say $t$, not occurring in any of the (finite number of) proper isotropy subgroups of $T$ on $K/H$. However, $t$ then has the same fixed set as does $T$ on $K/H$, namely one point, and this contradicts 2.3. (There are also other, more Lie group theoretic, proofs of this fact.) ▮

*Remark* The *singular* orbits for $N(T)$ on $M_0{}^T$ clearly correspond, under $\varphi$, to those orbits of $G$ on $M$ whose isotropy group has larger rank than does $H$. Thus $\varphi$ takes singular orbits into singular orbits. However, it may also take some exceptional orbits of $N(T)$ on $M_0{}^T$ to singular orbits of $G$ on $M$.

Since $T$ is the identity component of $H \cap N$, we see that the induced action of $N/T$ on $M_0{}^T$ has *finite* principal isotropy group $(H \cap N)/T$.

**Example** It is not generally true that $M_0{}^T$ coincides with $M^T$. This is shown by the action of $\mathrm{U}(2)$ on $\mathbf{CP}^2 = M$ given by the standard inclusion $\mathrm{U}(2) \subset \mathrm{U}(3)$; that is, $G = \mathrm{U}(2)$ acts on the first two homogeneous coordinates of points $z = (z_0 : z_1 : z_2)$ of $\mathbf{CP}^2$. Taking $\sum |z_i|^2 = 1$ as we may, we note that the orbits of $G$ correspond exactly to $|z_2|$ in $[0, 1]$. That is,

$z \mapsto |z_2|$ induces a homeomorphism $\mathbf{CP}^2/G \xrightarrow{\approx} [0, 1]$. The orbit $|z_2| = 0$ is just $\mathbf{CP}^1 \approx \mathbf{S}^2$ with isotropy group $\mathrm{U}(1) \times \mathrm{U}(1)$. The orbit $|z_2| = 1$ is just a point. The orbits with $0 < |z_2| < 1$ are just 3-spheres with standard $G = \mathrm{U}(2)$ action and isotropy group $\mathrm{U}(1) \times \{I\}$. The latter are the principal orbits and thus the principal isotropy group $H = T = \mathrm{U}(1) \times \{I\}$ is its own maximal torus. Note that $T$ has a circle of fixed points on each principal orbit $\mathbf{S}^3 \approx \mathrm{U}(2)/\mathrm{U}(1)$, and it is easy to see that $T$ has exactly two fixed points on the 2-sphere orbit $|z_2| = 0$. Thus it is not hard to conclude (or to observe directly) that the fixed set of $T$ on $\mathbf{CP}^2 = M$ consists of a 2-sphere (which is $M_0{}^T$) together with a disjoint point. Thus $M^T/N(T)$ is an arc (homeomorphic to $M/G$) together with one extra point.

Note, however, that if $M = \mathbf{S}^n$ or $M = \mathbf{R}^n$ (for example) then III.10.10 and 1.3 imply that $M^T$ is connected, so that $M_0{}^T = M^T$ in these cases.

**Example** Suppose that $G$ is connected and let $G$ act on itself $M = G$ by conjugation. The isotropy group of a point is its centralizer and this has maximal rank in $G$. Since there are points, called *regular* points, whose centralizer is a maximal torus, it follows that a principal isotropy group is a maximal torus $T$ of $G$, and also that regular points are just points on principal orbits. Now $M^T$ is the centralizer of $T$ and hence is just $T$ itself, and $N(T)/T$ is called the *Weyl group* of $G$ and is effective on $T$. Since $T = M^T$ is connected, we have $M_0{}^T = M^T$ and thus 5.2 implies that if two elements of $T$ are conjugate in $G$, then they are conjugate by an element of the Weyl group. Also 5.2 shows that a regular point of $T$ is just a point with trivial isotropy group under the action of the Weyl group. Of course, this is a well-known fact about compact Lie groups and can, and should, be proved directly. (Note, however, that since the principal isotropy group, and hence all isotropy groups, have maximal rank in this example, most of the complications in the proof of 5.2 are unnecessary for this result.) This example is mentioned here only to point out that 5.2 should be regarded as a generalization of this classical result.

Recall that $\dim M^G \leq \dim(B^* \cup E^*) \leq n - d - 1$. Also, if $T'$ is a maximal torus of $G$ and the action is *effective*, then an easy induction using III.10.12 and III.10.2 (or standard representation theory) applied to the unit sphere in a linear slice about a fixed point of $T'$, we see that $\dim M^{T'} \leq n - 2 \operatorname{rank} T'$. The following result, from Bredon [5], improves both of these inequalities (see 5.4 below) and illustrates the use of the reduction to finite principal isotropy groups.

**5.3. Theorem** *Let $H$ be the principal isotropy group of the locally smooth action of $G$ on the $n$-manifold $M$ with maximal orbit dimension $d = \dim G/H$. Put $r = \operatorname{rank} G - \operatorname{rank} H$. Let $T'$ be a maximal torus of $G$. Then*

$$\dim M^{T'} \leq n - d - r.$$

*If $M$ is a rational homology $n$-sphere, then this inequality also holds when $M^{T'} = \varnothing$, where $\dim M^{T'}$ is taken to be $-1$.*

***Proof*** We shall prove the inequality by induction on $n = \dim M$. First we shall reduce the inequality to the case in which $H$ is finite. Let $x$ be a point of the component of $M^{T'}$ having the largest dimension. We can take $H \subset G_x$ to be a principal isotropy group which occurs as an isotropy group of a point in a linear slice at $x$, and $T$ a maximal torus of $H$. Then $x$ is in $M^T_{(H)}$ so that the component of $M^{T'}$ containing $x$ is contained in $M_0{}^T$. Let $N = N(T)$. By applying the inductive assumption to the action of $N/T$ on $M_0{}^T$ we see that [since $(N \cap H)/T$ is finite]

$$\begin{aligned} \dim M^{T'} = \dim(M_0{}^T)^{T'/T} &\leq [n - d + \dim(N/T)] - \dim(N/T) - r \\ &= n - d - r \end{aligned}$$

as claimed. If $M^{T'}$ is empty and $M$ is a rational homology sphere, then $M^T = M_0{}^T$ is also a rational homology sphere by III.10.10 and the same remarks apply.

Thus we may assume that $H$ is finite and hence that $d = \dim G$ and $r = \operatorname{rank} G$. First suppose that $M^{T'} \neq \varnothing$ and let $x$ be a point in a component of $M^{T'}$ of maximal dimension. Let $S$ be a linear slice at $x$ and note that $T' \subset G_x$ so that $\operatorname{rank} G_x = r$. By the inductive assumption, applied to the unit sphere in $S$, we have

$$\dim S^{T'} \leq [n - \dim G(x)] - \dim G_x - \operatorname{rank} G_x = n - d - r.$$

Let $\pi_S \colon S \to S/G_x$ denote the orbit map and so on. If $K$ has maximal rank, then $N(T')$ is transitive on $(G/K)^{T'}$. Thus

$$\pi_S(S^{T'}) \approx S^{T'}/(N(T') \cap G_x)$$

and, since $N(T')/T'$ is finite, it follows that $\dim S^{T'} = \dim \pi_S(S^{T'})$. Similarly $\dim M^{T'} = \dim \pi_M(M^{T'})$. Now $\pi_M(M^{T'})$ consists exactly of those orbits whose isotropy group has rank $r$, and it follows that $\pi_S(S^{T'}) \approx \pi_{GS}((GS)^{T'})$

under the canonical homeomorphism $S/G_x \approx GS/G$. Thus

$$\begin{aligned}\dim M^{T'} = \dim(GS)^{T'} = \dim \pi_{GS}((GS)^{T'}) &= \dim \pi_S(S^{T'}) \\ &= \dim S^{T'} \le n - d - r\end{aligned}$$

as claimed.

Now suppose that $M^{T'} = \varnothing$ and that $M$ is a rational homology sphere. Then the case $r = 1$ is trivial and we may assume that $r > 1$. By III.10.12 (or as an easy consequence of III.8.1) there exists a subtorus $T''$ of $T'$ of codimension 1 such that $M^{T''} \neq \varnothing$. Let $x$ be a point of $M^{T''}$ and note that rank $G_x = r - 1$ with $T''$ a maximal torus. Let $S$ be a linear slice at $x$. Again $\pi_S(S^{T''}) \approx S^{T''}/(N(T'') \cap G_x)$ has the same dimension as does $S^{T''}$. The inductive assumption applied to $G_x$ on the unit sphere in $S$ gives

$$\dim S^{T''} \le [n - \dim G(x)] - \dim G_x - (r - 1) = n - d - r + 1.$$

Again, since $\pi_{GS}((GS)^{T''})$ consists exactly of those orbits in $GS$ with isotropy group of rank $r - 1$ ($=$ rank $G_x$), it is canonically homeomorphic to $\pi_S(S^{T''})$. Also

$$\pi_{GS}((GS)^{T''}) \approx (GS)^{T''}/N(T'')$$

and hence this has the same dimension as does $M^{T''}/N(T'')$. Thus

$$\begin{aligned}\dim M^{T''}/N(T'') = \dim \pi_{GS}((GS)^{T''}) &= \dim \pi_S(S^{T''}) \\ &= \dim S^{T''} \le n - d - r + 1.\end{aligned}$$

If $k$ is the maximum orbit dimension for $N(T'')$ on $M^{T''}$, then

$$\dim M^{T''} = \dim (M^{T''}/N(T'')) + k \le n - d - r + 1 + k.$$

Thus by the inductive assumption applied to the action of $N(T'') \supset T'$ on $M^{T''}$ we have

$$\begin{aligned}-1 &= \dim M^{T'} \\ &= \dim(M^{T''})^{T'} \le (n - d - r + 1 + k) - k - 1 = n - d - r\end{aligned}$$

as claimed. ∎

**5.4. Corollary** *If $G$ is effective on $G/H$, then*

$$\dim G/H \ge \operatorname{rank} G + \operatorname{rank} H.$$

*Thus, if G acts effectively on M in the situation of* 5.3, *we have*

$$n - d - r \leq n - 2\,\text{rank}\, G.$$

***Proof*** The second part is just a rewriting of the first inequality which holds since $G$ always acts effectively on a principal orbit in an *effective* action. (Otherwise, there would be a normal subgroup leaving an open dense set stationary.) For the first inequality, we apply 5.3 to the action of the maximal torus $T$ of $H$ on $G/H$. Since $(G/H)^T = N(T)H/H$ we have

$$\begin{aligned}\text{rank}\, G - \text{rank}\, H &\leq \dim \frac{N(T)}{N(T) \cap H} \\ &= \dim \frac{N(T)H}{H} \\ &= \dim\, (G/H)^T \\ &\leq \dim G/H - \dim T - \text{rank}\, T \\ &= \dim G/H - 2\,\text{rank}\, H. \quad \blacksquare\end{aligned}$$

## 6. ACTIONS ON $S^n$ WITH ONE ORBIT TYPE

We shall now consider the case of an action of a compact Lie group $G$ on a sphere having precisely one type of orbit. Although we shall assume local smoothness, it is used only minimally and can easily be dispensed with. Our immediate use (in the next section) of the results in this section will be in the case of orthogonal actions and in that case the proof simplifies somewhat. For this reason we shall first give the proof of the following result in the orthogonal case and then consider the modifications necessary for the general case.

**6.1. Theorem** *Let G be a compact connected Lie group acting effectively and locally smoothly on an n-manifold M which is a simply connected rational homology n-sphere. Assume that there is precisely one orbit type. Then G is either transitive on M or G is the circle group* $\mathbf{S}^1$, *the* 3*-sphere group* $\mathbf{S}^3$, *or* $\mathbf{SO}(3)$ *and acts freely on M. If M is also a* mod 2 *homology n-sphere, then* $G \neq \mathbf{SO}(3)$ *in the nontransitive case.*

***Proof*** As noted above, we shall first treat the case in which $G$ acts *orthogonally* on $M = \mathbf{S}^n$ and we shall use $S$ for $M$ in this case for clarity.

Suppose that the orbits have type $G/H$ and let $N = N(H)$ and $K = N/H$. By II.5.10 and II.5.11, $S^H/N \approx S/G = S^*$ and $S^H \to S^*$ is just the principal $K$-bundle associated with the $(G/H)$-bundle $S \to S^*$. Since $G/H$ is connected and since $S$ is simply connected, we see from the homotopy sequence of $S^H \to S^*$ that $K$ is *connected.* Since $K$ acts *freely* on the sphere $S^H$, it is either trivial, $\mathbf{S}^1$, or $\mathbf{S}^3$ by III.8.5. The free actions of $K$ on $G/H$ and on $S^H$ give a free $(K \times K)$-action on $(G/H) \times S^H$. The orbit space of the diagonal subgroup is just $(G/H) \times_K S^H \approx S$ (by II.5.9). Thus the inclusion of the (diagonal) $K$-orbits in the $(K \times K)$-orbits gives a map

$$\varphi\colon S \to \big((G/H)/K\big) \times (S^H/K) \approx G/N \times S^*$$

which is clearly a fiber bundle projection with fiber $K \times_K K \approx K$ and structure group $K \times K$ (see Chapter II, Exercise 7). If $K$ is trivial, then $S \approx G/N \times S^*$, which implies that $G = N = H$. If $K$ is $\mathbf{S}^1$, then a standard argument using the mod 2 Gysin sequence of $\varphi$ shows that the cohomology ring $H^*(G/N \times S^*; \mathbf{Z}_2)$ is generated by one element $u$ of degree 2; that is, $1, u, u^2, \ldots, u^r$ are the nonzero elements of $H^*(G/N \times S^*; \mathbf{Z}_2)$ and $u^{r+1} = 0$ (see Spanier [1, p. 264]). The Künneth Theorem then implies that (with $\mathbf{Z}_2$ coefficients)

$$\begin{aligned} H^2(G/N \times S^*) &\approx [H^2(G/N) \otimes H^0(S^*)] \oplus [H^0(G/N) \otimes H^2(S^*)] \\ &\approx H^2(G/N) \oplus H^2(S^*), \end{aligned}$$

where the summands are induced by the projections of $G/N \times S^*$ to its factors. If $H^2(G/N) \neq 0$, then $u$ is in the image of $H^2(G/N) \to H^2(G/N \times S^*)$. However, then $u^r$ is also in this image (in degree $2r$) which shows that $\dim G/N = \dim(G/N \times S^*)$. Thus $S^*$ must be a point and $G$ is transitive on $S$. If $G$ is not transitive, then the opposite argument shows that $G/N$ is a point. Thus $G = N$ and $H$, being normal and hence an isotropy group of every point, is trivial. Hence $G = K = \mathbf{S}^1$. The case $K = \mathbf{S}^3$ is exactly the same with $u$ of degree 4.

This completes the orthogonal case and we now consider the general case. In this case, $M^H$ need not be a sphere, or even connected, so that we must give another argument. Thus we shall first consider $M^T$ instead, where $T$ is a maximal torus of $H$. Now $N(T)/T$ has one type of orbit on $M^T$ by 5.2 and hence has finite isotropy groups. From III.10.12 it follows that $N(T)/T$ must have rank 0 or 1. That is, either rank $G =$ rank $H$ or rank $G = 1 +$ rank $H$, and it follows that $K = N(H)/H$ has rank 1 or is finite. If $K$ is finite, then $M^H$ is a covering space of $M^*$. Since $M^*$ is simply connected by II.6.3, a component of $M^H$ is a cross section for the orbit map

$M \to M^*$. However, then $H^*(M^*)$ would be a direct summand of $H^*(M)$, showing that dim $M^* =$ dim $M$. Thus $K = N(H)/H$ has rank 1. Let $M_1^H$ be a component of $M^H$ and let $K_1$ be that subgroup of $K$ preserving it. Then $M_1^H/K_1 \approx M^H/K \approx M^*$ (since this is connected). Let $N_1$ be the inverse image of $K_1$ in $N = N(H)$. Then the map $\varphi$ above can be modified as

$$\psi\colon M \to G/N_1 \times M^*$$

which is a bundle with fiber $K_1$. Now since $M^* \approx M_1^H/K_1$ is simply connected and $M_1^H$ is connected, $K_1$ is *connected.* Since $K_1$ has rank 1, $K_1$ must be $\mathbf{S}^1$, $\mathbf{S}^3$, or $\mathbf{SO}(3)$ (which is a *rational* homology 3-sphere). The homotopy sequence of the fibering $\psi$ shows that $G/N_1 \times M^*$ is simply connected. Thus $G/N_1$ and $M^*$ are orientable. Then the Gysin sequence of $\psi$ with *rational* coefficients can be used to show that $H^*(G/N_1 \times M^*; \mathbf{Q})$ is generated by one element $u$ as a ring. The same considerations as in the orthogonal case now imply that $G = N_1$ and hence that $H$ is trivial and $G = K_1$ is $\mathbf{S}^1$, $\mathbf{S}^3$, or $\mathbf{SO}(3)$. If $M$ is a mod 2 homology sphere, then $G \neq \mathbf{SO}(3)$ since $\mathbf{SO}(3)$ contains $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ which cannot act freely on $M$ by III.8.1. ∎

**Remarks** This theorem is due to Borel [5] (with somewhat weaker hypotheses). There is a generalization by Conner [7] (also see Borel [5]) in which the assumption of one type of orbit is weakened to the assumption that all orbits have the same dimension (i.e., $B = \varnothing$). The conclusion is also weaker in that it can only be asserted that all isotropy groups are finite. The proof of this generalization seems to require the use of considerably deeper results from algebraic topology, although it is quite close to the proof given for 6.1, and hence we shall not give it here. Another proof of Conner's result, which proves it without the assumption of simple connectivity, can be found in Bredon [6]. Even though the latter proof is completely different from Conner's proof and is conceptually simple (being in the spirit of 5.2), it is quite involved and also makes use of some deep results beyond our present scope.

We shall now show how to remove the connectedness condition from 6.1. Note that the following result also essentially generalizes III.8.5.

**6.2. Theorem** *Let the compact Lie group $G$ (possibly disconnected) act effectively and locally smoothly on $\mathbf{S}^n$. Assume that* dim $G > 0$ *and that there is precisely one orbit type. Then either $G$ is transitive on $\mathbf{S}^n$ or $G$ acts freely on $\mathbf{S}^n$. (Hence, in the latter case, $G$ is either $\mathbf{S}^1$, $\mathbf{S}^3$, or the normalizer $N(\mathbf{S}^1)$ of $\mathbf{S}^1$ in $\mathbf{S}^3$.)*

***Proof*** Put $M = S^n$. Assume that the action is not transitive. In view of III.8.5 it suffices to prove that $G$ acts *freely*. Let $G_0$ be the identity component of $G$. Let $H$ be an isotropy group of $G$. A tube about any orbit has the form $G/H \times V$. Since $G_0/(G_0 \cap H) \times V$ is open in this, it is a tube for the action of $G_0$ on $M$. Thus $G_0 \cap H$ is the principal isotropy group for the action of $G_0$. For any $g$ in $G$, $G_0 \cap (gHg^{-1})$ is also principal for $G_0$ for the same reason, and hence it is conjugate, *in* $G_0$, to $G_0 \cap H$. Thus $G_0$ acts with one type of orbit $G_0/(G_0 \cap H)$ and, by 6.1, $G_0 \cap H = \{e\}$. Similarly, $G_0H$ is a subgroup of $G$ and $(G_0H/H) \times V$ is a tube for the action of $G_0H$. Thus remarks similar to those above show that $G_0H$ acts with one type of orbit $G_0H/H \approx G_0 \approx S^1$ or $S^3$. Since we need only show that $H$ is trivial it suffices to consider the case $G = G_0H$, which we now assume.

Now $G/H \approx S^1$ or $S^3$ and $M^*$ is (now) simply connected by II.6.3. Thus the Gysin sequence of the spherical fibration $M \to M^*$ shows that $H^*(M^*)$ is generated as a ring by an element $u$ of degree 2 or 4 (according as $G/H = S^1$ or $S^3$). Let $p$ be any prime dividing ord $H$ and let $P$ be a $p$-Sylow subgroup of $H$. (We note here that if the action is orthogonal, then we can take $H$ rather than $P$ in the following argument.) The Sylow Theorems imply easily that

$$(G/H)^P = \frac{N(P)H}{H} \approx \frac{N(P)}{N(P) \cap H}$$

and this must be a mod $p$ homology sphere of dimension 0, 1, 2, or 3 inside $G/H = S^1$ or $S^3$. It follows that $N(P)/\big(N(P) \cap H\big) \approx S^i$ for some $0 \leq i \leq 3$. Now

$$M^P \to \frac{M^P}{N(P)} \approx M^*$$

is a bundle with fiber $N(P)/\big(N(P) \cap H\big)$ and its total space $M^P$ is a mod p homology sphere. Since $M^*$ is not a point, $M^P$ is connected. Since $M^*$ is simply connected, $N(P)/\big(N(P) \cap H\big)$ is connected. Thus $N(P) \cap G_0$ is transitive on $N(P)/\big(N(P) \cap H\big)$ and, since $H \cap G_0 = \{e\}$, we conclude that $N(P) \cap G_0 \approx N(P)/\big(N(P) \cap H\big)$ is either $S^1$ or $S^3$ (since it is a group). It also follows that $M^P$ is a principal $N(P) \cap G_0$-bundle over $M^*$. Now $N(P) \cap G_0$ must be $S^1$ when $G_0 \approx G/H$ is $S^1$ and $N(P) \cap G_0$ is either $S^1$ or $S^3$ when $G_0 \approx G/H$ is $S^3$. Suppose that $G/H$ is $S^3$ and that $N(P) \cap G_0$ is $S^1$. Then the mod $p$ Gysin sequence of the $N(P) \cap G_0$-bundle $M^P \to M^*$ would show that $H^2(M^*; \mathbf{Z}_p) \neq 0$ contrary to the previous remarks using the Gysin sequence of $M \to M^*$. Thus we conclude that $N(P) \cap G_0 = G_0$; that is, $G_0 \subset N(P)$. However, a connected group cannot act nontrivially by automorphisms on a finite group. Thus each element of $G_0$ commutes with every element of $P$, that is, $P$ is contained in the centralizer $Z$ of $G_0$. Since $Z \cap H$ contains every Sylow subgroup of $H$, it has the same order as does $H$, and hence equals $H$. Thus $H$ commutes with $G_0$ and hence $H$ is normal in $G_0H = G$. Since then $H$ is the isotropy group of each point of $M$ we conclude that $H$ is trivial, as desired. ▮

Of course 6.2 also holds when $G$ is *finite*. That is, if a finite group $G$ acts on a sphere with only one type of orbit, then it acts freely. However, this is trivial in this case, since the principal isotropy group of a *finite* group action is trivial (see 3.2).

**6.3. Corollary** *Suppose that $G$ is a compact Lie group acting locally smoothly on $M$. Let $x \in M$ and assume that there are precisely two types of orbits in some neighborhood of $G(x)$. Let $y$ be a point on a principal orbit in a linear slice $V$ at $x$. Put $K = G_x$ and $H = G_y$. Then one of the following possibilities must occur:*

(i) *$K$ is transitive on the unit sphere (homeomorphic to $K/H$) in the orthogonal complement to $V^K$ in $V$ and $M^*$ is a manifold with boundary $M^*_{(K)}$ near $x^*$.*

(ii) *$H$ is normal in $K$ and is the ineffective part of $K$ on $V$. Also, $K/H$ is either finite, $\mathbf{S}^1$, $\mathbf{S}^3$, or the normalizer of $\mathbf{S}^1$ in $\mathbf{S}^3$.*

*Moreover, the set of points $x$ in $M$ which satisfy the hypothesis is open and dense in $B \cup E$.*

*Proof* By 3.2, $(G \times_K V)_{(K)} = G \times_K (V^K)$ and $(G \times_K V)_{(H)} = G \times_K (V_{(H)})$ and, by assumption, these fill out $G(V)$. Hence $V = V^K \cup V_{(H)}$, which means that $K$ has only two types of orbits on $V$, one type consisting of fixed points. If $W$ is the orthogonal complement to $V^K$ in $V$, then $K$ acts with one type $(K/H)$ of orbit on the unit sphere in $W$. Thus 6.2 implies that either (i) or (ii) must hold. Clearly, the set of points $x$ in $M$ which satisfy the hypothesis is just the set of those points of $B \cup E$ whose orbit type is *locally maximal* (in $B \cup E$). Since the number of orbit types is locally finite, the set of such points is clearly open and dense in $B \cup E$. ▮

## 7. COMPONENTS OF $B \cup E$

In this section we shall apply 5.2 to obtain some results on the set $B \cup E$ of nonprincipal orbits of a locally smooth action on *euclidean space* $\mathbf{R}^n$. Our first result states that if a stationary point exists, then $B \cup E$ is connected. This is also shown to be the case when the principal isotropy group does not have maximal rank in $G$. The most general conditions under which such a result holds are not known, but it does not always hold, as we shall note below. We shall also study the question of whether $B \cup E$ can have a compact component. We shall show that such a component is a single orbit. Moreover, if this orbit is singular, then it is a stationary point and $G$ acts freely outside it when $\dim M^* > 1$.

As usual, we assume throughout that $M$ is an $n$-manifold on which the compact Lie group $G$ acts locally smoothly with maximum orbit dimension $d$.

We shall need the following lemma.

**7.1. Lemma** *Let $G$ be a compact Lie group and $T'$ a maximal torus of $G$. If $P \subset G$ is any $p$-group, then $P$ is conjugate to a subgroup of $N(T')$.*

*Proof* Put $N = N(T')$. Consider $(G/N)^{T'}$. If $T'gN = gN$, then $g^{-1}T'g \subset N$ which implies that $g$ is in $N$ since $T'$ is just the identity component of $N$. Thus $(G/N)^{T'} = N/N$ is a point. By III.10.9 we conclude that $\chi(G/N) = 1$. (We remark that, in fact, $G/N$ is acyclic over the rationals.) Now $\chi((G/N)^P) \equiv \chi(G/N) = 1$ (modulo $p$) and hence $(G/N)^P \neq \emptyset$, which is equivalent to the statement that $P$ is conjugate to a subgroup of $N$. ∎

**7.2. Theorem** *Let $M$ be acyclic over the integers* (e.g., $M = \mathbf{R}^n$) *and assume either that a stationary point exists for $G$ on $M$ or that the principal isotropy subgroup has nonmaximal rank in $G$. Then $B \cup E$ is connected.*

*Proof* Let $H$ be a principal isotropy group and assume that $\operatorname{rank} H < \operatorname{rank} G$. Let $T$ be a maximal torus of $H$ and $T' \supset T$, a maximal torus of $G$. Consider $M^{T'}$ which is acyclic, and hence connected, and is contained in $B$. Note that $G(M^{T'})$ consists exactly of points of $M$ whose isotropy group has maximal rank. However, if $G_0$ is the identity component of $G$, $G_0(M^{T'})$ also consists exactly of these points. Hence $G(M^{T'}) = G_0(M^{T'})$ is *connected.* Now suppose that the theorem is true when $H$ is finite. Then by 5.2 we would deduce that $(B \cup E) \cap M^T$ is connected. Thus

$$g[(B \cup E) \cap M^T] \cup G(M^{T'})$$

would be connected for each $g$ in $G$. The union of these over all $g$ is just $B \cup E \cup G(M^{T'}) = B \cup E$ (by 5.2) and hence $B \cup E$ is connected as claimed. Thus it suffices to prove the case in which $H$ is finite. In this case, suppose first that $x \in B$. Then $G_x$ contains a circle subgroup $S$ and this is contained in some maximal torus $T''$ of $G$. Then $M^S \supset M^{T''}$ and $M^S$ is connected. Then $M^S \cup G(M^{T'}) = M^S \cup G(M^{T''})$ is a connected subset of $B$ containing $x$ and $G(M^{T'})$. Now suppose that $x \in E$ so that $G_x$ is finite. Let $P$ be a Sylow $p$-group in $G_x$ whose order does not divide the order of $H$. Then $P$ is not conjugate to a subgroup of $H$ so that $M^P \subset B \cup E$. Now $P \subset N(T'')$ for some maximal torus $T''$ of $G$ by 7.1. Thus $M^{T''}$ is invariant under $N(T'')$ and hence under $P$. By III.7.11 (with $A = \emptyset$ and $n = 0$) we have that $(M^{T''})^P = M^{T''} \cap M^P$ is nonempty, and also $M^P$ is connected. Since $G(M^{T''}) = G(M^{T'})$ it follows that $M^P \cup G(M^{T'})$ is a connected subset of $B \cup E$ containing $x$ and $G(M^{T'})$. It follows that every component of $B \cup E$ contains the connected subset $G(M^{T'})$ and hence that $B \cup E$ is connected.

The case in which a stationary point $x_0$ exists (but $H$ may have maximal rank) can be proved in exactly the same way, but somewhat more easily, by replacing $G(M^{T'})$ by $x_0$. ∎

*Remark* This result does not hold without the existence of a stationary point when $G$ is finite. In fact, consider the example (I.8.3) of an action of $\mathbf{Z}_6 \approx \mathbf{Z}_2 \oplus \mathbf{Z}_3$ on $M = \mathbf{R}^n$ with no stationary points. By the Smith Theorem III.7.11, $M^{\mathbf{Z}_2}$ and $M^{\mathbf{Z}_3}$ are both nonempty. They are also disjoint, and their union is precisely the set of nonprincipal orbits. However, it does seem reasonable to expect that if stationary points do not exist, then $B \cup E$ can have no compact components. The following theorem is a partial result in this direction. (The case $d = n - 1$ is omitted from the following theorem for technical convenience. In Section 8 we shall analyze this case completely.)

**7.3. Theorem** *Let $M$ be acyclic over the integers and suppose that $d < n - 1$. Let $C$ be a compact component of $B \cup E$. Then $C$ consists of exactly one orbit $G(x)$ and* rank $G_x$ = rank $G$. *If $G(x)$ is singular* (e.g., *if $G$ has larger rank than that of the principal isotropy group*), *then $x$ is a stationary point, $G$ acts freely outside $x$, and $G$ is isomorphic to* $\mathbf{S}^1$, *to* $\mathbf{S}^3$, *or to the normalizer of* $\mathbf{S}^1$ *in* $\mathbf{S}^3$ (*assuming that $G$ acts effectively*).

*Proof* Let $x$ be in $C$. We may assume that there are precisely two types of orbits near $G(x)$ since, by 6.3, such points are dense in $B \cup E$ and hence in $C$. Let $K = G_x$ and let $H \subset K$ be a principal isotropy group for $K$ on a slice at $x$ (and hence for $G$ on $M$). Let $T$ be a maximal torus of $H$. We may assume that there are precisely two types of orbits of $N(T)$ near $N(T)(x)$ (by passing to a nearby point if necessary). By 5.2, to show that $C$ consists of exactly one $G$-orbit, it suffices to show that $C \cap M^T$ consists of exactly one $N(T)$-orbit, and hence it suffices to consider the case in which $H$ is finite (by 1.5). This case also clearly suffices to prove that rank $K =$ rank $G$.

If $H$ is finite and dim $K > 0$, consider $M^{T'}$, where $T'$ is a maximal torus of $K$. Since $M^{T'}$ is connected and does not touch principal orbits, it is contained in $C$, and hence is compact. Since $M^{T'}$ is acyclic, it consists of exactly one point. However, orbits in $C$ near $G(x)$ have type $G/K$ and hence contain fixed points of $T'$. Thus $C = G(x)$. Since

$$(G/K)^{T'} = \frac{N(T')K}{K} \approx \frac{N(T')}{N(T') \cap K}$$

is just one point and since $N(T')$ contains a maximal torus of $G$, we also conclude that $K$ has maximal rank in $G$ (and that it contains $N(T')$ in this case).

Now suppose that $H$ and $K$ are both finite. Let $P$ be a $p$-Sylow subgroup in $K$ whose order does not divide ord $H$. Just as above, we see that $C = G(x)$ and $(G/K)^P$ consists of one point. If dim $G > 0$, then the euler characteristic of $G/K$ is zero, since that of its covering space $G$ is zero. Thus

$$0 = \chi(G/K) \equiv \chi\big((G/K)^P\big) = 1 \pmod p$$

gives a contradiction. Thus $G$ must be finite, showing again that rank $K$ = rank $G$.

This completes the proof that $C = G(x)$ and that $K = G_x$ has maximal rank in $G$. Now suppose that $G(x)$ is *singular*. Let $V$ be a linear slice at $x$. Then, by 6.3, there are the following two possibilities:

(i) $K$ is transitive on the unit sphere in $V$.

(ii) $H$ is normal in $K$ and acts trivially on $V$. Also, $K/H$ acts freely on $V - \{x\}$ and $K/H \approx S^1$, $S^3$, or the normalizer of $S^1$ in $S^3$ (since dim $K/H > 0$).

In case (i), $V^*$ has dimension 1, so that dim $M^* = 1$ also. Thus $n - d = 1$, which is the case we have excluded. Thus case (ii) holds and hence rank $G$ = rank $K = 1 +$ rank $H$.

Suppose that $x$ is not stationary and that rank $G > 1$. Consider the action of a maximal torus $T'$ of $K$ (and hence of $G$) on $G/K$. Its fixed set is $N(T')/(N(T') \cap K)$, which is finite. By looking at a sphere about a fixed point and applying III.10.12 we see that there exists a subtorus $T_1$ of $T'$ of codimension 1, such that $\dim(G/K)^{T_1} > 0$. However, then $M^{T_1}$ is an acyclic manifold of positive dimension, hence noncompact. Since $M^{T_1}$ is connected, it must touch a principal orbit near $G(x)$. This means that $T_1$ is conjugate to the maximal torus $T$ of $H$, and we may as well take $T_1 = T$. Thus

$$\dim(G/K)^T > 0.$$

By 5.2, $N(T)$ is transitive on $(G/K)^T$. Since $\dim N(T)(x) > 0$, $x$ is not stationary for the action of $N(T)/T$ on $M^T$. Since rank $K = 1 +$ rank $H$, $x$ is also *singular* for this action. Thus to show that $x$ is stationary, it suffices to consider the case in which $H$ is finite. If $x$ is stationary for the identity component $G_0$ of $G$, then it is the *unique* such point, since $x$ is the unique fixed point of $T'$. Since $G$ permutes the fixed points of $G_0$ it would follow that $G$ would have to leave $x$ stationary. Thus it suffices to consider the case in which $G$ is connected in order to show that $x$ is stationary.

Thus suppose that $H$ is finite and that $G$ is connected (and hence rank $G = 1 =$ rank $K$). Supposing that $x$ is not fixed, and hence that $K \neq G$, we must then have that $G = \mathbf{S}^3$ or $G = \mathbf{SO}(3)$. Since $(G/K)^{T'}$ is one point, $K \supset N(T')$, and hence $K = N(T')$. If $G = \mathbf{S}^3$, consider its center $L = \{1, -1\} \subset K$. Then $(G/K)^L = G/K \approx \mathbf{P}^2$. Since $M^L$ is mod 2 acyclic, this implies that $M^L$ must touch a principal orbit near $G(x)$. However, then $L \subset H$, which implies that $L$ acts trivially on $M$, since $L$ is normal in $G$. Thus $G$, effectively, must be $\mathbf{SO}(3)$, $K = N(T')$, and $T' = \mathbf{SO}(2)$. If $k$ is in the component $K - T'$ of $K$, and if $t$ is in $T'$, then $kt^{-1}k^{-1} = t$. Thus $tkt^{-1} = t^2k$, which implies that $T'$ acts *transitively* by conjugation on the circle $K - T'$. Thus the *normal* finite subgroup $H \subset K$ must be contained in $T'$. However, $K/H$ is then easily seen to be isomorphic to $N(T')$ and this is *not* isomorphic to the normalizer of $\mathbf{S}^1$ in $\mathbf{S}^3$. (For example, it contains $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ whereas the normalizer of $\mathbf{S}^1$ in $\mathbf{S}^3$ does not.) Thus the hypothesis that $x$ is not stationary is untenable.

By 6.2, we now know that $G = K$ is $\mathbf{S}^1$, $\mathbf{S}^3$, or the normalizer of $\mathbf{S}^1$ in $\mathbf{S}^3$, and it acts *freely* outside $x$ *locally* near $x$. It remains to show that this is true globally; that is, that all other orbits are principal; but this now follows from 7.2. ▮

We shall now list some unsolved problems which have a vague connection with the material in this section. These problems are all interrelated and seem to be of a rather deep nature. Let $G$ denote a compact Lie group (possibly finite). The problems are of interest even for the case of cyclic $G$ (but not of prime power order).

*Problem 1* If $G$ acts on the disk $\mathbf{D}^n$ with the origin as a stationary point, then for $H \neq G$ with $\mathbf{D}^n_{(H)} \neq \emptyset$, is $\mathbf{S}^{n-1}_{(H)} \neq \emptyset$? Is this the case when the action on the boundary $\mathbf{S}^{n-1}$ is orthogonal?

*Problem 2* If $G$ acts on $\mathbf{D}^n$ and if $F(G, \mathbf{D}^n) \subset \operatorname{int} \mathbf{D}^n$, does $F(G, \mathbf{D}^n)$ contain *at most* one point?

*Problem 3* If $G$ acts on $\mathbf{S}^n \times \mathbf{I}$, then can the set $F$ of stationary points touch $\mathbf{S}^n \times \{0\}$ without also touching $\mathbf{S}^n \times \{1\}$? Can this happen when the action on both ends is orthogonal? (Note that if the latter can happen, then there would exist an action on $\mathbf{S}^{n+1}$ such that $F(G, \mathbf{S}^{n+1})$ has components of unequal dimension.)

*Problem 4* (Due to Raymond) Let $G$ act on $\mathbf{R}^n$ with a stationary point $x$. If $\mathbf{R}^n_{(H)} \neq \emptyset$, must $x$ be in the closure of $\mathbf{R}^n_{(H)}$?

## 8. ACTIONS WITH ORBITS OF CODIMENSION 1 OR 2

In this section we shall consider locally smooth actions on a connected $n$-manifold $M$ with principal orbits of codimension 1 or 2; that is, with orbit space of dimension 1 or 2. We shall show that such actions on euclidean space are equivalent to orthogonal actions.

Let us first discuss the case in which the maximal orbit dimension $d$ is $n - 1$. Then, by 4.1, $M^*$ is a 1-manifold, possibly with boundary. The space $U^*$ of principal orbits is open, connected, and dense and $U^*$ must also be connected *locally in* $M^*$ (by restricting one's attention to the part of $M$ over an open connected subset of $M^*$). Thus $U^*$ must consist exactly of the interior of the connected 1-manifold $M^*$ with boundary.

**8.1. Theorem** *Suppose that $M$ is noncompact and $d = n - 1$. If every orbit is principal, then $M \approx G/H \times \mathbf{R}$ with trivial action on $\mathbf{R}$. Otherwise, $M$ is equivalent to $G \times_K V$, where $V$ is a euclidean space and $K$ acts orthogonally on $V$ and transitively on the unit sphere in $V$. If $M$ is euclidean space, then $K = G$ and $M \approx V$.*

*Proof* By the above remarks, $M^* \approx \mathbf{R}$ iff every orbit is principal. In this case $M$ is a bundle over $\mathbf{R}$ with fiber $G/H$. Since such a bundle must be trivial, the first case follows. Otherwise, $M^*$ must be a ray $[0, \infty)$ and only the end point is nonprincipal. A linear tube about the nonprincipal orbit has the form $G \times_K V$ and, since $V^*$ is 1-dimensional, $K$ is transitive on the spheres in $V$ about the origin. Since $M^*$ is conical with vertex being this nonprincipal orbit, II.8.5 implies that $M$ is equivalent to this tube. Now $G \times_K \{0\} \approx G/K$ is a retract of $G \times_K V$, so that if $M$ is a euclidean space (or just mod 2 acyclic), then $G/K$ must be a point. ∎

***Remark*** Suppose that we are in the second case of 8.1 and let $D$ be the unit disk in $V$. Then the boundary of $D$ is an orbit of $K$; say of type $K/H$. Then $G \times_K D$ is a disk bundle over $G/K$ and the total space of the bounding sphere bundle is $G \times_K (K/H) \approx G/H$. The projection $G/H \to G/K$ in this bundle is just the canonical equivariant map. This shows that $G \times_K D$ is just the mapping cylinder of the equivariant map $G/H \to G/K$.

**8.2. Theorem** *Suppose that $M$ is compact and $d = n - 1$. If every orbit is principal, then $M$ is a $G/H$-bundle over $M^* \approx \mathbf{S}^1$ with structure group $N(H)/H$. Otherwise, there are two nonprincipal orbits of types $G/K_i$, $i = 0,1$, say, with $K_i \supset H$ ($H$ being the principal isotropy group). Moreover, the $K_i$ may be chosen so that $M$ is equivalent to the union of the two mapping cylinders of $G/H \to G/K_i$, $i = 0,1$.*

***Proof*** The first case is clear and coincides with the case $M^* \approx \mathbf{S}^1$. Otherwise, $M^* \approx [0,1]$. In this case, let $G/K_i$ be the type corresponding to $i = 0,1$ in $M^* = [0,1]$. The part of $M$ over $[0,1)$ satisfies 8.1 and it follows easily from the remark following 8.1, that the part of $M$ over $[0, \frac{1}{2}]$ is equivalent to a mapping cylinder $M_{f_0}$ of $f_0$: $G/H \to G/K_0$. Similarly the part of $M$ over $[\frac{1}{2}, 1]$ is equivalent to a mapping cylinder $M_{f_1}$ of $f_1$: $G/H \to G/K_1$. Thus $M$ is equivalent to

$$M_{f_0} \cup_\varphi M_{f_1},$$

where $\varphi$: $G/H \to G/H$ is some equivalence. Now $\varphi$ must be right translation $R_n^{H,H}$: $gH \mapsto gn^{-1}H$ by some element $n$ of $N(H)$; see I.4.3. Put $K_1' = n^{-1}K_1 n \supset H$ and define $f_1'$ by commutativity of the diagram

$$\begin{array}{ccc} G/H & \xrightarrow{f_1'} & G/K_1' \\ {\scriptstyle \varphi = R_n}\downarrow & & \downarrow{\scriptstyle R_n} \\ G/H & \xrightarrow{f_1} & G/K_1 \end{array}$$

(Recall that the vertical map on the right is $gK_1' \mapsto gK_1'n^{-1} = gn^{-1}K_1$.)

This gives an equivalence $\psi$: $M_{f_1'} \xrightarrow{\approx} M_{f_1}$ which equals $\varphi$ on the top face $G/H$. Thus

$$M \approx M_{f_0} \cup_\varphi M_{f_1} \approx M_{f_0} \cup M_{f_1'}$$

and we obtain the desired result upon replacing $K_1$ by its conjugate $K_1'$. ∎

Now we shall turn to the case $d = n - 2$. If $V$ is a linear slice at $x$ in $M$ and $S$ is the unit sphere in $V$, then, near $x$, $M^*$ is the open cone over $S^*$. Thus the orbit structure is "locally conical." Note that when $d = n - 2$ and $G(x)$ is singular, then $\dim S > 1$ so that $S^*$ must be an arc (since $S$ does not fiber over a circle). On the other hand, if $G(x)$ is exceptional, then $S$ is a circle and $G_x$ (effectively) acts as either a cyclic group of rotations or a dyhedral group (the symmetry group of a regular polygon), since these are precisely the finite subgroups of $\mathbf{O}(2)$. If $G_x$ acts cyclically on the circle $S$, then $S^*$ is a circle. Otherwise $S^*$ is an arc and it is clear that $x$ is in the *closure* of $SE$. Thus we have the following proposition.

**8.3. Proposition** *If $d = n - 2$, then the boundary of the 2-manifold $M^*$ consists exactly of $B^* \cup \overline{SE}^*$.* ∎

Let $C^*$ denote the set of those points $x^* \in \partial M^*$ such that the type of orbits in $\partial M^*$ near $x^*$ is *not* constant. Since the orbit structure is locally conical, we see that $C^*$ is discrete.

**8.4. Lemma** *If $x^* \in C^* - \overline{SE}^*$, then the dimension of the orbit $x^*$ is strictly smaller than that of any other nearby orbit.*

*Proof* By passing to a slice we may as well assume that $x$ is a stationary point and that $M$ is euclidean space with orthogonal $G$-action. Then let $G_0$ be the identity component of $G$. If the lemma is false, then $M^{G_0}$ is 1-dimensional and $M = M^{G_0} \times W$, where $W$ is the orthogonal complement to $M^{G_0}$ at $x$, with $G_0$ transitive on the unit sphere in $W$. Moreover, $G/G_0$ must act nontrivially on $M^{G_0}$. However, this clearly implies that $W$ consists of special exceptional orbits (outside of the origin) of $G$ on $M$, contrary to the assumption. ∎

We now come to the main result of this section.

**8.5. Theorem** *Suppose that $d = n - 2$ and that $M$ is acyclic over the integers. Then $M \approx \mathbf{R}^n$ and the action is equivalent to an orthogonal action.*

*Proof* Let $H$ be a principal isotropy group. We first claim that $B^* \cup E^*$ is connected. If rank $H <$ rank $G$, then this follows from 7.2. When rank $H$ = rank $G$ it suffices, by 1.5 and 5.2, to prove it in the case $n = 2$ and $G$ finite. In this case $M \approx \mathbf{R}^2$ by known characterizations of the plane, and we shall assume such results (see Wilder [1]). Suppose that $a^*$ and $b^*$ in $M^*$ are nonprincipal and let $\sigma$ be an arc from $a^*$ to $b^*$ whose interior lies in the space $U^*$ of principal orbits. (In this case, recall that "principal" means trivial isotropy group.) If $\pi$ is the orbit map, then $\omega = \pi^{-1}(\sigma)$ is a graph in the 2-plane $M$ with vertices being the elements of the orbits $a^*$ and $b^*$. Since $a^*$ and $b^*$ are nonprincipal, it follows that there are at least two edges of $\omega$ emanating out of each vertex. Thus $\omega$ must contain a simple closed curve. It follows that $\omega$ separates $M$ and that at least one component of $M - \omega$ is bounded. Since $G$ permutes the bounded components of $M - \omega$, it follows that the union $V$ of the bounded components and the union $W$ of the unbounded components of $M - \omega$ are saturated sets. Thus $M^* - \sigma = V^* \cup W^*$ and $V^* \cap W^* = \varnothing$. Thus $\sigma$ separates $M^*$. Since the interior of $\sigma$ lies in the interior of $M^*$ we conclude that both $a^*$ and $b^*$ must be boundary points. However, $V^* \cup \sigma$ is compact and this clearly implies that $a^*$ and $b^*$ lie on the *same* boundary component of $M^*$. Since $\partial M^*$ consists of nonprincipal orbits, this proves our contention.

We shall now continue the proof for the case in which $G$ is *connected*, and shall later show how to remove this restriction.

When $G$ is connected we have $H_1(M^*; \mathbf{Z}_2) = 0$ by 4.4. Also $M^*$ either has no boundary or has precisely one boundary component. This is enough to conclude from classical facts about 2-manifolds, that $M^*$ is homeomorphic to either the plane or the half plane. We shall assume this.

If $M^*$ is the plane, then, by the remarks above, it is clear that $B^* \cup E^*$ consists of exactly one orbit, say of type $G/K$. In this case, the orbit structure of $M^*$ is conical with vertex being this orbit, and it follows from II.8.5 that $M \approx G \times_K V$ whence $G = K$ acting orthogonally. By 8.3, this cannot happen when $G$ is connected.

Thus $M^*$ is the half plane ($y \geq 0$ in the $x$-$y$ plane) and $B^* \cup E^*$ consists of the boundary $y = 0$. By 8.3 and 3.12 we see that $E^* = \varnothing$.

Again, it suffices to show that $M^*$ has conical orbit structure. That is, we must show that $B^*$ contains a point $a^*$ such that the orbit type is constant on each of the two components of $B^* - \{a^*\}$.

Recall that $C^* \subset B^*$ consists of those points near which the orbit type on $B^*$ is *not constant*. Let $\sigma$ be an arc in $B^*$ whose end points $a^*$ and $b^*$ are in $C^*$ and whose interior does not meet $C^*$. Let $G/K$ be the type of orbits corresponding to interior points of $\sigma$, let $k = \dim G/K$, put $A = \pi^{-1}(\sigma)$, put $\partial A = \pi^{-1}(a^*) \cup \pi^{-1}(b^*)$, and let $A' = A - \partial A$. By 8.4 we have that $\dim \partial A < k$. Now $A'$ is a connected $(k+1)$-manifold so that

$$\check{H}^{k+1}(A, \partial A;\ \mathbf{Z}_2) \approx \mathbf{Z}_2.$$

Since $\dim \partial A \leq k - 1$ we have the diagram (coefficients in $\mathbf{Z}_2$)

$$\begin{array}{ccc} \check{H}^{k+1}(B_+, B_+ - A') & \xrightarrow{\ j^*\ } & \check{H}^{k+1}(B_+, \infty) \\ \Big\downarrow \approx & & \Big\downarrow i_A^* \\ \check{H}^{k+1}(A, \partial A) & \xrightarrow{\ \approx\ } & \check{H}^{k+1}(A) \end{array}$$

where all maps are induced by inclusions. [Since $A$ is compact we substitute $A$ for $(A_+, \infty)$.] Let $\mu_A \in \check{H}^{k+1}(B_+, \infty)$ denote the image of the nontrivial class in $\check{H}^{k+1}(B_+, B_+ - A')$ under $j^*$. Then $i_A{}^*(\mu_A) \neq 0$.

If $A_1$ is the inverse image of another such arc in $B^*$, then

$$\begin{array}{ccc} \check{H}^{k+1}(B_+, B_+ - A') & \longrightarrow & \check{H}^{k+1}(B_+, \infty) \\ \Big\downarrow & & \Big\downarrow i_{A_1}^* \\ 0 = \check{H}^{k+1}(A_1, A_1) & \longrightarrow & \check{H}^{k+1}(A_1) \end{array}$$

shows that $i_{A_1}^*(\mu_A) = 0$. Thus the elements $\mu_A$ are all distinct.

Now, by Lefschetz duality (with $q \leq n - 2$)

$$\check{H}^q(B_+, \infty) \approx \check{H}^{q+1}(M_+, B_+) \approx H_{n-q-1}(M - B) \approx H_{n-q-1}(G/H)$$

since $M - B$ fibers over $M^* - B^* \approx \mathbf{R}^2$ with fiber $G/H$ and hence has $G/H$ as a deformation retract. Since $H_*(G/H;\ \mathbf{Z}_2)$ is finitely generated, it now follows that $C^*$ is *finite*.

Since $C^*$ is finite, there is an arc (or a point) $\tau$ in $B^*$ whose end points are in $C^*$ and which *contains* $C^*$.

Let $D$ be a 2-cell neighborhood of $\tau$ in $M^*$; see Figure IV-1. Then $D$ is a strong deformation retract of $M^*$ by a deformation which preserves the orbit structure. By the Covering Homotopy Theorem II.7.3, $\pi^{-1}D$ is a deformation retract of $M$ and hence is acyclic. Since, for varying $D$, the $\pi^{-1}D$ form a fundamental neighborhood system of $\pi^{-1}(\tau)$, it follows that $\check{H}^q(\pi^{-1}(\tau);\ \mathbf{Z}_2) = 0$ for $q > 0$. By the above remarks, this implies that $C^*$

is either a point or is empty. Thus the orbit structure of $M^*$ is conical and the theorem follows from II.8.5.

We still must treat the case for which $G$ is disconnected. Consider the action of $G/G_0$ on $M/G_0$. Since this preserves the local conical structure of $M/G_0$ and since the actions of a finite group on arcs and circles are easily

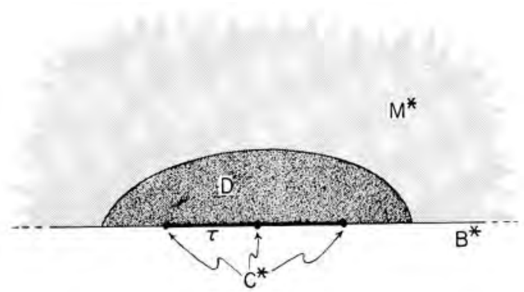



FIGURE IV-1

analyzed (see Exercise 3), we see that this action is locally smooth. First, let us assume that $G_0$ acts nontrivially, so that $M/G_0$ is a halfplane. On the boundary of $M/G_0$, $G/G_0$ must effectively be either trivial or a reflection. If an element $g \in G$ leaves $\partial(M/G_0)$ fixed, then it must leave all of $M/G_0$ fixed by an easy argument using local smoothness (or by extending it trivially to the other halfplane and using Newman's Theorem III.9.5). Thus $G/G_0$ is effectively either trivial or $\mathbf{Z}_2$ on $M/G_0$. If this action is nontrivial, then there is a unique fixed point on $\partial(M/G_0)$. (Note that $M^{G_0}$ is either a point or is all of $\partial(M/G_0)$ and, in the first case, $G$ must preserve this point.) Local smoothness shows that $(M/G_0)^G$ is then a 1-manifold with boundary being one point, its intersection with $\partial(M/G_0)$. Since $M/G = (M/G_0)/(G/G_0)$ is simply connected by II.6.3, it is a halfplane and its boundary is clearly the union of two rays, one the orbit space of $\partial(M/G_0)$ under $G/G_0$ and the other, the fixed set of $G/G_0$ on $M/G_0$ (either by the Smith Theorems or by the fact that $B^* \cup E^*$ is connected). From this it is again clear that the orbit structure of $M/G$ is conical, which implies the desired result. This is also clear when $G/G_0$ acts trivially on $M/G_0$.

If $G_0$ acts trivially on $M$ so that, effectively, $G$ is finite and $M \approx \mathbf{R}^2$, then a similar analysis can be made by first considering the action of the subgroup $G_+$ preserving orientation, and then the action of $G/G_+$ (trivial or $\mathbf{Z}_2$) on $M/G_+$. (Note that $M/G_+$ is a plane and that $G_+$ acts freely outside of a unique point, by our previous remarks.) The details of this case are left to the reader as Exercise 4. ∎

*Remarks* The case $d = n - 1$ is treated in Hofmann and Mostert [2], Montgomery and Zippin [2, 4], and Mostert [1] (without local smoothness). Theorem 8.5 was proved in Montgomery, Samelson, and Yang [2]. A similar investigation of the case $d = n - 3$ on $\mathbf{S}^n$ with a stationary point is carried out in Montgomery and Yang [3].

We now focus our attention on actions on *compact* manifolds with $d = n - 2$. The proof of the following result is essentially from Montgomery, Samelson and Yang [2] and could be used as an alternative approach to part of the proof of 8.5.

**8.6. Theorem** *Suppose that $G$ is connected and that $d = n - 2$. Also suppose that $M$ is compact and connected with $H_1(M; \mathbf{Z}) = 0$, and that a singular orbit exists. Then $E^* = \emptyset$ and $M^*$ is a 2-disk with boundary $B^*$.*

*Proof* By II.6.5 $M^* \approx \mathbf{D}^2$, $\mathbf{S}^2$, or $\mathbf{P}^2$. From 3.12 and 8.3 we know that $\partial M^* = B^* \neq \emptyset$. Thus $M^* \approx \mathbf{D}^2$ and it suffices to show that $E^* = \emptyset$. Let $\tau$ be an arc in $M^*$ from a point of $B^*$ to a point of $E^*$ and otherwise passing through principal orbits. Consider the set $A = \pi^{-1}(\tau)$. It is clear from the proof of 8.2 that $A$ is the union of mapping cylinders

$$A = M_\varphi \cup M_\psi,$$

where $\varphi\colon P \to Q$ and $\psi\colon P \to S$ are equivariant maps from a principal orbit $P$ to an exceptional orbit $Q$ and a singular orbit $S$. By 4.5, $P$ and $Q$ are orientable. If $\varphi$ is a $k$-fold covering map, then we have the exact sequence

$$\cdots \to \check{H}^{n-2}(Q;\mathbf{Z}) \xrightarrow{\varphi^*} \check{H}^{n-2}(P;\mathbf{Z}) \to \check{H}^{n-1}(M_\varphi,P;\mathbf{Z}) \to \check{H}^{n-1}(Q;\mathbf{Z}) = 0$$

of $\varphi$ [i.e., of the pair $(M_\varphi,P)$]. The first two groups displayed are infinite cyclic, since $P$ and $Q$ are connected, orientable $(n - 2)$-manifolds, and the map between them is essentially multiplication by $k$ since $\deg \varphi = k$. Thus $\check{H}^{n-1}(A, M_\psi; \mathbf{Z}) \approx \check{H}^{n-1}(M_\varphi,P;\mathbf{Z}) \approx \mathbf{Z}_k$. However, $M_\psi$ has $S$ as a deformation retract and $\dim S \leq n - 3$. Thus $\check{H}^{n-1}(A; \mathbf{Z}) \approx \mathbf{Z}_k$. Since $\check{H}^{n-1}(M;\mathbf{Z}) \approx H_1(M; \mathbf{Z}) = 0$, the exact sequence

$$0 = \check{H}^{n-1}(M; \mathbf{Z}) \to \check{H}^{n-1}(A; \mathbf{Z}) \to \check{H}^n(M, A; \mathbf{Z}) \to \cdots$$

shows that $H_0(M - A; \mathbf{Z}) \approx \check{H}^n(M, A; \mathbf{Z})$ has torsion, which is impossible. ∎

By similar arguments, we shall now show that, for general compact orientable $M$ with $M^G \neq \emptyset$ (for example), quite strong information on the set of exceptional orbits can be obtained.

**8.7. Theorem** *Let $G$ be connected and let $M$ be compact, connected, and orientable. Suppose that the inclusion of a principal orbit $P$ in $M$ induces the trivial homomorphism $H_1(P; \mathbf{Z}) \to H_1(M; \mathbf{Z})$. (This holds if $H_1(Q; \mathbf{Z}) = 0$ for some orbit $Q$; e.g., if $M^G \neq \emptyset$.) Also suppose that $d = n - 2$. Then we have the following facts:*

(i) *If $H_1(M; \mathbf{Z})$ has no 2-torsion, then there are no special exceptional orbits.*

(ii) *If there is a singular orbit but no special exceptional orbits, then* rank $H_1(M; \mathbf{Z}) = \dim H_1(M^*; \mathbf{Z}_2)$, *and, if $Q_1, \ldots, Q_m$ are the exceptional orbits (in* int $M^*$*) and the canonical map $P \to Q_i$ is a $k_i$-fold covering, then the torsion subgroup of $H_1(M; \mathbf{Z})$ is $\mathbf{Z}_{k_1} \oplus \cdots \oplus \mathbf{Z}_{k_m}$.*

***Proof*** The parenthetical remark holds because the inclusion $P \subset M$ factors through *any* orbit $Q$, since $P$ can be assumed to lie in a linear tube about $Q$ and this tube deforms into $Q$. Note that the theorem has no content unless the 2-manifold $M^*$ has a boundary, i.e., $B^* \cup \overline{SE^*} \neq \emptyset$. Consider the set $A^* \subset M^*$ consisting of the boundary curves of $M^*$, together with $s$ arcs with end points in $\partial M^*$ and together with $m$ arcs from $\partial M^*$ to the $m$ exceptional orbits in int $M^*$ with the interiors of these $m + s$ arcs lying in the set of principal orbits. Furthermore, we choose the $s$ arcs so that $M^* - A^* \approx \mathbf{R}^2$. Then $s = \dim H_1(M^*; \mathbf{Z}_2)$, as is well known and easily computed, whether $M^*$ is orientable or not. (See Figure IV-2.) Let $A = \pi^{-1}(A^*)$. Then $M - A \approx (M^* - A^*) \times P \approx \mathbf{R}^2 \times P$ and, by hypothesis, this implies that $H_1(M - A; \mathbf{Z}) \to H_1(M; \mathbf{Z})$ is trivial. By Poincaré–

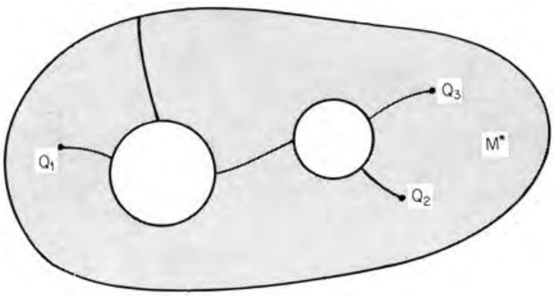



FIGURE IV-2 $m = 3,\ s = 2$.

Lefschetz duality, $\check{H}^{n-1}(M, A; \mathbf{Z}) \to \check{H}^{n-1}(M; \mathbf{Z})$ is trivial. Similarly, $\check{H}^{n}(M, A; \mathbf{Z}) \to \check{H}^{n}(M; \mathbf{Z})$ is an isomorphism, since $M - A$ is connected. Thus the restriction

$$H_1(M; \mathbf{Z}) \approx \check{H}^{n-1}(M; \mathbf{Z}) \to \check{H}^{n-1}(A; \mathbf{Z})$$

is an isomorphism.

If $SE \neq \varnothing$, then by 3.11 we see that $A$ contains a set of the form $\mathbf{I} \times Q$; $Q$ special exceptional, *nonorientable*. By the remark following 3.9 this implies that $\check{H}^{n-1}(A; \mathbf{Z})$ contains 2-torsion; proving (i).

In the situation of (ii) we have $B^* = \partial M^*$. If $A_1$ denotes the portion of $A$ over $B^*$ and $A_2$, that over the added arcs, then $\dim(A_1 \cap A_2) \leq n - 3$ and $\dim A_1 \leq n - 2$. Thus the Mayer–Vietoris sequence implies that $\check{H}^{n-1}(A; \mathbf{Z}) \approx \check{H}^{n-1}(A_2; \mathbf{Z})$ and, similarly, this is the direct sum of the pieces corresponding to each arc. Recall that $P$ is orientable by 3.11. From the *proof* of 4.5, using the fact that each $Q_i$ is isolated in $E^*$ and that $M^*$ is orientable in its neighborhood, we see that each $Q_i$ is orientable. Thus it follows from the proof of 8.6 that the arc from $B^*$ to $Q_i$ contributes a summand $\mathbf{Z}_{k_i}$ to $\check{H}^{n-1}(A; \mathbf{Z})$. It is easy to see, in a similar manner, that each arc with end points in $B^*$ contributes an infinite cyclic summand, and the theorem follows. ∎

*Remark* It should be clear that, for explicit groups $G$, the analysis of actions with $d = n - 2$ can be carried considerably further. In fact, Raymond [2] and Orlik and Raymond [1] have completely classified actions of the circle group on 3-manifolds, and in Orlik and Raymond [3] a similar study is made of actions of the 2-torus on 4-manifolds.

Let us now discuss briefly the case of a locally smooth action of a connected, compact Lie group $G$ on $M = \mathbf{S}^n$ $(n > 2)$ with $d = n - 2$. If $n \neq 3$, it can be shown that a singular orbit exists, and we will assume that this is the case. Then, by 8.6, $M^* \approx \mathbf{D}^2$, $B^* \approx \partial\mathbf{D}^2$, and $E^* = \varnothing$. Consider the set $C^*$ of points in $B^* = \partial\mathbf{D}^2$ near which the orbit type *on* $B^*$ is not constant, and let $c$ be the number of points in $C^*$.

The examples of Chapter I, Section 7 of $\mathbf{SO}(n)$ on $\Sigma_k^{2n-1}$ $(n \geq 3)$ show that $c$ can be zero, and that the action may be nonorthogonal.

For an example with $c = 1$, let $G = \mathbf{U}(2)$ and recall that there is a homomorphism $G \to \mathbf{SO}(3)$ whose kernel is precisely the center of $\mathbf{U}(2)$. Through this homomorphism, $G$ acts on $\mathbf{R}^3$ and, through the usual representation of $\mathbf{U}(2)$, it also acts on $\mathbf{R}^4 = \mathbf{C}^2$. Thus we have the diagonal action of $G = \mathbf{U}(2)$ on $\mathbf{S}^6 \subset \mathbf{R}^3 \times \mathbf{R}^4$. The action on $\mathbf{R}^3$ is transitive on $\mathbf{S}^2$ with isotropy group a maximal torus $\mathbf{U}(1) \times \mathbf{U}(1)$ of $\mathbf{U}(2)$. The action on $\mathbf{R}^4$ is transitive on $\mathbf{S}^3$ with isotropy group $\mathbf{U}(1) = \mathbf{U}(1) \times \{I\} \approx \mathbf{S}^1$. Since $G_{(x,y)} = G_x \cap G_y$ we see that this is either finite or equals $\mathbf{S}^1$ when $x \neq 0$. However, $G$ is transitive on $\{0\} \times \mathbf{S}^3 \subset \mathbf{S}^6$ with isotropy group $\mathbf{U}(1) \times \mathbf{U}(1)$. Since some conjugate of $\mathbf{U}(1)$ can be seen to have trivial intersection with $\mathbf{U}(1) \times \mathbf{U}(1)$ we see that the principal isotropy group is trivial. By the general results 8.3 and 8.4 and the fact that $E^* = \emptyset$, it now follows that the orbit structure is as shown in Figure IV-3 (in which the isotropy groups are indicated).

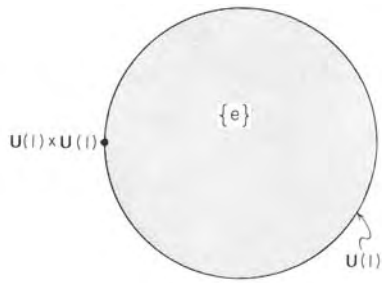



FIGURE IV-3

An example with $c = 2$ is given by $\mathbf{SO}(n) \times \mathbf{SO}(m)$ on $\mathbf{S}^{n+m} \subset \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ as is easily seen. Another example is the compactification of the example given in Chapter I, Section 4.

An example with $c = 3$ is given by $\mathbf{SO}(n) \times \mathbf{SO}(m) \times \mathbf{SO}(r)$ on $\mathbf{S}^{n+m+r-1} \subset \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r$.

It was conjectured by Montgomery and Yang that $c \leq 3$ in general. We also conjecture that the examples of Chapter I, Section 7 are the only locally smooth, but nonorthogonal, actions on $\mathbf{S}^n$ with $d = n - 2$. (If $c = 0$, then this was shown to be the case in Bredon [11, 12]. We shall discuss special cases of this in Chapter V.)

## 9. ACTIONS ON TORI

Although actions on spheres can be quite complicated, we shall show, in this section, that an action on a torus $\mathbf{T}^k$ has a very simple nature. Specifically, if $G$ is a compact, connected, Lie group acting on $\mathbf{T}^k$, then we shall show that $G$ is a toral group, that $G$ acts freely, and that the orbit map is a *trivial* principal fibration. These results are due to Conner and Montgomery [1], who treated the more general case of actions on compact aspherical manifolds (vanishing higher homotopy groups). More recently, Conner and Raymond [1] have extended these investigations considerably, and have repaired a gap in the original work of Conner and Montgomery. We shall restrict our attention to actions on tori in order to make available a relatively simple approach to these matters using Newman's Theorem (Chapter III, Section 9). We need not assume local smoothness in this section, and, in fact, this section is independent of the rest of this chapter.

**9.1. Proposition** *Let $G$ be an arcwise connected group acting on a space $X$ and let $H \subset G$ have a stationary point $x \in X$. Then the action of $H$ on $\pi_1(X, x)$ is an action by inner automorphisms.*

*Proof* If $\mu$ is a loop at $x$, and $h \in H$, then, since $G$ is connected, $h\mu$ is *freely* homotopic to $\mu$. But this is equivalent to the statement that the homotopy classes $h_{\#}[\mu] = [h\mu]$ and $[\mu]$ are *conjugate* in $\pi_1(X, x)$. ▮

**9.2. Theorem** *If $G$ is a compact Lie group acting effectively on $M = \mathbf{T}^k$ with a stationary point $x \in M$, then $G$ acts effectively on $\pi_1(M, x)$.*

*Proof* Recall that the identity component of $G$ acts *trivially* on $\pi_1(M, x)$, and thus the subgroup acting trivially on $\pi_1(M, x)$ is open and closed. Hence we may assume that $G$ acts trivially on $\pi_1$ and must prove that $G$ is trivial. Now $\mathbf{R}^k$ is the universal covering space of $M$ and we may assume that 0 projects to $x$. The $G$-action may be lifted to an action on $\mathbf{R}^k$ with 0 stationary, by I.9.2. The action of $G$ on $\pi_1(M, x)$ may be identified with conjugation by $G$ on the group $\Delta \approx \mathbf{Z}^k$ of deck transformations. Thus the assumption implies that $G$ commutes with $\Delta$. If $\mathbf{I} = [0,1]$, then $\mathbf{I}^k \subset \mathbf{R}^k$ is a fundamental domain for $\Delta$. Let $A = G(\mathbf{I}^k) \subset \mathbf{R}^k$ which is a compact invariant subset. Let $y \in \mathbf{R}^k$. Then $y = D(z)$ for some $z \in \mathbf{I}^k$ and $D \in \Delta$. Then

$$G(y) = G(D(z)) = D(G(z)) \subset D(A).$$

Thus

$$\operatorname{diam} G(y) \leq \operatorname{diam} D(A) = \operatorname{diam} A$$

for *all* $y$. Since this gives a uniform bound for the diameters of the orbits in $\mathbf{R}^k$, it contradicts Newman's Theorem III.9.7. ▮

**9.3. Theorem** *If $G$ is a compact, connected, Lie group acting effectively on $M = \mathbf{T}^k$, then $G$ is a torus and the action is free.*

*Proof* If $H$ is the isotropy group of a point $x \in M$, then, by 9.1, $H$ acts trivially on $\pi_1(M, x)$, since this is abelian. Then, by 9.2, we conclude that $H$ is trivial. Hence $G$ acts freely. The action may be lifted to an effective action of some covering group $G'$ of $G$ on $\mathbf{R}^k$, by I.9.1. If $G'$ contains a compact subgroup, then it contains a cyclic subgroup $H$ of prime order. Then $H$ has a stationary point on $\mathbf{R}^k$ by III.7.11, and hence has a stationary point on $M$. Thus $H$ acts trivially on $M$, since $G$ acts freely, and hence is in the kernel of $G' \to G$ which is contained in $\Delta \approx \mathbf{Z}^k$; a contradiction. Since the simply connected covering group of a semisimple compact Lie group is again compact, it follows that $G$ can contain no such subgroup, and hence that $G$ is a toral group; see 0.6.10. ▌

**9.4. Lemma** *If $S$ is a circle group acting effectively on $M = \mathbf{T}^k$, then, for any $x \in M$, the inclusion $S(x) \to M$ induces a monomorphism $\pi_1(S(x)) \to \pi_1(M)$.*

*Proof* Let $S'$ acting on $\mathbf{R}^k$ be the unique effective covering action as given in I.9.1. Then $S'$ cannot be a circle, by the proof of 9.3, and hence $S' \approx \mathbf{R}$. Then the kernel $\mathbf{Z}$ of $\mathbf{R} \to S$ is a subgroup of the group $\Delta$ of deck transformations of $\mathbf{R}^k \to M$, by I.9.1. However, the inclusion $\mathbf{Z} \subset \Delta$ can be identified with the map $\pi_1(S) \to \pi_1(M)$ [via $S \to S(x)$] once the base point $x$ is given, and hence this map is a monomorphism. ▌

*Remark* The preceding results clearly hold, more generally, when $M$ is any space such that $\mathbf{T}^r \times M$ is a torus for some $r$ (or, indeed, if $X \times M \approx \mathbf{T}^k$ for any $X$), since the results can be applied to the action on $\mathbf{T}^r \times M$, trivial on $\mathbf{T}^r$. This is just a minor generalization, but it will be convenient for use in inductive arguments below. We also note that singular and Čech theories coincide for such spaces, since they are ANR's. We shall assume this. If the reader wishes, he may assume that the action is locally smooth, so that these spaces will be manifolds.

**9.5. Theorem** *If $G$ is a toral group acting on $M = \mathbf{T}^k$, then the orbit map $M \to M^*$ is a trivial principal $G$-fibration; that is, $M \approx G \times M^*$ with trivial action on $M^*$.*

*Proof* First let us reduce the theorem to the case in which $G$ is a circle group. Let $G = G' \times S$, where $S$ is a circle factor. Suppose we know that there is an equivalence $M \approx G' \times M'$, where $M' = M/G'$. Then $S$ acts freely on $M'$ (by inspection, or by the above results and the remark, since $G' \times M'$ is a torus). Thus if $M' \approx S \times (M'/S)$, then $M \approx G' \times S \times (M'/S) \approx G \times M^*$.

Thus we may assume that $G = \mathbf{S}^1$ is a circle group and that $M$ becomes a torus upon multiplying by some torus. In particular, $\pi_1(M)$ is free abelian and $\pi_i(M) = 0$ for $i > 1$. Since $\mathbf{S}^1$ acts freely on $M$, $M \to M^*$ is a principal $\mathbf{S}^1$-fibration. By 9.4 and the exact homotopy sequence

$$0 \to \pi_2(M^*) \to \pi_1(\mathbf{S}^1) \to \pi_1(M) \to \pi_1(M^*) \to 0$$

we see that $\pi_i(M^*) = 0$ for $i > 1$. Since $\pi_1(M)$ is abelian, so is $\pi_1(M^*)$. Now the universal covering space of $M^*$ is acyclic and finite-dimensional, and hence, by Smith theory, its group of deck transformations contains no elements of finite order. Thus $\pi_1(M^*)$ is *free* abelian and

$$0 \to \pi_1(\mathbf{S}^1) \to \pi_1(M) \to \pi_1(M^*) \to 0$$

must *split*. Since these are abelian, we may substitute $H_1$ for $\pi_1$. Applying $\mathrm{Hom}(\cdot, \mathbf{Z})$ to this gives a split exact sequence and, using $H^1(\cdot\,; \mathbf{Z}) = \mathrm{Hom}(H_1(\cdot), \mathbf{Z})$, we see that

$$0 \to H^1(M^*; \mathbf{Z}) \to H^1(M; \mathbf{Z}) \to H^1(\mathbf{S}^1; \mathbf{Z}) \to 0$$

is exact. In particular, the inclusion $\mathbf{S}^1 \to M$ of a fiber induces an *epimorphism* on cohomology.

Now consider the Gysin sequence (coefficients in $\mathbf{Z}$) and the diagram induced by inclusion of a fiber

$$\begin{array}{ccccccc} H^1(M^*) & \to & H^1(M) & \to & H^0(M^*) & \xrightarrow{\omega} & H^2(M^*) \to \\ \downarrow & & \downarrow & & \downarrow \approx & & \downarrow \\ 0 = H^1(\star) & \to & H^1(\mathbf{S}^1) & \to & H^0(\star) & \to & H^2(\star) = 0. \end{array}$$

This shows that $\omega = 0$.

Now consider a universal $\mathbf{S}^1$-bundle $\mathbf{S}^\infty \to \mathbf{CP}^\infty$ (or use $\mathbf{S}^{2N+1}$ for large $N$). There is a classifying map $M^* \to \mathbf{CP}^\infty$ and a natural diagram

$$\begin{array}{ccc} M & \longrightarrow & \mathbf{S}^\infty \\ \downarrow & & \downarrow \\ M^* & \xrightarrow{\varphi} & \mathbf{CP}^\infty \end{array}$$

The induced map of Gysin sequences gives a diagram

$$\begin{array}{ccc} H^0(\mathbf{CP}^\infty) & \xrightarrow[\approx]{\omega} & H^2(\mathbf{CP}^\infty) \\ \downarrow & & \downarrow \varphi^* \\ H^0(M^*) & \xrightarrow{0} & H^2(M^*) \end{array}$$

which shows that $\varphi^* = 0$. By standard obstruction theory, since $\mathbf{CP}^\infty$ is a $\mathbf{K}(\mathbf{Z}, 2)$, homotopy classes of maps of $M^*$ into $\mathbf{CP}^\infty$ correspond to elements of $H^2(M^*)$ and, in fact, the correspondence takes $[\varphi]$ to $\varphi^*(u)$, where $u$ is some generator. Thus $\varphi$ is homotopic to a constant map which means that $M \to M^*$ is a trivial bundle. ∎

*Remark* Clearly a large part of the above proof is standard material, and the reader with an extensive background on topology should see how to condense it. A more efficient version of essentially the same proof is given in Conner and Montgomery [1].

## 10. FINITENESS OF NUMBER OF ORBIT TYPES

We shall now prove a theorem due to Mann [2] which states that, on an orientable manifold $M$ having finitely generated homology, an action has only finitely many orbit types. As usual, we shall restrict our attention to the locally smooth case, although the result can be proved without this restriction. We will say that $M$ has **finite type** if each $H_i(M; \mathbf{Z})$ is finitely generated.

**10.1. Lemma** *Let $M$ be orientable and of finite type. If $C$ is a compact neighborhood of infinity in $M_+$, then there is a neighborhood $C' \subset C$ of infinity with $\check{H}^*(C, \infty; \mathbf{Z}) \to \check{H}^*(C', \infty; \mathbf{Z})$ trivial. (That is, $M_+$ is clc at $\infty$.)*

*Proof* If $C' \subset \operatorname{int} C$, then it is well known that the image of $\check{H}^*(M_+, C) \to \check{H}^*(M_+, C')$ is finitely generated. [This holds since $M$ is cohomologically locally connected in each degree (clc), and a proof can be found, for example, in Bredon [13, p.77]. Here we shall be satisfied with the remark that this is easily proved when $M$ is triangulable. For this case, a subdivision will provide a finite polyhedron $K$ in $M$ with $K \subset M_+ - C'$ and $M_+ - C \subset \operatorname{int} K$.

Then $\check{H}^*(M_+, M_+ - \text{int } K) \approx \check{H}^*(K, \partial K)$ is finitely generated, and the homomorphism in question factors through this.]

Now $\check{H}^i(M_+, \infty) \approx H_{n-i}(M)$ is finitely generated, by assumption, and the diagram

$$\begin{array}{ccccc} \check{H}^i(M_+, \infty) & \to & \check{H}^i(C, \infty) & \to & \check{H}^{i+1}(M_+, C) \\ \downarrow \approx & & \downarrow & & \downarrow \\ \check{H}^i(M_+, \infty) & \to & \check{H}^i(C', \infty) & \to & \check{H}^{i+1}(M_+, C') \end{array}$$

implies that $\check{H}^i(C, \infty) \to \check{H}^i(C', \infty)$ has finitely generated image. Now

$$\varinjlim \check{H}^i(C, \infty) = \check{H}^i(\infty, \infty) = 0$$

by continuity of Čech theory. Thus any element of the image of $\check{H}^i(C, \infty) \to \check{H}^i(C', \infty)$ can be killed by further restriction to a smaller neighborhood of $\infty$. By finite generation of this image we can, in fact, kill all of it at once. ∎

**10.2. Corollary** *The conclusion of* 10.1 *holds for coefficients in* $\mathbf{Z}_p$, *uniformly in p.*

*Proof* Suppose that 10.1 holds for the two inclusions $C'' \subset C' \subset C$. The coefficient sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0$ gives the diagram

$$\begin{array}{ccccc} & & \check{H}^i(C, \infty; \mathbf{Z}_p) & \to & \check{H}^{i+1}(C, \infty; \mathbf{Z}) \\ & & \downarrow & & \downarrow 0 \\ \check{H}^i(C', \infty; \mathbf{Z}) & \to & \check{H}^i(C', \infty; \mathbf{Z}_p) & \to & \check{H}^{i+1}(C', \infty; \mathbf{Z}) \\ \downarrow 0 & & \downarrow & & \\ \check{H}^i(C'', \infty; \mathbf{Z}) & \to & \check{H}^i(C'', \infty; \mathbf{Z}_p) & & \end{array}$$

and a diagram chase shows that $\check{H}^i(C, \infty; \mathbf{Z}_p) \to \check{H}^i(C'', \infty; \mathbf{Z}_p)$ is trivial. ∎

We remark that, since the universal coefficient sequence is valid in the present context, a similar proof gives 10.2 for *arbitrary* coefficients, but we shall not use this fact.

**10.3. Lemma** *Let M be an arbitrary n-manifold. Suppose that* $G = \mathbf{Z}_p$ *acts on M and that* $C_0 \supset C_1 \supset \cdots \supset C_{n+1}$ *are compact invariant neighborhoods of infinity in* $M_+$ *such that* $\check{H}^*(C_i, \infty; \mathbf{Z}_p) \to \check{H}^*(C_{i+1}, \infty; \mathbf{Z}_p)$ *is* *trivial for all i. Then*

$$\check{H}^*(C_0{}^G, \infty; \mathbf{Z}_p) \to \check{H}^*(C_{n+1}^G, \infty; \mathbf{Z}_p)$$

*is trivial.*

***Proof*** In fact, we claim that (with coefficients in $\mathbf{Z}_p$)

$$\check{H}_\varrho^{n-i+1}(C_0, \infty) \oplus \check{H}^{n-i+1}(C_0{}^G, \infty) \to \check{H}_\varrho^{n-i+1}(C_i, \infty) \oplus \check{H}^{n-i+1}(C_i{}^G, \infty)$$

is trivial for both $\varrho = \tau$ and $\varrho = \sigma$. This holds vacuously for $i = 0$ and will be proved in general by induction on $i$. The Smith sequences give the diagram

$$\begin{array}{ccccc}
 & & \check{H}_\varrho^{n-i}(C_0, \infty) \oplus \check{H}^{n-i}(C_0{}^G, \infty) & \to & \check{H}_{\bar\varrho}^{n-i+1}(C_0, \infty) \\
 & & \downarrow & & \downarrow{\scriptstyle 0} \\
\check{H}^{n-i}(C_i, \infty) & \to & \check{H}_\varrho^{n-i}(C_i, \infty) \oplus \check{H}^{n-i}(C_i{}^G, \infty) & \to & \check{H}_{\bar\varrho}^{n-i+1}(C_i, \infty) \\
\downarrow{\scriptstyle 0} & & \downarrow & & \\
\check{H}^{n-i}(C_{i+1}, \infty) & \to & \check{H}_\varrho^{n-i}(C_{i+1}, \infty) \oplus \check{H}^{n-i}(C_{i+1}^G, \infty) & &
\end{array}$$

from which the conclusion follows by a diagram chase. ∎

**10.4. Theorem** *Let $G$ be a compact Lie group acting locally smoothly on the orientable manifold $M$ of finite type. Then there is a compact subset $K$ of $M$ which touches every component of the fixed point set of every finite $p$-group in $G$, uniformly for all primes $p$.*

***Proof*** Let $M_+ = C_0 \supset C_1 \supset \cdots$ be compact neighborhoods of infinity in $M_+$ such that $\check{H}^*(C_i, \infty; \mathbf{Z}_p) \to \check{H}^*(C_{i+1}, \infty; \mathbf{Z}_p)$ is trivial for all $i$ and all $p$. By 10.2 these exist, and it is clear that they may be chosen so as to be *invariant* under $G$. Let $P \subset G$ be a $p$-group. It will suffice to show that no component $F$ of $M^P$ is contained in $C_{(n+1)^n}$. Let $M_0$ be the component of $M$ containing $F$. Since $P$ has a composition series consisting of normal subgroups, it follows that there is a normal subgroup $P_1$ of $P$ which is *effectively* of order $p$ on $M_0$. Let $M_1$ be the component of $M_0^{P_1}$ containing $F$. Then $P$ acts on $M_1$ (with $P_1$ acting trivially) and we repeat this process to obtain a normal subgroup $P_2 \supset P_1$ effectively of order $p$ on $M_1$, and let $M_2$ be the component of $M_1^{P_2}$ containing $F$. Since the $M_i$ have decreasing dimension, this process stops with $M_k = F$ for some $k \leq n$. Now an obvious induction on 10.3 shows that

$$\check{H}^*(M_{i,+}, \infty; \mathbf{Z}_p) \to \check{H}^*(C_{(n+1)^i} \cap M_{i,+}, \infty; \mathbf{Z}_p)$$

is trivial. Hence

$$\check{H}^*(F_+, \infty; \mathbf{Z}_p) \to \check{H}^*(C \cap F_+, \infty; \mathbf{Z}_p)$$

is trivial, where $C = C_{(n+1)^n}$. Since $F$ is orientable (or $p = 2$) by 2.1, we have $\check{H}^m(F_+, \infty; \mathbf{Z}_p) \approx H_0(F; \mathbf{Z}_p) \neq 0$, where $m = \dim F$. Thus $F \not\subset C$. ∎

**10.5. Theorem** *Let $T$ be a toral group acting locally smoothly on the orientable manifold $M$ of finite type. Then $T$ has only finitely many orbit types on $M$.*

*Proof* By induction on $\dim T$ we may assume that each proper subtorus $S$ of $T$ has only finitely many orbit types on $M$ and, from III.10.13 and Poincaré duality, this implies that $F(S, M)$ has finite type. Suppose that $K \subset M$ is as in 10.4. Let $P_1 \subset P_2 \subset \cdots$ be $p$-groups ($p$ fixed) converging to a torus $S \subset T$. Then $F(P_1, M) \supset F(P_2, M) \supset \cdots$ is a sequence of manifolds with intersection $F(S, M)$. Since each component of $F(P_i, M)$ touches $K$, it follows that the $F(P_i, M)$ are eventually constant. In particular, each component of $F(S, M)$ touches $K$. The proof now follows the lines of that of 1.3 (and 1.4), with only minor modifications which the reader should be able to supply without difficulty. ∎

*Remark* There is a purely Lie group theoretic theorem of Mostow [2] which implies that results such as 10.5 also hold for actions of any compact Lie group $G$ when they hold for tori. This result states that if $\mathscr{C}$ is any class of subgroups of a given compact Lie group $G$, if $\mathscr{C}$ is closed under conjugation, and if $\{C \cap T \mid C \in \mathscr{C}\}$ is finite, where $T$ is the maximal torus of $G$, then $\mathscr{C}$ contains only finitely many conjugacy classes. An exposition of this result may be found in Borel [5, Chapter VII] and will not be repeated here.

*Remark* If $M$ is nonorientable, then for 10.4 (and hence 10.5) to hold on $M$ it would suffice for $M$ *and* its orientable double covering $M'$ to have finite type. To see this, first note that 10.4 holds for the prime 2 since orientability does not enter the proof in that case. If $p$ is an odd prime and $P$ is a $p$-group acting on $M$, consider the lifting of $P$ to an action on $M'$ (given by I.9.4). Since ord $P$ is odd it is clear that $M'^P$ is the full inverse image of $M^P$. The result then follows from 10.4 applied uniformly to all *odd* primes and to this lifted action. Theorem 10.5 also follows, since its proof out of 10.4 makes no essential use of orientability.

## EXERCISES FOR CHAPTER IV

**1.** Show that the principal isotropy group of a locally smooth action of a compact abelian Lie group is trivial.

**2.** Suppose that the compact Lie group $G$ acts locally smoothly on $\mathbf{S}^{n-1}$ or $\mathbf{R}^n$ with principal isotropy group $H$. Let $r = \operatorname{rank} G - \operatorname{rank} H$. Show that $\dim G \leq [(n-r)(n-r+1)]/2$.

**3.** If $G$ is a finite group acting locally smoothly on $\mathbf{S}^1$, show that the action is equivalent to an orthogonal action (and, in particular, that $G$ is either cyclic or dyhedral).

**4.** Show that a locally smooth action on $\mathbf{R}^2$ is equivalent to an orthogonal action.

**5.** If $\mathbf{S}^1$ acts locally smoothly on a compact 3-manifold $M$ having the homology of $\mathbf{S}^3$, show that either $M \approx \mathbf{S}^3$ or that there are no stationary points.

**6.** Let $G$ be a compact Lie group acting locally smoothly on the connected orientable manifold $M$ of finite type. Show that there is a compact subset of $M$ which touches each component of the set $B \cup E$ of points on nonprincipal orbits and hence that there are only finitely many such components.

**7.** Suppose that $\mathbf{SO}(3)$ acts locally smoothly on $\mathbf{S}^4$ with principal orbits of dimension 2. Show that the action is equivalent to the orthogonal action on $\mathbf{S}^4 \subset \mathbf{R}^5 = \mathbf{R}^3 \times \mathbf{R}^2$ which is the sum of the standard representation on $\mathbf{R}^3$ and the trivial representation on $\mathbf{R}^2$.

**8.** Suppose that $\mathbf{SO}(3)$ acts locally smoothly on $\mathbf{S}^4$ with principal orbits of dimension 3. Show that the principal isotropy group is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and that there are precisely two singular orbits, both projective planes.

**9.** Suppose that $\mathbf{SO}(3)$ acts locally smoothly on $\mathbf{S}^5$ with principal orbits of dimension 2. Show that $F(\mathbf{SO}(3), \mathbf{S}^5) \approx \mathbf{S}^2$ and that all nonfixed orbits are 2-spheres.

**10.** If $G$ is a compact, connected, Lie group acting locally smoothly on the $n$-manifold $M$ and if $H_1(M; \mathbf{Z}_2) = 0$, show that $n - \dim E$ is even.

**11.** Suppose that $G$ is a compact, connected, Lie group acting locally smoothly on an orientable $n$-manifold $M$ with principal orbits of dimension $n - 2$. Assume that $M^G \neq \emptyset$ and that $M^*$ is an annulus. If $\pi_1(M)$ is abelian, show that there are no exceptional orbits in int $M^*$. (Hint: Consider covering spaces of $M^*$ and the associated pull-backs of $M$.)

**12.** If $H_1(M^n; \mathbf{Z}_2) = 0$ and if the compact Lie group $G$ acts locally smoothly on $M$ with principal orbit $\mathbf{S}^{2k}$, show that all nonprincipal orbits are stationary points and that each component of $M^G$ has dimension $n - 2k - 1$.

**13.** Let $G$ be a compact Lie group acting locally smoothly on $\mathbf{S}^n$ or $\mathbf{R}^n$ with principal isotropy group $H$. Let $T$ be a maximal torus of $H$ and assume that $(G/H)^T$ is connected. Show that $Q^T$ is connected for *any* orbit $Q$.

# CHAPTER V

## ACTIONS WITH FEW ORBIT TYPES

In this chapter we shall prove and apply several classification theorems concerning, basically, locally smooth actions with two or three orbit types.

In the first section we prove an equivariant version of the collaring theorem of Brown. However, only the nonequivariant version is used in subsequent sections.

In Section 2 we study actions on spheres such that the fixed point set has the largest possible dimension for a given maximal orbit dimension.

We deal with some background material on reduction of structure groups of bundles in Section 3. This is applied in Section 4 to prove the "Tube Theorem" which states that, for a $G$-space with a manifold with boundary as orbit space and with two orbit types (one corresponding to the boundary points), the part over a collar has the structure of an equivariant mapping cylinder. This result in the locally smooth case is analogous to the relatively simple Tubular Neighborhood Theorem in the smooth case, and is basic to the remainder of the chapter.

In Sections 5 and 6 we prove two classification theorems concerning actions with two types of orbits and with a manifold with boundary as orbit space. The second of these is analogous to a theorem of Jänich and of W.-C. and W.-Y. Hsiang in the smooth case. Classification of self-equivalences of such actions is studied in Section 7. These results are applied in Sections 8 and 9 to study equivariant plumbing and actions on Brieskorn manifolds.

Actions with three types of orbits, one consisting of fixed points, over a contractible manifold as orbit space, are studied in Sections 10 and 11.

## 1. THE EQUIVARIANT COLLARING THEOREM

A well-known result of Brown [1] states, in particular, that the boundary of a topological (paracompact) manifold with boundary has a collar; that is, a product neighborhood. We shall prove an equivariant version of this result in this section.

Recall that a $G$-orbit structure on a space $X$ is just a function from $X$ to the set $\mathscr{E}_G$ of $G$-orbit types. In general, we will use the term **structured space** for a space $X$ and a function, the "structure function," from $X$ to a set $S$ which is fixed once and for all. We shall denote a structured space by the same symbol as its underlying space, the structure function being understood. A map of structured spaces is just a continuous map which commutes with the structure functions. Let $\mathbf{I} = [0,1]$ and $\mathbf{J} = [0,1)$. If $X$ is a structured space, then we understand $X \times \mathbf{J}$ to have the structure induced by the projection to $X$ composed with the structure function on $X$.

If $X$ is a structured space and $B \subset X$ (with the induced structure), then $B$ is said to be **collared** in $X$ if there is a homeomorphism (of structured spaces) $h$ of $B \times \mathbf{J}$ onto an open neighborhood of $B$ in $X$ such that $h(b,0) = b$. Next, $B$ is said to be **locally collared** in $X$ if each point of $B$ has an open neighborhood in $B$ which is collared in $X$. We shall show that a locally collared closed subspace $B$ of a paracompact structured space $X$ is collared in $X$. Our proof follows Brown [1] with a few minor modifications, and the "structure" adds no complications whatever.

We *assume* throughout that $X$ is a paracompact structured space and that $B$ is a closed subspace.

For a map $\mu\colon B \to \mathbf{I}$ we put

$$U_\mu = \{b \in B \mid \mu(b) > 0\}$$

and

$$S_\mu = \{(b, t) \in B \times \mathbf{J} \mid t < \mu(b)\}.$$

Then $S_\mu$ is called a **spindle neighborhood** of $U_\mu \times \{0\}$. Note that $U_\mu$ must be an $F_\sigma$ (a countable union of closed sets). Moreover, if $U$ is an open $F_\sigma$ set in $B$, then $U$ is paracompact by Dugundji [1, p. 165] and is the nonzero set of *some* map $B \to \mathbf{I}$ by Dugundji [1, p. 148]. Thus it is easy to see that the spindle neighborhoods of $U \times \{0\}$ form a neighborhood basis of $U \times \{0\}$ in $B \times \mathbf{J}$ when $U$ is an open $F_\sigma$.

Given $\mu$ as above and putting $(\mu/2)(b) = \mu(b)/2$, we define a map $\varphi_\mu$: $B \times \mathbf{J} \to B \times \mathbf{J}$ by putting

$$\varphi_\mu(b, t) = \begin{cases} (b, t) & \text{if } (b, t) \notin S_\mu, \\ (b, 0) & \text{if } (b, t) \in S_{\mu/2}, \\ (b, 2t - \mu(b)) & \text{if } (b, t) \in S_\mu - S_{\mu/2}. \end{cases}$$

Note that $\varphi_\mu$ preserves "verticals" and hence preserves structure. Also note that

$$\varphi_\mu\colon B \times \mathbf{J} - S_{\mu/2} \xrightarrow{\approx} B \times \mathbf{J}.$$

**1.1. Lemma** *Let $U$ be an open $F_\sigma$ set in $B$ and let $N_1$ and $N_2$ be open neighborhoods of $U \times \{0\}$ in $B \times \mathbf{J}$. Suppose that $f\colon \bar{N}_1 \to \bar{N}_2$ is a homeomorphism (of structured spaces) such that $f \mid \bar{U} \times \{0\}$ is the identity. Then there exists a homeomorphism $f'\colon \bar{N}_1 \to \bar{N}_2$ and an open neighborhood $V$ of $U \times \{0\}$ in $N_1 \cap N_2$ such that*

(a) *$f' = f$ on $\bar{N}_1 - N_1$,*
(b) *$f'$ is the identity on $V$.*

***Proof*** There exists a map $\mu\colon B \to \mathbf{I}$ such that $U_\mu = U$ and with $S_\mu \subset N_1 \cap N_2$. Put $V = S_{\mu/2}$ and let

$$f'(x) = \begin{cases} x & \text{for } x \in \overline{S_{\mu/2}}, \\ \varphi_\mu^{-1} f \varphi_\mu(x) & \text{for } x \in \bar{N}_1 - S_{\mu/2}. \end{cases} \quad \blacksquare$$

**1.2. Lemma** *Let $U_1$ and $U_2$ be open $F_\sigma$ sets in $B$ and let $K \subset U_1 \cap U_2$ be closed relative to $U_1 \cup U_2$. Suppose that $h_i\colon U_i \times \mathbf{J} \xrightarrow{\approx} W_i$, where $W_i$ is some open neighborhood of $U_i$ in $X$ $(i = 1, 2)$ such that $h_i(u, 0) = u$. Then there exists a homeomorphism $h_2'\colon U_2 \times \mathbf{J} \xrightarrow{\approx} W_2$ such that*

(a) *$h_2'(u, 0) = u$ for $u \in U_2$,*
(b) *$h_2' \mid V = h_1 \mid V$ for some open neighborhood $V$ of $K \times \{0\}$ in $(U_1 \cap U_2) \times \mathbf{J}$.*

***Proof*** Put $B' = U_1 \cup U_2$ (not closed in $X$). We shall forget about the rest of $B$, for convenience, so that closures will be *relative* to $B'$ (or $B' \times \mathbf{J}$) below. Since $B'$ is paracompact, there exists an open $F_\sigma$ set $U$ in $B'$ with

$$K \subset U \qquad \text{and} \qquad \bar{U} \subset U_1 \cap U_2.$$

Put $W = W_1 \cap W_2$ and note that $h_2^{-1}(W) \subset B \times \mathbf{J}$ is an open neighborhood of $\bar{U} \times \{0\}$. It follows that there is an open (spindle) neighborhood $N_1$ of $U \times \{0\}$ in $B' \times \mathbf{J}$ such that $f = h_1^{-1} h_2$ is a well-defined homeomorphism on $\bar{N}_1 \subset \bar{U} \times \mathbf{J}$ to $\bar{N}_2 = f(\bar{N}_1)$, where $N_2 = f(N_1)$. By 1.1 there is a homeomorphism $f'\colon \bar{N}_1 \xrightarrow{\approx} \bar{N}_2$ such that $f' = f$ on $\bar{N}_1 - N_1$ and $f' \mid V$ is the identity for some open neighborhood $V$ of $U \times \{0\}$ in $B' \times \mathbf{J}$. Put

$$h_2'(x) = \begin{cases} h_1 f'(x) & \text{for } x \in \bar{N}_1, \\ h_2(x) & \text{for } x \in (U_2 \times \mathbf{J}) - N_1. \end{cases}$$

It is easily checked that $h_2'$ has the desired properties (including preservation of structure). $\blacksquare$

*Remark* The reader will find more details of the proofs of 1.1 and 1.2 in Brown [1].

Suppose that $k\colon U \times \mathbf{J} \to X$ $[k(u, 0) = u]$ is a collar, where $U \subset B$ is open. Let $b_0 \in U$ and let $W$ be an open neighborhood of $b_0$ in $X$ such that $\overline{W} \subset k(U \times \mathbf{J})$. There is a map $\mu\colon B \to \mathbf{I}$ such that $b_0 \in U_\mu$ [i.e., $\mu(b_0) \neq 0$] and $k(S_\mu) \subset W$. Then we see that $k(\overline{S_\mu}) = \overline{k(S_\mu)}$. Then the map $h\colon U_\mu \times \mathbf{J} \to X$ given by

$$h(b, t) = k\big(b, \mu(b)t\big)$$

defines a collar of $U_\mu$ with the added property that for any open $V \subset U_\mu$ we have

$$\overline{h(V \times \mathbf{J})} \cap B = \overline{V}.$$

Let us call a collar (of an open set in $B$) with this property, a **normal collar.**

**1.3. Lemma** *Let $U_1$ and $U_2$ be open $F_\sigma$ sets in $B$ with $U = U_1 \cup U_2$ and let $h_i\colon U_i \times \mathbf{J} \xrightarrow{\approx} W_i \subset X$ be normal collars. Then there exists a normal collar $h\colon U \times \mathbf{J} \xrightarrow{\approx} W \subset W_1 \cup W_2$ such that $h \mid (U_1 - U_2) \times \mathbf{J}$ $= h_1 \mid (U_1 - U_2) \times \mathbf{J}$.*

*Proof* As in the proof of 1.2 we put $B' = U_1 \cup U_2 = U$ (which is paracompact) and take closures relative to $B'$. We can write $B' = O_1 \cup O_2$ (open in $B'$), where $\overline{O}_i \subset U_i$, and we put $K = \overline{O}_1 \cap \overline{O}_2$. By 1.2 we may as well assume that $h_1 = h_2$ on an open neighborhood $V$ of $K \times \{0\}$. Put $V_1 = (O_1 - K) \times \mathbf{J}$. Then $\overline{h_1(V_1)} \cap B' = \overline{O_1 - K}$ which does not meet $\overline{O}_2 - O_1$. That is, $((\overline{O}_2 - O_1) \times \mathbf{J}) \cap h_2^{-1}\big(\overline{h_1(V_1)}\big)$ does not meet $(\overline{O}_2 - O_1) \times \{0\}$. Since $B' \times \mathbf{J}$ is paracompact, and hence normal, and since $\overline{O}_2 - O_1 \supset O_2 - K$ it follows that there is an open neighborhood $V_2$ of $(O_2 - K) \times \{0\}$ in $B' \times \mathbf{J}$ such that $h_2(V_2) \cap h_1(V_1) = \varnothing$.

Define $k$ on $V_1 \cup V \cup V_2$ to be $h_1$ on $V_1$, $h_2$ on $V_2$, and $h_1 = h_2$ on $V$. Then $k$ is continuous, open, and one–one, hence it is a homeomorphism into. Since $B'$ is paracompact, there is a map $\eta\colon B' \to \mathbf{I}$ which is nowhere zero and is such that $S_\eta \subset V_1 \cup V \cup V_2$. Also there is a map $\tau\colon B' \to \mathbf{I}$ such that

$$\tau(b) = \begin{cases} 1 & \text{for } b \in U_1 - U_2, \\ 0 & \text{for } b \in \overline{O}_2. \end{cases}$$

Put $\mu = \max(\tau, \eta)$. Then $S_\mu \subset V_1 \cup V \cup V_2$, $\mu$ is nowhere zero, and $\mu(b) = 1$ for $b \in U_1 - U_2$. The map $h$ defined by

$$h(b, t) = k(b, \mu(b)t)$$

clearly has the desired properties. ▮

**1.4. Theorem** *If $X$ is a paracompact structured space and $B$ is a closed subspace which is locally collared in $X$, then $B$ is collared in $X$.*

***Proof*** The subspace $B$ is covered by open sets having normal collars in $X$. The images of the collars form an open covering of $B$ *in* $X$. Together with $X - B$ we have an open covering of $X$ which can be refined to a locally finite covering. Passing to the covering associated with a partition of unity subordinate to this, we obtain a locally finite covering by open $F_\sigma$ sets in $X$. Thus there is a collection $\{V_\alpha\}$ of open sets in $X$ which is locally finite in $X$ and such that each $U_\alpha = V_\alpha \cap B$ is an $F_\sigma$ which is normally collared in $X$. The (image of the) collar of $U_\alpha$ may also be assumed to be in $V_\alpha$. Well-order the index set and define

$$V_\alpha' = \bigcup_{\beta<\alpha} V_\beta \qquad \text{and} \qquad U_\alpha' = \bigcup_{\beta<\alpha} U_\beta = V_\alpha' \cap B.$$

We shall construct inductively a *normal* collar $h_\beta$: $U_\beta' \times \mathbf{J} \to X$ with *image in* $V_\beta'$. As an inductive assumption, we decree that for $\beta < \alpha$ we have

$$h_\beta = h_\alpha \qquad \text{on} \qquad (U_\beta' - \bigcup_{\beta\le\tau<\alpha} U_\tau) \times \mathbf{J}.$$

Assume that $h_\beta$ has been defined for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, then $U_\alpha' = \bigcup_{\beta<\alpha} U_\beta'$. If $b \in U_\alpha'$, then there is a neighborhood $N$ of $b$ in $U_\alpha' \subset B$ touching only finitely many $U_\beta$ for $\beta < \alpha$. If $\tau, \tau' < \alpha$ are both larger than any of *these* $\beta$, then $h_\tau$ and $h_{\tau'}$ coincide on $N \times \mathbf{J}$. Thus the definition $h_\alpha = \lim_{\tau\to\alpha} h_\tau$ makes sense. As is easily seen $h_\alpha$ is continuous, open, and one–one, and hence is a homeomorphism onto its image which is inside $V_\alpha'$. Using the fact that $\{V_\alpha\}$ is locally finite *in* $X$, it follows that $h_\alpha$ is normal. (Also note that the $U_\alpha'$ are $F_\sigma$ sets.)

If $\alpha$ is the successor of $\beta$, then put $U_1 = U_\beta'$ and $U_2 = U_\beta$ so that $U_1 \cup U_2 = U_\alpha'$. Using 1.3 we may define $h_\alpha$ on $U_\alpha' \times \mathbf{J}$ so as to coincide with $h_\beta$ on $(U_1 - U_2) \times \mathbf{J} = (U_\beta' - U_\beta) \times \mathbf{J}$ and this finishes the induction. (Compare the proof of II.7.1, part C.) ▮

*Remark* Brown [1] gives the proof for metric $X$ and general $B$. Note, however, that local collaring implies that $B$ is locally closed. Thus $B$ is always a closed subset of an open set in $X$ (which is paracompact when $X$ is metric), so that Brown's version follows from the present one.

**1.5. Theorem** *Suppose that $G$ is a compact Lie group acting locally smoothly on the (paracompact) manifold $M$ with boundary $B$. Then there exists an equivariant homeomorphism $h$ of $B \times \mathbf{I}$ onto a neighborhood of $B$ in $M$ with $h(b, 0) = b$ and where $G$ acts on $B \times \mathbf{I}$ by the product action (trivial on $\mathbf{I}$).*

*Proof* Local smoothness implies easily that this holds locally and hence that $B/G$ is locally collared in $M/G$ *with* the given orbit structure on $M/G$. Thus $B/G$ is collared in $M/G$ and the theorem now follows from the Covering Homotopy Theorem II.7.1 of Palais. ▮

We shall take this opportunity to state the following theorem, which is unrelated to the preceding material. It is quite well known but we know of no convenient reference for it. It will be used several times in this chapter.

**1.6. Theorem** *A (paracompact) manifold $M$, with or without boundary, has the homotopy type of a CW-complex.*

*Proof* We shall only outline the proof of this well-known fact. The manifold $M$ may be taken to be connected and can then be embedded as a closed subset of some euclidean space $\mathbf{R}^n$. Since $M$ is an ANR (see Hanner [1]) there is a neighborhood $U$ of $M$ in $\mathbf{R}^n$ and a retraction $r\colon U \to M$. Then $\mathbf{R}^n$ may be triangulated so that $M$ is contained in some subcomplex $L$ which is inside $U$. Cells may be added (inductively) to $L$ to kill $\ker r_\#\colon \pi_*(L) \to \pi_*(M)$ and in such a way that $r$ extends to the new CW-complex $K$. That is, there is a CW-complex $K \supset L$ such that $r$ extends to $s\colon K \to M$ and such that $s_\#\colon \pi_*(K) \to \pi_*(M)$ is a monomorphism. Since $s_\# i_\# = 1$ (where $i\colon M \to K$ is the inclusion), $s_\#$ is an isomorphism with inverse $i_\#$. Thus $is\colon K \to K$ is a weak homotopy equivalence (an isomorphism on homotopy) and hence is a homotopy equivalence since $K$ is a CW-complex (see Spanier [1, p. 405]). If $\varphi\colon K \to K$ is a homotopy inverse to $is$, then $is\varphi \simeq 1_K$ so that $is \simeq isis\varphi = is\varphi \simeq 1_K$. Thus $i$ and $s$ are inverse homotopy equivalences between $M$ and $K$. ▮

Thus standard results from obstruction theory can be applied to $M$. For example, if $M$ is simply connected and homologically acyclic over the integers, then $M$ is contractible.

## 2. THE COMPLEMENTARY DIMENSION THEOREM

Suppose that $G$ acts locally smoothly on an $n$-sphere $M$ with principal orbits of dimension $d$. Then from IV.3.8 we have that $\dim M^G \leq n - d - 1$. In this section we study the case in which equality is achieved and shall obtain quite complete results.

**2.1. Theorem** *Let $M$ be a compact integral homology $n$-sphere and let the compact Lie group $G$ act locally smoothly on $M$ with principal orbits of dimension $d$. Assume that* $\dim M^G = n - d - 1$. *Then*

(a) *All nonfixed orbits are principal and are equivalent to $\mathbf{S}^d$ with an orthogonal $G$-action.*

(b) *$M^G$ is an integral homology $(n - d - 1)$-sphere.*

(c) *$M^*$ is an acyclic $(n - d)$-manifold with boundary $M^G$.*

(d) *There is a cross section in $M^H$ for the orbit map $M \to M^*$, where $H$ is a principal isotropy group.*

*Moreover, if the orthogonal action of $G$ on $\mathbf{S}^d$ is given, then the assignment $M \mapsto M^*$ induces a one–one correspondence between equivalence classes of such actions on homology $n$-spheres and homeomorphism classes of acyclic $(n - d)$-manifolds with boundary. Also $M$ is simply connected iff $M^*$ is simply connected (and hence contractible).*

*Proof* Let $F$ be a component of $M^G$ of dimension $n - d - 1$ and let $x \in F$. By local smoothness, a neighborhood of $x$ has the form $\mathbf{R}^{n-d-1} \times \mathbf{R}^{d+1}$, where $G$ acts orthogonally on $\mathbf{R}^{d+1}$ and trivially on $\mathbf{R}^{n-d-1}$. Then each sphere about the origin in $\mathbf{R}^{d+1}$ is an orbit of $G$ and the action on $\mathbf{R}^{d+1}$ is the cone over the action on $\mathbf{S}^d$. Thus $G$ has only principal orbits $\mathbf{S}^d \approx G/H$ and fixed points near $x$. Moreover, $(\mathbf{R}^{n-d-1} \times \mathbf{R}^{d+1})/G \approx \mathbf{R}^{n-d-1} \times [0, \infty)$ so that $M^*$ is an $(n - d)$-manifold with boundary $M^G$ in the vicinity of $x^*$.

In particular, we see that $F$ is a component of $B \cup E$ (in the notation of Chapter IV). By IV.7.2 (upon removing a point of $F$), $B \cup E$ is connected, which shows that $F = M^G$ and that all other orbits are principal. This proves part (a) and also shows that $M^*$ is an $(n - d)$-manifold with boundary $M^G$.

If rank $G >$ rank $H$ and if $T$ is a maximal torus of $G$, then $(G/H)^T = \varnothing$ so that $M^G = M^T$ which is a homology sphere by III.10.2.

If rank $G =$ rank $H$, then $(G/H)^T = N/T$ is finite, where $N = N(T)$. Since it is a homology sphere, it is just $\mathbf{Z}_2$. Since $(G/H)^N = \varnothing$ we have that $M^G = M^N = (M^T)^{N/T}$. Now $\dim M^T = \dim M^* + \dim(G/H)^T = n - d$ and $\dim(M^T)^{N/T} = n - d - 1$. Thus it follows from IV.2.6 that $M^G = (M^T)^{N/T}$ is an integral homology $(n - d - 1)$-sphere, which proves part (b).

Let $U \approx \mathbf{R}^n$ be an open $n$-disk in $M$ about a point of $M^G$ on which $G$ acts orthogonally (so that $U \cap M^G \approx \mathbf{R}^{n-d-1}$) and let $y \in M^G - U$. Since $M$ and $M^G$ are homology spheres, the inclusions induce isomorphisms

$$\check{H}^i(M, M - U) \xrightarrow{\approx} \check{H}^i(M, y),$$
$$\check{H}^i(M^G \cup (M - U), M - U) \approx \check{H}^i(M^G, M^G - U) \xrightarrow{\approx} \check{H}^i(M^G, y).$$

The 5-lemma applied to the map from the cohomology sequence of the triple $(M, M^G \cup (M - U), M - U)$ to that of $(M, M^G, y)$ implies that

$$\check{H}^i(M, M^G \cup (M - U)) \xrightarrow{\approx} \check{H}^i(M, M^G).$$

By Poincaré–Lefschetz duality, the inclusion $U - U^G = M - (M^G \cup (M - U)) \to M - M^G$ induces an isomorphism

$$H_j(U - U^G) \xrightarrow{\approx} H_j(M - M^G).$$

There is an orbit $\mathbf{S}^d \subset U - U^G$ which is clearly a deformation retract of $U - U^G$. Thus we conclude that the inclusion $\mathbf{S}^d \subset M - M^G$ of a principal orbit in $M - M^G$ induces an isomorphism $H_j(\mathbf{S}^d) \xrightarrow{\approx} H_j(M - M^G)$; that is,

$$H_*(M - M^G, \mathbf{S}^d) = 0. \tag{1}$$

Now $M - M^G$ is a fiber bundle with fiber $\mathbf{S}^d$ over $M^* - M^G$. The bundle is orientable (i.e., a loop in $M^* - M^G$ cannot reverse orientation of $\mathbf{S}^d$) since otherwise $\mathbf{S}^d \subset M - M^G$ would not give an isomorphism in homology. Thus there is the relative Gysin sequence

$$\cdots \to H_i(M - M^G, \mathbf{S}^d) \to H_i(M^* - M^G, x) \to H_{i-d-1}(M^* - M^G, x)$$
$$\to H_{i-1}(M - M^G, \mathbf{S}^d) \to \cdots$$

(where $x \in M^* - M^G$ corresponds to the orbit $\mathbf{S}^d$ in $M - M^G$). By (1) and an easy induction, this implies that $H_i(M^* - M^G, x) = 0$ for all $i$; that is, $M^* - M^G$ is acyclic. By the Collaring Theorem, $M^* \approx M^* \cup (M^G$

$\times$ **I**) and $M^*$ has the same homotopy type as does its interior $M^* - M^G$, so that (c) is proved.

Now the principal bundle associated with $M - M^G \to M^* - M^G$ is just $M^H - M^G \to M^* - M^G$ which is an $N(H)/H$-bundle. Note that $N(H)/H = (G/H)^H$ is a sphere since $G$ operates orthogonally on $G/H \approx \mathbf{S}^d$. Let $K = N(H)/H = \mathbf{S}^0$, $\mathbf{S}^1$, or $\mathbf{S}^3$, and let $B_K$ be the classifying space for $K$. Then the given bundle is induced from a map $M^* - M^G \to B_K$. Obstructions to deforming this map to a constant lie in $H^i(M^* - M^G; \pi_i(B_K)) = 0$ [since $M^* - M^G$ has the homotopy type of a CW-complex and since $\pi_1(B_K)$ operates trivially on the $\pi_i(B_K)$]. Thus the bundle $M^H - M^G \to M^* - M^G$ is trivial. If $C' \subset M^H \subset M$ is a cross section of this bundle, then the closed set $C = C' \cup M^G$ is a cross section of the orbit map $M \to M^*$, by I.3.2, proving (d).

To prove the statements in the last paragraph of the theorem, suppose that we have two such actions on $M_1$ and $M_2$, say, with $M_1^* \approx M_2^*$. Let $C_i \subset M_i^H$ $(i = 1, 2)$ be the cross sections given by part (d), so that the homeomorphism of orbit spaces corresponds to a homeomorphism $C_1 \approx C_2$. By I.3.4, this extends uniquely to an equivariant homeomorphism $M_1 \approx M_2$. Now suppose that $X$ is any acyclic $(n-d)$-manifold with boundary and put $M = \partial(\mathbf{D}^{d+1} \times X)$ with $G$ acting orthogonally on the factor $\mathbf{D}^{d+1}$ and trivially on $X$. Since $M = \partial(\mathbf{D}^{d+1} \times X) = (\mathbf{S}^d \times X) \cup (\mathbf{D}^{d+1} \times \partial X)$, and, since $\mathbf{D}^{d+1}/G \approx [0, 1]$ with $\mathbf{S}^d/G$ corresponding to $\{0\}$, we see that

$$M^* \approx (\{0\} \times X) \cup ([0,1] \times \partial X) \approx X$$

by the Collaring Theorem.

If $M$ is simply connected, then so is $M^*$ by II.6.3. If $X \approx M^*$ is simply connected, then the Van Kampen Theorem implies that $\pi_1(M)$ is generated by the images of $\pi_1(\mathbf{S}^d \times X) \approx \pi_1(\mathbf{S}^d)$ and of $\pi_1(\mathbf{D}^{d+1} \times \partial X) \approx \pi_1(\partial X)$. Since $\pi_1(\mathbf{S}^d) \to \pi_1(\mathbf{D}^{d+1})$ and $\pi_1(\partial X) \to \pi_1(X)$ are trivial, this implies that $\pi_1(M)$ is trivial. ▌

From Connell [1] it is known that if $X$ is a compact contractible $k$-manifold with boundary $\mathbf{S}^{k-1}$ and if $k = \dim X \neq 3, 4$, then $X \approx \mathbf{D}^k$. (Also, if $k > 5$, then the boundary of a contractible manifold is $\mathbf{S}^{k-1}$ iff it is simply connected.) Since the $(n - d)$-disk clearly corresponds to the orthogonal case of 2.1, we have the following corollary.

**2.2. Corollary** *Let $G$ act locally smoothly on the homotopy $n$-sphere $M$ with principal orbits of dimension $d$. Suppose that* $\dim M^G = n - d - 1$

$\neq 2, 3$. *Then the action is equivalent to an orthogonal action iff* $M^G \approx \mathbf{S}^{n-d-1}$. *If* $n - d - 1 \geq 5$, *then* $M^G \approx \mathbf{S}^{n-d-1}$ *iff it is simply connected. In general, for* $n - d \geq 5$, *such actions embed in an orthogonal action of* $G$ *on* $\mathbf{S}^{n+1}$.

*Proof* For the last statement, note that $M^*$ embeds as $M^* \times \{\frac{1}{2}\}$ in $X = M^* \times \mathbf{I}$. Then $X$ is a contractible $(n - d + 1)$-manifold whose boundary $\partial X = (\partial M^* \times \mathbf{I}) \cup (M^* \times \partial \mathbf{I})$ is the double of $M^*$ and hence is simply connected by the Van Kampen Theorem. Thus $X \approx \mathbf{D}^{n-d+1}$. The corresponding $G$-action on an $(n + 1)$-manifold is then orthogonal, has orbit space $X$, and its restriction to the part over $M^* \times \{\frac{1}{2}\}$ is clearly equivalent to the original action. ∎

**2.3. Corollary** *If* $G = \mathbf{S}^1$ *acts on a homotopy n-sphere* $M$ *with* $n \neq 4,5$ *and if* $M^G \approx \mathbf{S}^{n-2}$, *then* $M \approx \mathbf{S}^n \subset \mathbf{R}^{n+1}$ *and the action is equivalent to the orthogonal action by rotations in the first two coordinates.* ∎

*Remarks* Theorem 2.1 was proved in Bredon [2] (also see Bredon [4]) in the nonlocally smooth case. The proof there is considerably more difficult since $M^G$ is not, a priori, a manifold (and, in fact, turns out to be only a generalized manifold) and it is more difficult to identify the principal orbit. There are obvious analogs of 2.1 for actions on homology disks and euclidean space, and which have essentially the same proofs.

## 3. REDUCTION OF STRUCTURE GROUPS

In this section we shall develop some background material on the reduction of structure groups of bundles and shall prove some results, vital to us later, which are not available in the standard literature.

Let $T$ be a topological group and suppose that $p_Y\colon Y \to B$ is a principal $T$-bundle. Let $S$ be a closed subgroup of $T$. By an $S$-**reduction** of $Y$ we mean a principal $S$-bundle $p_X\colon X \to B$ *together with an* $S$*-equivariant map* $f$ *over* $B$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle p_Y} & \\ & B & \end{array}$$

If $X$ is given, then $f$ is called an $S$-**reduction of** $Y$ **to** $X$.

Two $S$-reductions $(X,f)$ and $(X',f')$ of $Y$ are said to be **equivalent** if there is an $S$-equivariant map $\varphi$ such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & Y & \end{array}$$

commutes. [Since $\varphi$ must cover the identity on $B$ it follows that $\varphi$ is a homeomorphism.]

Since it is convenient, in dealing with associated bundles, to use both left and right actions, we shall use $S\backslash X$ for the orbit space of a left $S$-action on $X$ (for example) in this section, contrary to our usual notation.

Recall from II.2.2 that the canonical map $T \times_T Y \to Y$ is a homeomorphism. Also $T \to S\backslash T$ is open, so that the induced map

$$T \times_T Y \to (S\backslash T) \times_T Y$$

is also open by II.2.1. The obvious induced map

$$S\backslash Y \approx S\backslash(T \times_T Y) \to (S\backslash T) \times_T Y$$

is then one–one, continuous, and open, and hence is a homeomorphism. Thus the projection $S\backslash Y \to T\backslash Y \approx B$ can be regarded as the projection in the associated $(S\backslash T)$-bundle $(S\backslash T) \times_T Y \to B$. Since $S\backslash X \approx B$, an $S$-reduction $f\colon X \to Y$ induces a commutative diagram

$$\begin{array}{ccc} S\backslash X & \xrightarrow{f'} & S\backslash Y \\ & {}_{\approx}\searrow \quad \swarrow & \\ & B & \end{array}$$

Thus $\sigma_f\colon B \approx S\backslash X \xrightarrow{f'} S\backslash Y$ is a cross section of the $(S\backslash T)$-bundle $S\backslash Y \to B$.

Note that $T \to S\backslash T$ is a (principal) bundle if $S$ is *compact Lie* (see II. 5.8).

**3.1. Theorem** *Suppose that the natural map $T \to S\backslash T$ is a bundle. Then the assignment of the cross section*

$$\sigma_f\colon B \approx S\backslash X \xrightarrow{f'} S\backslash Y$$

*to an $S$-reduction $f\colon X \to Y$ gives a one–one correspondence between equiv-*

*alence classes of S-reductions of the principal T-bundle $Y \to B$ and cross sections of the associated $(S\backslash T)$-bundle*

$$(S\backslash T) \times_T Y \approx S\backslash Y \xrightarrow{q} B.$$

*Proof* The fact that $T \to S\backslash T$ is a principal $S$-bundle implies that $T \times_T Y \to (S\backslash T) \times_T Y$ is also a principal $S$-bundle, since locally it is just $T \times U \to (S\backslash T) \times U$. That is, the orbit map $Y \to S\backslash Y$ is a bundle projection. Let $\sigma$: $B \to S\backslash Y$ be a cross section and let $\sigma^* Y$ be the $S$-bundle over $B$ induced by $\sigma$ from the bundle $Y \to S\backslash Y$. Then we have the pull-back diagram

$$\begin{array}{ccccc} \sigma^* Y & \xrightarrow{g} & Y & & \\ \downarrow & & \downarrow & \searrow^{p_Y} & \\ B & \xrightarrow{\sigma} & S\backslash Y & \xrightarrow{q} & B. \end{array}$$

Since $q\sigma$: $B \to B$ is the identity, by assumption, we see that

$$\begin{array}{ccc} \sigma^* Y & \xrightarrow{g} & Y \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutes. Thus $(\sigma^* Y, g)$ is an $S$-reduction of $Y \to B$ with $\sigma_g = \sigma$. If $f$: $X \to Y$ is any $S$-reduction with $\sigma_f = \sigma$, then by the pull-back property we have the diagram of $S$-spaces

$$\begin{array}{ccccc} X & & \xrightarrow{f} & & Y \\ \downarrow & \searrow & & \nearrow^{g} & \downarrow \\ & & \sigma^* Y & & \\ \downarrow & \swarrow & & & \downarrow \\ B & & \xrightarrow[\sigma = \sigma_f]{} & & S\backslash Y \end{array}$$

which gives an equivalence between the $S$-reductions $(X, f)$ and $(\sigma^* Y, g)$. ▮

*Remark* If $f$: $X \to Y$ is an $S$-reduction and $F$: $X \times \mathbf{I} \to Y$ is an $S$-equivariant homotopy over $B$, then $F$ clearly induces a homotopy of cross sections of $S\backslash Y \to B$. Conversely, since a bundle over $B \times \mathbf{I}$ is the product of some bundle over $B$ with $\mathbf{I}$ (when $B$ is paracompact), it is easily seen that any homotopy of cross sections is induced by such an $S$-equivariant homotopy $X \times \mathbf{I} \to Y$ over $B$; $B$ paracompact.

**3.2. Theorem** *Let $G$ be a compact group and let $H \subset G$ be a closed subgroup. Let $T = N(H)/H$ and $S \subset T$ a closed subgroup. Let $X \to B$ and $Y \to B$ be principal $S$ and $T$ bundles, respectively. To each reduction $f\colon X \to Y$ of $Y$ to $X$ we associate a $G$-equivariant map*

$$f'\colon (G/H) \times_S X \to (G/H) \times_T Y$$

*over $B$ by $f'[gH, x] = [gH, f(x)]$. This is a one–one correspondence of $S$-reductions of $Y$ to $X$ and $G$-equivariant maps over $B$ of the associated $G/H$-bundles.*

*Proof* Given $f'$ note that restriction to the fixed sets of $H$ gives a $T$-equivariant map $T \times_S X \to T \times_T Y$. The inclusion $X \to T \times_S X$ given by $x \mapsto [e, x]$ is $S$-equivariant and thus so is the composition $X \to T \times_S X \to T \times_T Y \approx Y$. ∎

Now let $A$ be a locally compact space which is an effective *right* $T$-space, where $T$ is an arbitrary subgroup of $\mathbf{Map}(A, A)$ with the compact-open topology. Let $S$ be a subgroup of $T$, let $X$ and $X'$ be principal $S$-bundles over $B$, and consider the associated $A$-bundles $A \times_S X$ and $A \times_S X'$. (Note that the structure group can be *extended* to $T$ since $A \times_S X \approx (A \times_T T) \times_S X \approx A \times_T (T \times_S X)$.) Consider a map $\varphi$ over $B$

$$\begin{array}{ccc} A \times_S X & \xrightarrow{\varphi} & A \times_S X' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

and recall, from II.2.7, that $\varphi$ corresponds to a cross section $\bar{\varphi}$ of the associated $\mathbf{Map}(A, A)$-bundle

$$\mathbf{Map}(A,A) \times_{S \times S} \Delta \to B.$$

We shall say that $\varphi$ is a $T$-**equivalence** if its associated section $\bar{\varphi}$ lies in the subbundle

$$T \times_{S \times S} \Delta \to B.$$

(Using charts coming from charts of $X$ and $X'$, one easily checks that this means precisely that $\varphi$ is fiberwise operation by elements of $T$.) Similarly, $\varphi$ is an $S$-equivalence if $\bar{\varphi}$ is a cross section of $S \times_{S \times S} \Delta \to B$.

*Remark* Since $T = \mathbf{Map}^T(T,T)$ it is easily checked that $\varphi$ is a $T$-equivalence iff it is induced from a $T$-equivariant map $T \times_S X \to T \times_S X'$ over

$B$. Similarly, it is an $S$-equivalence iff it comes from an $S$-equivariant map $X \to X'$ over $B$. It is clear that a $T$- (or $S$-) equivalence is invertible to a $T$- (or $S$-) equivalence.

The following result will be of primary importance to us in the remainder of this chapter.

**3.3. Theorem** *Let $T$ be a topological group and $S$ a closed subgroup such that $T \to S\backslash T$ is a bundle. Suppose that the inclusion $S \subset T$ is a weak homotopy equivalence. Also assume that $B$ is paracompact and of the homotopy type of a* CW-*complex. Then*:

(a) *Every $T$-bundle over $B$ has an $S$-reduction and any two such reductions are $S$-equivariantly homotopic to equivalent $S$-reductions.*

(b) *If $A$ is a right $T$-space and $X$ and $X'$ are principal $S$-bundles over $B$, then every $T$-equivalence $\varphi$: $A \times_S X \to A \times_S X'$ is homotopic (through $T$-equivalences) to an $S$-equivalence. Here we assume that $A$ is locally compact and that $T$ has the compact-open topology.*

*Proof* Suppose that $B$ is a CW-complex. Since $S\backslash T$ has trivial homotopy groups, there are no obstructions to constructing a section of an $(S\backslash T)$-bundle over $B$ and no obstructions to deforming one section to another. Thus (a) follows from 3.1 and the remarks following it. Similarly, there are no obstructions to deforming a section of $T \times_{S\times S} \Delta \to B$ to a section of $S \times_{S\times S} \Delta \to B$. Such a homotopy of sections corresponds to a homotopy of $T$-equivalences $A \times_S X \to A \times_S X'$, by Chapter II, Section 2, and this implies (b). The use of obstruction theory in these arguments is easily justified when $B$ has only the homotopy type of a CW-complex $C$ and is paracompact. For instance, let $h$: $C \to B$ be a homotopy equivalence with homotopy inverse $k$: $B \to C$ and let $W = S\backslash Y$ be the $S\backslash T$-bundle associated with a principal $T$-bundle $Y$ over $B$. Then $h^*W$ has a section and it follows that $k^*h^*W$ has a section. But $W \approx k^*h^*W$ since $hk \simeq 1$ and $B$ is paracompact. Similarly with the bundle pair $(T \times_{S\times S} \Delta, S \times_{S\times S} \Delta)$ over $B$. ▌

*Remark* The hypothesis that $T \to S\backslash T$ be a bundle is not used in (b). It could also be dropped from (a) by utilizing the theory of classifying spaces in the proof, but we shall not need this fact.

## 4. THE STRAIGHTENING LEMMA AND THE TUBE THEOREM

Let $G$ be a compact Lie group and let $H \subset K \subset G$ be closed subgroups. Consider the canonical projection $\pi$: $G/H \to G/K$ and let

$$M_\pi = (G/H) \times \mathbf{I} \bigcup_{\pi \times \{0\}} (G/K)$$

be the mapping cylinder of $\pi$. Then $M_\pi$ is a $G$-space with orbit space $M_\pi^*$ canonically homeomorphic to $\mathbf{I}$ via the projection $(G/H) \times \mathbf{I} \to \mathbf{I}$. We shall, in fact, identify $M_\pi^*$ with $\mathbf{I}$.

Let $\mathbf{Homeo}_\mathbf{I}{}^G(M_\pi)$ denote the group, under composition, of self-equivalences of $M_\pi$ *over* $\mathbf{I}$ (i.e., $G$-equivariant homeomorphisms inducing the identity on the orbit space). This is given the compact-open topology which, since $M_\pi$ is compact metric, is the same as the topology given by the uniform metric. It is clearly a topological group.

Recall that $N(H) \cap N(K)$ acts by right translation $[R_n(gH) = gn^{-1}H$ etc.] on $G/H$ and on $G/K$, and the diagram

$$\begin{array}{ccc} G/H & \xrightarrow{\pi} & G/K \\ {\scriptstyle R_n}\downarrow & & \downarrow{\scriptstyle R_n} \\ G/H & \xrightarrow{\pi} & G/K \end{array}$$

clearly commutes. Thus there is an induced action of $N(H) \cap N(K)$ on $M_\pi$ whose kernel is just $H$. This gives a *monomorphism*

$$\frac{N(H) \cap N(K)}{H} \to \mathbf{Homeo}_\mathbf{I}{}^G(M_\pi)$$

which we shall regard as *inclusion*.

The next result is basic for the remainder of this chapter. The reason for its name will become apparent later.

**4.1. The Straightening Lemma** *The subgroup* $[N(H) \cap N(K)]/H$ *is a strong deformation retract of* $\mathbf{Homeo}_\mathbf{I}{}^G(M_\pi)$.

*Proof* First, let us interpret the group $\mathbf{Homeo}_\mathbf{I}{}^G(M_\pi)$ in an instructive geometric way. Note that the $G$-space $M_\pi$ has a cross section $J$ given by the identity cosets; that is,

$$J = (\{eH\} \times (0,1]) \cup \{eK\}.$$

Let $J_t$ denote the point of the cross section corresponding to $t \in \mathbf{I} = M_\pi{}^*$; that is, $J_t = (eH, t)$ for $t \neq 0$ and $J_0 = eK$. If $\varphi \in \mathbf{Homeo_I}^G(M_\pi)$, then $\varphi(J)$ is another cross section and, since $\varphi$ preserves isotropy groups, we see that

$$\varphi(J)_t = \varphi(J_t) \in \begin{cases} \dfrac{N(H)}{H} \times \{t\} & \text{for} \quad t \neq 0, \\ \dfrac{N(K)}{K} & \text{for} \quad t = 0. \end{cases}$$

Conversely, any such cross section gives rise to a self-equivalence by I.3.4. Moreover the correspondence between the self-equivalences $\varphi$ and the cross sections $\varphi(J)$ is clearly bicontinuous, where the space of cross sections is given the obvious uniform metric. To simplify notation we shall now put $\varphi(t) = \varphi(J_t)$ and we display its coordinates by putting $\varphi(t) = (\varphi_t, t)$ *for* $t \neq 0$, where $\varphi_t \in N(H)/H$.

Let $Q$ denote the subspace

$$Q = \left(\frac{N(H)}{H} \times (0, 1]\right) \cup \frac{N(K)}{K} \subset M_\pi,$$

which is generally noncompact. Put $Q_0 = N(K)/K \subset Q$. We may regard $\mathbf{Homeo_I}^G(M_\pi)$ as a subspace of the function space

$$(Q, Q_0)^{(\mathbf{I},0)} \subset (M_\pi, G/K)^{(\mathbf{I},0)}.$$

Note that

$$\frac{N(H) \cap N(K)}{H} = \frac{N(H)}{H} \cap \pi^{-1}\left(\frac{N(K)}{K}\right).$$

Now $[N(H) \cap N(K)]/H$ has a tubular neighborhood $U$ in $N(H)/H$ (a linear tube about this orbit by left translation of this subgroup on $N(H)/H$). Clearly there exists a smaller tubular neighborhood $V$ and a deformation

$$F\colon \frac{N(H)}{H} \times \mathbf{I} \to \frac{N(H)}{H}$$

such that

$$\begin{aligned} &F(x, 0) = x, \\ &F(x, t) = x && \text{for} \quad x \notin U, \\ &F(x, t) = x && \text{for} \quad x \in \frac{N(H) \cap N(K)}{H}, \\ &F(v, 1) \in \frac{N(H) \cap N(K)}{H} && \text{for} \quad v \in V. \end{aligned}$$

Define the deformation

$$D\colon Q \times \mathbf{I} \to Q$$

by

$$D\big((x, s), t\big) = \big(F(x, t), s\big) \quad \text{for} \quad x \in N(H)/H \quad \text{and} \quad s \neq 0,$$
$$D(y, t) = y \qquad \text{for} \quad y \in N(K)/K.$$

Then $D$ induces a deformation

$$D'\colon (Q, Q_0)^{(\mathbf{I},0)} \times \mathbf{I} \to (Q, Q_0)^{(\mathbf{I},0)}$$

in the function space by $D'(\psi, t)(s) = D\big(\psi(s), t\big)$. Clearly this *restricts* to a deformation

$$D_1\colon \mathbf{Homeo_I}^G(M_\pi) \times \mathbf{I} \to \mathbf{Homeo_I}^G(M_\pi).$$

Now for $\varphi \in \mathbf{Homeo_I}^G(M_\pi)$ we put

$$g(\varphi) = \sup\{s > 0 \mid 0 < t < s \Rightarrow \varphi_t \in V\}.$$

Then $g$ is clearly a lower semicontinuous, positive, real-valued function on the metric space $\mathbf{Homeo_I}^G(M_\pi)$. By a theorem of Dowker (see Dugundji [1, p. 170]), there is a *continuous* function $f$ such that

$$0 < f(\varphi) < g(\varphi)$$

for all $\varphi$. In particular, note that

$$D_1(\varphi, 1)(s) = D\big(\varphi(s), 1\big) = \big(F(\varphi_s, 1), s\big) \in \frac{N(H) \cap N(K)}{H} \times \{s\}$$

for $0 < s \le f(\varphi)$.

We shall multiply elements $x \in M_\pi$ by numbers $r \in \mathbf{I}$ by simply multiplying the $\mathbf{I}$ parameter by $r$, meaning the canonical projection $M_\pi \to G/K \subset M_\pi$ when $r = 0$. Note that this (for $r = 0$) does *not* restrict to $Q$ and this is basically the reason for the first deformation $D_1$. With this notation we define another deformation

$$D_2\colon \mathbf{Homeo_I}^G(M_\pi) \times \mathbf{I} \to \mathbf{Homeo_I}^G(M_\pi)$$

by

$$D_2(\varphi, t)(s) = \begin{cases} D_1(\varphi, 1)(s) & \text{for} \quad s \ge f(\varphi)t, \\ \dfrac{s}{f(\varphi)t}\, D_1(\varphi, 1)\big(f(\varphi)t\big) & \text{for} \quad s \le f(\varphi)t. \end{cases}$$

Then

$$D_2(\varphi, 0) \;\; = D_1(\varphi, 1),$$
$$D_2(\varphi, 1)(s) = \frac{s}{f(\varphi)} D_1(\varphi, 1)\big(f(\varphi)\big) \qquad \text{for} \quad s \leq f(\varphi).$$

The latter equation means that the $N(H)/H$ coordinate is constant, and is in $[N(H) \cap N(K)]/H$, for *small* $s$. Because of this, we can follow this by the "Alexander deformation":

$$D_3\colon \mathbf{Homeo_I}^G(M_\pi) \times \mathbf{I} \to \mathbf{Homeo_I}^G(M_\pi)$$

given by

$$D_3(\varphi, t)(s) = \begin{cases} \dfrac{1}{1-t} D_2(\varphi, 1)\big((1-t)s\big) & \text{for} \quad t \neq 1, \\[2ex] \dfrac{s}{f(\varphi)} D_2(\varphi, 1)\big(f(\varphi)\big) & \text{for} \quad t = 1. \end{cases}$$

Then

$$D_3(\varphi, 0) \;\; = D_2(\varphi, 1),$$
$$D_3(\varphi, 1)(s) = \big(n(\varphi), s\big) \qquad \text{where} \quad n(\varphi) \in \frac{N(H) \cap N(K)}{H}$$

is independent of $s$.

The latter fact clearly means that $D_3(\varphi, 1) \in [N(H) \cap N(K)]/H$ regarded as a subgroup of $\mathbf{Homeo_I}^G(M_\pi)$. Thus $D_1$, $D_2$, and $D_3$ patch together to give the desired strong deformation retraction of $\mathbf{Homeo_I}^G(M_\pi)$ onto $[N(H) \cap N(K)]/H$. ▌

Although we have been taking $H$ to be a subgroup of $K$ and $\pi\colon G/H \to G/K$ the canonical projection, it will be convenient for the statement, and applications, of the next theorem to free ourselves of this restriction. Thus suppose now that

$$\pi\colon G/H \to G/K$$

is *any* $G$-equivariant map with arbitrary $H$ and $K$. Then, by I.4.2, $\pi$ is of the form $\pi = R_a^{K,H}$ for some $a \in G$ determined up to its coset $Ka$. Thus with $K' = a^{-1}Ka \supset H$ and $\pi' = R_e^{K',H}$ (the canonical projection) there is the commutative diagram

$$\begin{array}{ccc} & & G/K' \\ & \overset{\pi'}{\nearrow} & \Big\downarrow {\scriptstyle R_a^{K,K'}} \\ G/H & & \\ & \underset{\pi}{\searrow} & \\ & & G/K \end{array}$$

This induces a $G$-equivariant homeomorphism $M_{\pi'} \xrightarrow{\approx} M_\pi$ over $\mathbf{I}$ and hence yields an isomorphism

$$\mathbf{Homeo}_{\mathbf{I}}{}^G(M_{\pi'}) \xrightarrow{\approx} \mathbf{Homeo}_{\mathbf{I}}{}^G(M_\pi).$$

This carries the subgroup $[N(H) \cap N(K')]/H$ onto a subgroup

$$S(\pi) \subset \mathbf{Homeo}_{\mathbf{I}}{}^G(M_\pi)$$

which is a deformation retract. In fact, it is clear that $S(\pi)$ may be regarded as

$$S(\pi) = \frac{N(H) \cap N(K')}{H} = \frac{N(H) \cap a^{-1}N(K)a}{H}$$

which acts on $M_\pi$ via the commutative diagram

$$\begin{array}{ccc} G/H & \xrightarrow{\pi=R_a} & G/K \\ {\scriptstyle R_n}\downarrow & & \downarrow{\scriptstyle R_{ana^{-1}}} \\ G/H & \xrightarrow{\pi=R_a} & G/K \end{array}$$

where $n \in N(H) \cap a^{-1}N(K)a$.

**4.2. Theorem** (The Tube Theorem) *Let $G$ be a compact Lie group and let $W$ be a $G$-space with orbit space $\mathbf{I} \times B$, where $B$ is connected, locally connected, paracompact, and of the homotopy type of a* CW-*complex. Suppose that the orbit type on $\{0\} \times B$ is* type$(G/K)$ *and that on $(0,1] \times B$ is* type$(G/H)$. *Then there exists an equivariant map $\pi\colon G/H \to G/K$ and, with $S = S(\pi)$, there exists a principal $S$-bundle $X \to B$ (unique to equivalence) and a $G$-equivariant homeomorphism*

$$M_\pi \times_S X \xrightarrow{\approx} W$$

*commuting with the canonical projections to $\mathbf{I} \times B$. Moreover, the map $\varphi = \pi \times_S X\colon G/H \times_S X \to G/K \times_S X$ gives rise to a $G$-equivariant homeomorphism*

$$f\colon M_\varphi \xrightarrow{\approx} M_\pi \times_S X \approx W$$

*over $\mathbf{I} \times B$. (Thus, if $W_0$ and $W_1$ are the subspaces of $W$ lying over $\{0\} \times B$ and $\{1\} \times B$, respectively there is an equivariant map $\psi\colon W_1 \to W_0$ and an equivalence $M_\psi \xrightarrow{\approx} W$ of $G$-spaces over $\mathbf{I} \times B$ extending the identity on $W_0$ and $W_1$.)*

*Proof* In the proof it will be convenient to choose $H \subset K$ in a special way so that $\pi$ can be taken to be the canonical projection. To do this, let $b \in B$ and let $w \in W$ be a point projecting to $(0,b) \in \mathbf{I} \times B = W^*$. Assume that $G_w = K$ and let $A$ be a slice at $w$ in $W$. Choose $H$ to be an isotropy group at a point $a \in A$. [Note that since $w$ can be altered by action of any element of $N(K)$ we are free to alter $H$ by conjugating it by an element of $N(K)$.] If $a'$ is near $a$ in $A$, then $K_{a'}$ is conjugate in $K$ to a subgroup of $K_a = H$, but since $K_{a'} \sim H$ in $G$ it follows that $K_{a'} \sim H$ in $K$. This shows that $H$ "works" for all points in $B$ near $b$. Since $B$ is connected, the point $b \in B$ is irrelevant to the choice of $H$. We now fix such $H \subset K$ and let $\pi$: $G/H \to G/K$ be the canonical projection.

Now we may choose the slice $A$ to lie over a set of the form $[0, 2\varepsilon) \times U \subset \mathbf{I} \times B$, where $U$ is a connected neighborhood of $b$ in $B$. Over $(0, 2\varepsilon) \times U$ the $K$-orbit type is locally constant (as noted above) and hence it is constant by connectivity. Thus the part of $A$ over $(0, 2\varepsilon) \times U$ is a $K/H$-bundle. Now $U$ may be taken to be so small that this bundle is trivial over $\{\varepsilon\} \times U$, and then, by the Covering Homotopy Theorem, it is trivial over $(0, 2\varepsilon) \times U$. That is, there is a cross section of this bundle in $A$ with each point having isotropy group $H$. Adding the part of $A$ over $\{0\} \times U$ (which is just $A^K$) to this, we obtain a cross section of $W$ over $[0, 2\varepsilon) \times U$ such that each point has isotropy group $H$ or $K$. Restricting this cross section to $[0, \varepsilon] \times U$ and applying the Covering Homotopy Theorem to the part of $W$ over $[\varepsilon, 1] \times U$ we finally obtain a cross section $C$ of $W$ over $\mathbf{I} \times U$ such that points of $C$ have isotropy groups $K$ or $H$.

Consider the composition

$$p\colon W \to \mathbf{I} \times B \to B.$$

Now the $G$-space $M_\pi \times U$ over $\mathbf{I} \times U$ clearly has the same type of cross section as that constructed above for $p^{-1}U$. By I.3.4 the canonical map between these cross sections yields an equivalence $M_\pi \times U \approx p^{-1}U$ which commutes with the projections to $\mathbf{I} \times U$ (as well as to $U$).

This shows that $p$: $W \to B$ is a *bundle* with fiber $M_\pi$ and *structure group*

$$T = \mathbf{Homeo}_{\mathbf{I}}{}^G(M_\pi).$$

Let $Y \to B$ be the associated principal $T$-bundle, so that $W \approx M_\pi \times_T Y$ with the projection to $\mathbf{I} \times B$ being induced by the canonical map $M_\pi \to \mathbf{I}$ (i.e., $M_\pi \times_T Y \to \mathbf{I} \times_T Y \approx \mathbf{I} \times B$).

By 3.3 and the Straightening Lemma 4.1, there is an $S$-reduction $X \to Y$ where

$$S = \frac{N(H) \cap N(K)}{H}$$

and $X \to B$ is a principal $S$-bundle. Then we have the induced $G$-equivariant homeomorphism

$$M_\pi \times_S X \to M_\pi \times_T Y \approx W$$

which clearly commutes with the projection to $\mathbf{I} \times B$.

To prove the remainder of the theorem, note that the group $S$ *preserves the mapping cylinder structure of* $M_\pi$ (which is the whole point). In particular, the $S$-action commutes with $\pi$. Thus $\pi \times X$: $G/H \times X \to G/K \times X$ is $S$-equivariant and induces

$$\varphi = \pi \times_S X\colon\ G/H \times_S X \to G/K \times_S X.$$

Now we can define a $G$-equivariant function

$$f\colon\ M_\varphi \to M_\pi \times_S X$$

by $f([gH, x], t) = [(gH, t), x]$ for $t \neq 0$ and $f([gK, x]) = [gK, x]$ for $t = 0$. Over a small open set $U \subset B$ this is just the obvious function taking the mapping cylinder $M_{\pi \times U}$ (of the map $\pi \times U$: $G/H \times U \to G/K \times U$) to $M_\pi \times U$. Now if we put

$$P = (G/H \times [0,1]) + (G/K)$$

(disjoint union), then $M_\pi$ has the identification topology from the canonical map $P \to M_\pi$. Now $M_{\pi \times U}$ is just $M_\pi \times U$ as a *set*, but with the identification topology from the map $P \times U \to M_\pi \times U$. However, these spaces are Hausdorff and $P$ is compact, from which it is easily deduced that $P \times U \to M_\pi \times U$ is a *closed* map. It follows that $M_\pi \times U$ *has* the identification topology from $P \times U \to M_\pi \times U$. Thus $M_{\pi \times U} \approx M_\pi \times U$ and it follows that $f$ is a homeomorphism. ∎

Clearly the map $\pi$: $G/H \to G/K$ of 4.2 is completely determined up to equivalence (i.e., up to composition with equivalences of the domain and target) by the $G$-space $W$. It is natural to ask to what extent $\pi$ is determined by only the orbit types of $G/H$ and $G/K$. In many cases of interest it turns out that there is precisely one such map up to equivalence. The following result gives some useful elementary information about this.

**4.3. Proposition** *Let $H$ and $K$ be closed subgroups of the compact Lie group $G$. There is a natural one–one correspondence between the equivalence classes of equivariant maps $G/H \to G/K$ and orbits of the action of $N(H)/H \times N(K)/K$ on $(G/K)^H$. These are also in one–one correspondence with the simultaneous conjugacy classes in $G$ of pairs $(H', K')$, where $H' \subset K'$ with $H'$ conjugate to $H$ and $K'$ conjugate to $K$.*

*If $H \subset K$, then there is precisely one such class iff*

$$(G/K)^H = \frac{N(H)\,N(K)}{K}.$$

*In particular, this holds when $N(H)$ is transitive on $(G/K)^H$ and this always holds when $(G/K)^H$ is connected.*

*Proof* Recall that $R_b^{K,H} \mapsto b^{-1}K$ is a one–one correspondence between $\mathbf{Map}^G(G/H, G/K)$ and $(G/K)^H$ (see Chapter I, Exercise 12). By composition $N(H)/H = \mathbf{Map}^G(G/H, G/H)$ acts on the right of $\mathbf{Map}^G(G/H, G/K)$ and $N(K)/K$ acts on the left. Since

$$R_a^{K,K} R_b^{K,H} R_c^{H,H} = R_{abc}^{K,H} \mapsto c^{-1}b^{-1}a^{-1}K$$

this $N(K)/K \times N(H)/H$-action on $\mathbf{Map}^G(G/H, G/K)$ corresponds to the canonical action of $N(H)/H \times N(K)/K$ on $(G/K)^H$ (where $N(H)/H$ acts on the *left* and $N(K)/K$ acts on the *right*). This proves the first statement. If $H \subset K$, then the orbit of $eK$ under this action on $(G/K)^H$ is just $N(H)N(K)/K$ so that the first part of the second paragraph holds. If $(G/K)^H$ is connected, then $N(H)$ is transitive on it since $N(H)$ always has only finitely many orbits on $(G/K)^H$ by II.5.7.

For the correspondence with pairs $(H', K')$ we associate to such a pair the class of the canonical projection $R_e^{K',H'}\colon G/H' \to G/K'$. By I.4.2 we obtain every map $G/H \to G/K$, up to equivalence, this way. If $H'' = aH'a^{-1}$ and $K'' = aK'a^{-1}$, then

$$\begin{array}{ccc} G/H' & \xrightarrow{R_e} & G/K' \\ {\scriptstyle R_a}\downarrow & & \downarrow{\scriptstyle R_a} \\ G/H'' & \xrightarrow{R_e} & G/K'' \end{array}$$

commutes so that the two projections $R_e$ are equivalent. Conversely, if

$(H'', K'')$ and $(H', K')$ are pairs giving equivalent maps, then there exist elements $a$ and $b$ in $G$ such that

$$\begin{array}{ccc} G/H' & \xrightarrow{R_e} & G/K' \\ {\scriptstyle R_a}\downarrow & & \downarrow{\scriptstyle R_b} \\ G/H'' & \xrightarrow{R_e} & G/K'' \end{array}$$

commutes. However, then $R_a^{K'',H'} = R_b^{K'',H'}$ which means that $b = ka$ for some $k \in K''$ by I.4.2. Thus $H' = a^{-1}H''a$ and $K' = b^{-1}K''b = a^{-1}k^{-1}K''ka = a^{-1}K''a$ so that $(H', K')$ and $(H'', K'')$ are simultaneously conjugate. ∎

The case of main interest to us later is that for which $G \subset \mathbf{O}(n)$, $K = G \cap \mathbf{O}(n-1)$, $H = G \cap \mathbf{O}(n-2)$ and for which $G$ is transitive on $\mathbf{S}^{n-1}$. In this case we have $(G/K)^H = (\mathbf{S}^{n-1})^H$ which is a sphere containing $(\mathbf{S}^{n-1})^{\mathbf{O}(n-2)} = \mathbf{S}^1$. Thus $(G/K)^H$ is connected and there is just one equivariant map $G/H \to G/K$ up to equivalence in this case.

## 5. CLASSIFICATION OF ACTIONS WITH TWO ORBIT TYPES

Let $X$ be a given (paracompact) manifold with boundary $B$. Let $G$ be a compact Lie group and $H \subset K$ closed subgroups of $G$. Let $X$ have the orbit structure which assigns type$(G/K)$ to $B$ and type$(G/H)$ to $X - B$. In this section we shall classify $G$-spaces $W$ *over* $X$; that is, $G$-spaces $W$ together with an orbit structure preserving homeomorphism $W/G \xrightarrow{\approx} X$. Of course, we wish to classify such actions up to equivalence *over* $X$.

Note that in the present context $W$ need not be a manifold. Even though the case of locally smooth actions on manifolds $W$ is our main concern here, such an assumption does not simplify the discussion. (See the remarks in Example 5.2.)

By the Collaring Theorem, we can regard $X$ as the union $X_2 \cup ([0, 2] \times B)$ with intersection $\{2\} \times B$ and with $\{0\} \times B$ identified with $B$. Let $X_1 = X_2 \cup ([1,2] \times B)$ so that $X = X_1 \cup ([0,1] \times B)$. (The interval $[1,2]$ will be used later to support a homotopy.) Put $B_1 = \{1\} \times B$.

Let $W$ be a $G$-space over $X$ and $p\colon W \to X$ the given projection. Then $p^{-1}B \to B$ is a $G/K$-bundle with structure group $N(K)/K$. We shall denote this bundle by $\sigma$ (for "singular"). Similarly $p^{-1}X_1 \to X_1$ is a $G/H$-bundle with structure group $N(H)/H$ and will be denoted by $\varrho$ (for "regular"). By the Tube Theorem 4.2, there is an equivariant map $\psi\colon p^{-1}(B_1) \to p^{-1}(B)$ over $B$ and an equivalence $M_\psi \approx p^{-1}(\mathbf{I} \times B)$ over $\mathbf{I} \times B$, which restricts to the canonical identification of the top of $M_\psi$ with $p^{-1}(B_1)$ (and similarly for the bottom). Thus $W$ is equivalent over $X$ to the union of the total space $E(\varrho) = p^{-1}(X_1)$ of $\varrho$ and the mapping cylinder $M_\psi$ of the equivariant map $\psi$: $E(\varrho \mid B_1) = p^{-1}B_1 \to p^{-1}B = E(\sigma)$ over $B$. (In this section we use $E(\sigma)$ to denote the total space of $\sigma$, etc.)

Therefore every such $G$-space $W$ over $X$ can be constructed, to equivalence over $X$, as follows: We are given a $G/H$-bundle $\varrho$ over $X_1$, a $G/K$-bundle $\sigma$ over $B$, and an equivariant map $\psi$: $E(\varrho \mid B_1) \to E(\sigma)$ over $B$. Then construct

$$W = W(\varrho, \sigma, \psi) = E(\varrho) \cup M_\psi,$$

where the union is via the canonical identification of $E(\varrho \mid B_1) \subset E(\varrho)$ with the top of the mapping cylinder $M_\psi$.

Let $\mathbf{Map}^G(\varrho \mid B_1, \sigma)$ denote the set of all equivariant maps $\psi$: $E(\varrho \mid B_1) \to E(\sigma)$ *over* $B$. By II.2.8, since $\mathbf{Map}^G(G/H, G/K) \approx (G/K)^H$, there is a bundle ($\mu$ for "map")

$$\mu = \mathscr{M}\!ap^G(\varrho \mid B_1, \sigma)$$

over $B$ (depending on $\varrho \mid B_1$ and $\sigma$) with fiber $(G/K)^H$ and structure group $N(H)/H \times N(K)/K$ (made effective) and a canonical one–one correspondence

$$\mathbf{Map}^G(\varrho \mid B_1, \sigma) \leftrightarrow \Gamma(\mu)$$

of equivariant maps over $B$ with cross sections of $\mu$. Also, by the remarks following II.2.8, equivariant homotopies of maps correspond to homotopies of sections. Thus the set

$$\pi_0\Gamma(\mu)$$

of homotopy classes of sections of $\mu$ corresponds to the set of equivariant homotopy classes of maps $E(\varrho \mid B_1) \to E(\sigma)$ over $B$.

Thus, given $\sigma, \varrho$ (which determine $\mu$) and $\psi \in \Gamma(\mu)$, we have shown how to construct a $G$-space $W(\varrho, \sigma, \psi)$ over $X$. Let us show now that $W(\varrho, \sigma, \psi)$ depends, to equivalence, only on the homotopy class $[\psi] \in \pi_0\Gamma(\mu)$ of $\psi$.

Suppose that $\psi_0$ and $\psi_1$ are equivariantly homotopic maps, via the homotopy

$$F\colon E(\varrho \mid B_1) \times \mathbf{I} \to E(\sigma)$$

over $B$. Put $F'(x, t) = \big(F(x, t), t\big)$ so that

$$F'\colon E(\varrho \mid B_1) \times \mathbf{I} \to E(\sigma) \times \mathbf{I}$$

is an equivariant map over $B \times \mathbf{I}$. Put

$$Y = \big(E(\varrho) \times \mathbf{I}\big) \cup M_{F'}$$

which is a $G$-space over $X \times \mathbf{I}$ with orbit structure induced by the projection to $X$. However, the part of $Y$ over $X \times \{0\}$ is $W(\varrho, \sigma, \psi_0)$ and that over $X \times \{1\}$ is $W(\varrho, \sigma, \psi_1)$. Thus these are equivalent over $X$ by the Covering Homotopy Theorem of Palais in the form of II.7.1.

There are some other obvious redundancies in the data $(\varrho, \sigma, \psi)$. Indeed, suppose that $\varphi$: $E(\varrho) \to E(\varrho)$ is a self-equivalence over $X_1$ and that $\theta$: $E(\sigma) \to E(\sigma)$ is a self-equivalence over $B$. Let $\varphi'$ be the restriction of $\varphi$ to $E(\varrho \mid B_1)$. Then the commutative diagram

$$\begin{array}{ccc} E(\varrho \mid B_1) & \xrightarrow{\varphi'\psi\theta} & E(\sigma) \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\theta^{-1}} \\ E(\varrho \mid B_1) & \xrightarrow{\psi} & E(\sigma) \end{array}$$

gives rise to an equivalence $M_{\varphi'\psi\theta} \xrightarrow{\approx} M_\psi$. The union of this with the equivalence $\varphi$: $E(\varrho) \to E(\varrho)$ produces an equivalence

$$W(\varrho, \sigma, \varphi'\psi\theta) \xrightarrow{\approx} W(\varrho, \sigma, \psi)$$

of $G$-spaces over $X$.

*Note.* Here we are using the convention that the composition $\varphi'\psi\theta$ acts in the left to right order (as if we write these as maps on the right). This is contrary to our usual convention and we apologize to any traditionalist readers for it. The reason for it is that equivariant maps $G/H \to G/K$, etc., are *right* translations. If we did not use this convention, it would tend to confuse our later remarks.

Thus we consider the composition action

$$\Theta: \mathbf{Map}^G(\varrho, \varrho) \times \mathbf{Map}^G(\varrho \mid B_1, \sigma) \times \mathbf{Map}^G(\sigma, \sigma) \to \mathbf{Map}^G(\varrho \mid B_1, \sigma).$$

Thinking of equivariant maps as sections of bundles via II.2.8, this is

$$\Theta: \Gamma\big(\mathscr{M}\!\mathit{ap}^G(\varrho, \varrho)\big) \times \Gamma(\mu) \times \Gamma\big(\mathscr{M}\!\mathit{ap}^G(\sigma, \sigma)\big) \to \Gamma(\mu),$$

and, fiberwise, $\Theta$ corresponds to the usual action

$$\frac{N(H)}{H} \times (G/K)^H \times \frac{N(K)}{K} \to (G/K)^H.$$

This action on $\Gamma(\mu)$ induces one on $\pi_0\Gamma(\mu)$, and, since operating on a map by homotopic maps gives homotopic maps, this is essentially an action

$$\Theta_0 : \pi_0\Gamma(\mathcal{M}ap^G(\varrho, \varrho)) \times \pi_0\Gamma(\mu) \times \pi_0\Gamma(\mathcal{M}ap^G(\sigma, \sigma)) \to \pi_0\Gamma(\mu).$$

For $[\psi] \in \pi_0\Gamma(\mu)$, the homotopy class of $\psi \in \Gamma(\mu)$, let $[\psi]^*$ denote its orbit under the action $\Theta_0$.

We have shown that $W(\varrho, \sigma, \psi)$ depends, up to equivalence over $X$, only on $\varrho$, $\sigma$, and $[\psi]^*$. Thus we let

$$W(\varrho, \sigma, [\psi]^*)$$

denote this *equivalence class over $X$* of $G$-spaces.

***Remark*** It is clear, of course, that a change of $\varrho$ and $\sigma$ by equivalences produces canonical changes in $\mu$ and the action $\Theta_0$. Thus the dependence of $W(\varrho, \sigma, [\psi]^*)$ on $\varrho$ and $\sigma$ is actually only on the ($G$-equivariant) equivalence classes $[\varrho]$ and $[\sigma]$ of $\varrho$ and $\sigma$ over $X_1$ and $B$, respectively. Also, since the principal bundle associated with the $G/H$-bundle $\varrho$ is retrieved simply by passing to the fixed point set of $H$ on $E(\varrho)$ (by II.5.11), we may regard $[\varrho]$ as the equivalence class of this associated principal $N(H)/H$-bundle or, by classification, as a homotopy class

$$[\varrho] \in [X_1, B_{N(H)/H}],$$

where $B_{N(H)/H}$ is a classifying space for $N(H)/H$. (Also note that $X_1$ could be replaced by $X$ here, if we wish, and $\varrho$ may be regarded as a bundle over $X$.) We may similarly regard

$$[\sigma] \in [B, B_{N(K)/K}].$$

**5.1. Theorem** (First Classification Theorem) *For given regular and singular bundles $\varrho$ and $\sigma$, the assignment $[\psi]^* \mapsto W(\varrho, \sigma, [\psi]^*)$ is a one–one correspondence between the set of orbits of the action $\Theta_0$ on $\pi_0\Gamma(\mu)$ [where $\mu = \mathcal{M}ap^G(\varrho \mid B_1, \sigma)$] and the set of equivalence classes over $X$ of $G$-spaces over $X$ with these regular and singular bundles.*

***Proof*** First let us remark that it may very well happen that for given $\varrho$, $\sigma$ there are no sections of $\mu$. This simply means that $\varrho$, $\sigma$ are inconsistent in the sense that there exist no $G$-spaces over $X$ with these particular regular and singular bundles.

We have already shown that the indicated assignment is well defined and onto. Thus we must show that if $W(\varrho, \sigma, \psi)$ and $W(\varrho, \sigma, \varphi)$ are equivalent over $X$, then $[\psi]^* = [\varphi]^*$. Thus we are given an equivariant map

$$f\colon E(\varrho) \cup M_\psi \to E(\varrho) \cup M_\varphi$$

over $X$ (necessarily an equivalence of $G$-spaces), where

$$\varphi, \psi\colon E(\varrho \mid B_1) \to E(\sigma)$$

are $G$-equivariant maps over $B$.

If $b \in B$, then the part of $M_\varphi$ above $\mathbf{I} \times \{b\}$ is equivalent as a $G$-space over $\mathbf{I}$ to a mapping cylinder $M_\pi$, where $\pi\colon G/H \to G/K$ is some equivariant map. (In fact, by conjugating $H$ and $K$ (separately) it could be assumed that $\pi$ is the canonical projection.) Since $f$ is an equivalence of this to the part of $M_\varphi$ over $\mathbf{I} \times \{b\}$, the latter is also equivalent to $M_\pi$. Thus $M_\psi$ and $M_\varphi$ may both be regarded as $M_\pi$-bundles over $B$ with structure group $S = S(\pi)$ in the notation of Section 4. (Thus $S = [N(H) \cap N(K)]/H$ if we arrange $\pi$ to be the projection.)

Since the restriction $f'\colon M_\psi \to M_\varphi$ of $f$ is an equivalence of $G$-spaces over $\mathbf{I} \times B$, it is a $T$-equivalence in the terminology of Section 3, where $T = \mathbf{Homeo}_{\mathbf{I}}{}^G(M_\pi)$. By 3.3 and 4.1, $f'$ is homotopic *through* $T$-equivalences to an $S$-equivalence $f_1'\colon M_\psi \to M_\varphi$. Let

$$F\colon \mathbf{I} \times M_\psi \to M_\varphi$$

be this homotopy, so that $F(0, x) = f'(x)$ and $F(1, x) = f_1'(x)$. Now a $T$-equivalence is necessarily $G$-equivariant, so that $F$ is an *equivariant* homotopy. Now $\varrho \mid ([1,2] \times B)$ is equivalent to $[1,2] \times (\varrho \mid B_1)$. Using this parametrization, and regarding $\varrho \mid B_1$ as the top of $M_\psi$, we define

$$F'\colon [1,2] \times E(\varrho \mid B_1) \to [1,2] \times E(\varrho \mid B_1)$$

by

$$F'(t, x) = \big(t, F(2 - t, x)\big).$$

Then $F'$ can be regarded as an equivalence of $E(\varrho \mid [1,2] \times B)$ with itself over $[1,2] \times B$ which equals $f$ $(= f')$ over $\{2\} \times B$ and equals $f_1'$ over $\{1\} \times B$. Thus $f$ over $X - X_2$, $F'$ over $[1,2] \times B$, and $f_1'$ over $[0,1] \times B$ match together to define an equivalence

$$E(\varrho) \cup M_\psi \xrightarrow{\approx} E(\varrho) \cup M_\varphi$$

*which extends* the $S$-equivalence $f_1'\colon M_\psi \to M_\varphi$.

This shows that we may as well *assume* that the restriction $f'\colon M_\psi \to M_\varphi$ of $f$ is an $S$-equivalence. However, this simply means that $f'$ is the map of mapping cylinders induced by a commutative diagram

$$\begin{array}{ccc} E(\varrho \mid B_1) & \xrightarrow{\psi} & E(\sigma) \\ {\scriptstyle f\mid B_1}\downarrow & & \downarrow{\scriptstyle f\mid B} \\ E(\varrho \mid B_1) & \xrightarrow{\varphi} & E(\sigma) \end{array}$$

(That is, the commutativity of this is *implied* by the fact that $f'$ is an $S$-equivalence, the point being that $S$ preserves the mapping cylinder structure of the fibers.) This shows that $\varphi$ and $\psi$ are in the same orbit of the action $\Theta$ and, in particular, that $[\varphi]^* = [\psi]^*$. ∎

*Remark* Note that the proof shows that if $\varphi$, $\psi$: $E(\varrho \mid B_1) \to E(\sigma)$ are equivariantly homotopic (and hence lead to equivalent $G$-spaces), then they are, in fact, in the same orbit of $\Theta$. That is, $\varphi$ differs from $\psi$ only by composition with self-equivalences of $E(\varrho)$ and $E(\sigma)$. In fact, one can show that equivariantly homotopic $\varphi$ and $\psi$ differ only by composition with a self-equivalence of $E(\varrho)$. This is due to the fact (II.5.7) that each component of $(G/K)^H$ is contained in an $N(H)/H$-orbit.

We shall now apply 5.1 to some specific examples.

**5.2. Example** Let $(G, K, H) = (\mathbf{U}(n), \mathbf{U}(n-1), \mathbf{U}(n-2))$. Note that the diagonal action of $\mathbf{U}(n)$ on $\mathbf{S}^{4n-1} \subset \mathbf{C}^n \times \mathbf{C}^n$ has the given orbit types and has orbit space $\mathbf{D}^3$ with the boundary $\mathbf{S}^2$ consisting exactly of the singular orbits $\mathbf{U}(n)/\mathbf{U}(n-1) \approx \mathbf{S}^{2n-1}$ (which the reader may verify; compare Chapter I, Section 7).

Now consider any $G$-space $W$ over $\mathbf{D}^3$ with this orbit structure. By 4.3 there is precisely one equivariant map $\pi\colon \mathbf{U}(n)/\mathbf{U}(n-2) \to \mathbf{U}(n)/\mathbf{U}(n-1)$ up to equivalence. If $\pi$ is the canonical projection, then $M_\pi$ is a *linear* (closed) tube about its orbit $\mathbf{U}(n)/\mathbf{U}(n-1)$. Thus it follows that *any* such $G$-space $W$ is a $(4n-1)$*-manifold with locally smooth $G$-action.* Clearly $N(H)/H \approx \mathbf{U}(2)$ and $N(K)/K \approx \mathbf{U}(1) \approx \mathbf{S}^1$. Now there is exactly one element $[\varrho] \in [\mathbf{D}^3, B_{\mathbf{U}(2)}]$. Also

$$[\sigma] \in [\mathbf{S}^2, B_{\mathbf{U}(1)}] \approx H^2(\mathbf{S}^2;\ \pi_1(\mathbf{S}^1)) \approx H^2(\mathbf{S}^2;\ \mathbf{Z}) \approx \mathbf{Z}$$

is characterized by the Chern class $c_1(\sigma) = [\sigma] \in H^2(\mathbf{S}^2; \mathbf{Z}) \approx \mathbf{Z}$.

Since $(\mathbf{U}(n)/\mathbf{U}(n-1))^{\mathbf{U}(n-2)} = (\mathbf{S}^{2n-1})^{\mathbf{U}(n-2)} = \mathbf{S}^3$ we have that $\mu = \mathcal{M}ap^G(\varrho \mid \mathbf{S}^2, \sigma)$ is some $\mathbf{S}^3$-bundle over $\mathbf{S}^2$ (for any choice of $\sigma$). By obstruction theory, there is a one–one correspondence

$$\pi_0\Gamma(\mu) \leftrightarrow H^2(\mathbf{S}^2; \pi_2(\mathbf{S}^3)) = 0.$$

Thus such $\mathbf{U}(n)$-spaces $W$ over $\mathbf{D}^3$ are completely determined by the characteristic class $c_1(\sigma)$ of the principal $\mathbf{S}^1$-bundle associated with the $(\mathbf{U}(n)/\mathbf{U}(n-1))$-bundle $\sigma$ of singular orbits over $\mathbf{S}^2$. Now if $T = \{\mathrm{diag}(1, z_2, \ldots, z_n) \in \mathbf{U}(n)\}$, which is a maximal torus of $\mathbf{U}(n-1)$, then we see that

$$(\mathbf{U}(n)/\mathbf{U}(n-2))^T = \varnothing,$$
$$(\mathbf{U}(n)/\mathbf{U}(n-1))^T = (\mathbf{U}(n)/\mathbf{U}(n-1))^{\mathbf{U}(n-1)} = \mathbf{S}^1.$$

Thus $W^T \to \mathbf{S}^2$ is precisely the principal $\mathbf{S}^1$-bundle associated with $\sigma$. Since $c_1(\sigma)$ is just the euler class of this $\mathbf{S}^1$-bundle over $\mathbf{S}^2$, one computes from the Gysin sequence that

$$H^2(W^T; \mathbf{Z}) \approx \mathbf{Z}_k,$$

where $k = |\, c_1(\sigma) \,|$. Thus $W^T$ is a homology 3-sphere (and, if so, then an actual 3-sphere) iff $c_1(\sigma) = \pm 1$. Therefore, if $W$ is a homology sphere, then so is $W^T$ by III.10.2, and hence $c_1(\sigma) = \pm 1$.

Now if we reflect $\mathbf{D}^3$ across $\mathbf{D}^2$ by $f$: $\mathbf{D}^3 \to \mathbf{D}^3$, then we have the pull-back

$$\begin{array}{ccc} f^*W & \longrightarrow & W \\ \downarrow & & \downarrow \\ \mathbf{D}^3 & \xrightarrow{\ f\ } & \mathbf{D}^3. \end{array}$$

Clearly the invariant $[\sigma] = c_1(\sigma)$ attached to $f^*W$ is just the negative of that for $W$.

Thus, up to reflection of $\mathbf{D}^3$, the only such $\mathbf{U}(n)$-action on an integral homology sphere $W$ is precisely the linear action described at the beginning.

A completely analogous discussion shows that an action of $\mathbf{Sp}(n)$ over $\mathbf{D}^5$ with isotropy types $\mathbf{Sp}(n-1)$ and $\mathbf{Sp}(n-2)$ is completely characterized by the characteristic class

$$[\sigma] \in H^4(\mathbf{S}^4; \pi_3(\mathbf{S}^3)) \approx \mathbf{Z}$$

of the principal $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ associated with the bundle of singular orbits. Again, up to reflection of $\mathbf{D}^5$, there is precisely one such action on an integral homology sphere, and this is the diagonal action of $\mathbf{Sp}(n)$ on $\mathbf{S}^{8n-1} \subset \mathbf{Q}^n \times \mathbf{Q}^n$ [twice the standard representation of $\mathbf{Sp}(n)$].

*Remark* The case, analogous to 5.2, of $\mathbf{O}(n)$-actions over $\mathbf{D}^2$ with isotropy types $\mathbf{O}(n-1)$ and $\mathbf{O}(n-2)$ is actually the main case of interest to us, since it includes the exotic examples discussed in Chapter I, Section 7. However, it is more convenient to treat this case from another point of view and we shall do this at some length in Sections 6 and 7. Nevertheless, it is of interest to carry out the analysis of this case from the present point of view, since it illustrates the action $\Theta_0$ on $\pi_0\Gamma(\mu)$ quite well, and the reader is invited to do this for himself. Exercise 1 gives another case in which the action $\Theta_0$ plays an important role.

*Remark* If $B$ is disconnected, then it may be necessary to choose different maps $\pi\colon G/H \to G/K$ over each component of $B$ for the proof of 5.1. This does not affect the statement of 5.1 itself, however. Moreover, it is not even necessary that type$(G/K)$ be the same for each component of $B$; that is, $K$ may be allowed to be different (nonconjugate) for different components of $B$. The necessary modifications in the proof are obvious. The statement of 5.1 covers this generalization if it is interpreted correctly in the obvious way.

## 6. THE SECOND CLASSIFICATION THEOREM

The Classification Theorem 5.1 contains the singular bundle $\sigma$ explicitly. Sometimes this is an advantage, particularly in examples (such as 5.2) for which $\sigma$ carries all or most of the relevant information. In other cases, however, this can be a disadvantage. In this section, we shall prove an alternative classification theorem in which the singular bundle is hidden, but which possesses other advantages, particularly in that it takes a very simple form when $X$ is contractible.

We retain the notation of Section 5. In particular, $H \subset K \subset G$ are fixed once and for all. The manifold $X$ with boundary $B$ and its orbit structure are as in Section 5.

First, we must discuss a technical point necessary to the present approach. If $Y$ is a $G$-space over $X$ with projection $p\colon Y \to X$, then, for each component $B_i$ of $B$, $p^{-1}(\mathbf{I} \times B_i) \to B_i$ is an $M_\pi$-bundle for some equivariant map $\pi\colon G/H \to G/K$ (depending on $i$), by the Tube Theorem 4.2. This map $\pi$ is determined only up to self-equivalences of $G/H$ and $G/K$; that is, up to its orbit in $(G/K)^H$ under $N(H)/H \times N(K)/K$ (see 4.3). We shall say that $Y$ is **proper** [with respect to $(H, K)$] if $\pi$ (for each $i$) can be chosen to be the canonical projection $G/H \to G/K$. By 4.3, if $(G/K)^H$ is connected, then *all* $G$-spaces over $X$ are proper, and this will be true for all specific cases with which we will deal. If $B$ is connected, then $K$ can be changed by a conjugation so that any given $Y$ will *become* proper for the *new* $(H, K)$, but the distinction must be made here.

If $B$ is disconnected, then one should allow $K$ to vary over the components and the notion of properness should be generalized in the obvious way. Since we shall not need this generalization, we shall not do this, so as not to unduly clutter the exposition. However, it will be clear how to carry this modification along.

A restriction to proper $G$-spaces $Y$ over $X$ is tantamount to the selection of what Jänich [1] calls a *fine orbit structure* on $X$. Also note (from the proof of 4.2) that $Y$ is proper over $X$ iff, for some point $y \in Y$ with $G_y = K$, $H$ occurs as an isotropy group in a slice at $y$ (over each component of $B$).

From now on, in this section, $\pi\colon G/H \to G/K$ will denote the *canonical projection.* Also, we put

$$S = \frac{N(H) \cap N(K)}{H} \qquad \text{and} \qquad N = \frac{N(H)}{H}.$$

Let $Y$ be a proper $G$-space over $X$ with projection $p\colon Y \to X$. Let $\varrho$ be the $G/H$-bundle $p^{-1}X_1 \to X_1$ as in Section 5. It will be convenient to display the principal bundles explicitly in this section, and thus we let $P \to X_1$ be the principal $N$-bundle associated with $\varrho$. Thus we have an equivalence

$$h\colon G/H \times_N P \xrightarrow{\approx} p^{-1}X_1$$

of $G$-spaces over $X_1$. By the Tube Theorem 4.2 and properness, there exists a principal $S$-bundle $Q \to B$ and an equivalence

$$k'\colon M_\pi \times_S Q \xrightarrow{\approx} p^{-1}(\mathbf{I} \times B)$$

of $G$-spaces over $\mathbf{I} \times B$. Since $G/H$ may be regarded as the top of $M_\pi$, $k'$ restricts to an equivalence

$$k\colon G/H \times_S Q \xrightarrow{\approx} p^{-1}B_1$$

of $G$-spaces over $B_1$. Let us denote by $P_1$ the part of $P$ over $B_1 \subset X_1$. Then

$$f' = h^{-1}k\colon G/H \times_S Q \to G/H \times_N P_1$$

is a $G$-equivariant map (equivalence) over $B_1$. By 3.2, $f'$ corresponds uniquely to an $S$-reduction

$$f\colon Q \to P_1$$

of $P_1$ to $Q$.

Thus the $G$-space $Y$ can be constructed, to equivalence over $X$, as follows: We are given the $G/H$-bundle $\varrho$ over $X_1$ which is associated with a principal $N$-bundle $P \to X_1$. We are also given an $S$-reduction $(Q, f)$ of $P_1$ (the part of $P$ over $B_1$). To this corresponds the $G$-equivariant map

$$f'\colon G/H \times_S Q \xrightarrow{\approx} G/H \times_N P_1$$

over $B_1$. Then we construct the (proper) $G$-space

$$Y(\varrho, (Q, f)) = (M_\pi \times_S Q) \cup_{f'} (G/H \times_N P)$$

over $X$.

It is clear that if the $S$-reduction $(Q, f)$ is changed by an equivalence, then $Y(\varrho, (Q, f))$ is changed by an equivalence of $G$-spaces over $X$.

Suppose now that $\varphi \in \mathbf{Map}^G(\varrho, \varrho)$. [Note that this corresponds (by 3.2, for example) uniquely to a self-equivalence $P \to P$ of principal $N$-bundles over $X_1$ which we shall continue to denote by $\varphi$.] If $\varphi_1\colon P_1 \to P_1$ is the restriction of $\varphi$, then the union of the identity on $M_\pi \times_S Q$ and $\varphi$ on $G/H \times_N P$ clearly gives an equivalence

$$Y(\varrho, (Q, f)) \approx Y(\varrho, (Q, f\varphi_1)).$$

(Here, as in Section 5, the composition $f\varphi_1$ acts from left to right.)

Note that

$$S \backslash N = (N(H) \cap N(K)) \backslash N(H)$$

and recall that by 3.1, the $S$-reduction $(Q, f)$ gives rise to a cross section

$$\tau \in \Gamma(\eta \mid B_1),$$

where $\eta$ is the $S \backslash N$-bundle associated with $\varrho$. (Thus the total space of $\eta$ is $(S \backslash N) \times_N P$.) Also $\tau$ depends only on the equivalence class of $(Q, f)$. The composition with the restriction $\varphi_1$ of $\varphi$ defines an action

$$\Phi\colon \Gamma(\eta \mid B_1) \times \mathbf{Map}^G(\varrho, \varrho) \to \Gamma(\eta \mid B_1).$$

Regarding elements of $\mathbf{Map}^G(\varrho, \varrho)$ as sections of the bundle $\mathscr{M}\!\mathit{ap}^G(\varrho, \varrho)$ with fiber $N = N(H)/H$, then

$$\Phi\colon \Gamma(\eta \mid B_1) \times \Gamma(\mathscr{M}\!\mathit{ap}^G(\varrho, \varrho)) \to \Gamma(\eta \mid B_1)$$

is, fiberwise, just the canonical right translation action

$$(S\backslash N) \times N \to S\backslash N.$$

By the remark preceding 3.2, a homotopy of $\tau$ is induced by an $S$-equivariant homotopy $Q \times \mathbf{I} \to P_1$ of $f$; that is, by a $G$-equivariant homotopy

$$(G/H \times_S Q) \times \mathbf{I} \to G/H \times_N P_1$$

of $f'$ over $B_1$. It is easily seen, much as in the proof of 5.1, that this change of $f'$ changes $Y(\varrho, (Q, f))$ only by an equivalence over $X$. (Alternatively, this can be seen by showing that homotopic sections of $\eta \mid B_1$ lie in the same orbit of $\Phi$.) Denote the action on homotopy classes induced by $\Phi$ by

$$\Phi_0 \colon \pi_0\Gamma(\eta \mid B_1) \times \pi_0\Gamma(\mathcal{M}ap^G(\varrho, \varrho)) \to \pi_0\Gamma(\eta \mid B_1).$$

Then we have shown that $Y(\varrho, (Q, f))$ depends to equivalence over $X$ only on $\varrho$ and the orbit $[\tau]^*$ of $[\tau] \in \pi_0\Gamma(\eta \mid B_1)$ under $\Phi_0$, where $\tau \in \Gamma(\eta|B_1)$ is the section corresponding to the $S$-reduction $(Q, f)$. Let

$$Y(\varrho, [\tau]^*)$$

denote this equivalence class of $G$-spaces over $X$. (As in Section 5, an equivalence of $\varrho$ with $\varrho'$ produces canonical changes in $\eta$, $\Phi_0$, and $[\tau]^*$, so that only the equivalence class $[\varrho] \in [X_1, B_N]$ matters.)

Now suppose that $Y(\varrho, (Q_0, f_0))$ and $Y(\varrho, (Q_1, f_1))$ are equivalent over $X$ via the map

$$(M_\pi \times_S Q_0) \cup_{f_0'} (G/H \times_N P) \xrightarrow{\approx} (M_\pi \times_S Q_1) \cup_{f_1'} (G/H \times_N P)$$

which, as in the proof of 5.1, may be taken as an $S$-equivalence over $\mathbf{I} \times B$. If $\varphi$ is the restriction of this to $G/H \times_N P$ and if $\psi$ is the restriction of it to $G/H \times_S Q_0 \to G/H \times_S Q_1$, then we have the commutative diagram

$$\begin{array}{ccc} G/H \times_S Q_0 & \xrightarrow{\psi} & G/H \times_S Q_1 \\ {\scriptstyle f_0'}\downarrow & & \downarrow{\scriptstyle f_1'} \\ G/H \times_N P & \xrightarrow{\varphi} & G/H \times_N P \end{array}$$

This shows that the $S$-reductions $(Q_1, f_1)$ and $(Q_0, f_0\varphi)$ are equivalent. Thus the sections $\tau_0$ and $\tau_1$ of $\eta \mid B_1$ associated with $(Q_0, f_0)$ and $(Q_1, f_1)$ are in the same orbit under $\Phi$. This proves the following theorem.

**6.1. Theorem** (Second Classification Theorem) *Let $H \subset K \subset G$ be given and let $X$ be as above. For a given regular bundle $\varrho$ (a $G/H$-bundle over $X_1$ with structure group $N(H)/H$), let $\eta$ be the associated bundle with fiber $(N(H) \cap N(K))\backslash N(H)$ and let $\Phi_0$ be the action of $\pi_0\Gamma(\mathcal{M}ap^G(\varrho, \varrho))$ on $\pi_0\Gamma(\eta \mid B_1)$ defined above. Then the assignment $[\tau]^* \mapsto Y(\varrho, [\tau]^*)$ is a one–one correspondence between the set $\pi_0\Gamma(\eta \mid B_1)/\Phi_0$ and the set of equivalence classes over $X$ of proper $G$-spaces over $X$ having regular bundle $\varrho$.* ∎

*Remark* Theorems similar to 6.1 were proved by Jänich [1] and Hsiang and Hsiang [4] in the differentiable case (which is easier). The reader should beware of some mistakes in these references, particularly the latter. For example, in the Hsiang paper, $[\tau] \in \pi_0\Gamma(\eta \mid B_1)$ is called the "twist invariant" and is treated as if it were an invariant, which it is not. A strong differentiable version of 6.1 will be proved in Chapter VI, Section 6.

This result takes a particularly simple form, which is due to Jänich, when $X$ is contractible. In this case $\varrho$ is a trivial bundle (and the trivialization is essentially unique) and hence so is $\eta$ and $\eta \mid B_1$. Thus

$$\pi_0\Gamma(\eta \mid B_1) = [B_1, (N(H) \cap N(K))\backslash N(H)].$$

Also $\mathcal{M}ap^G(\varrho, \varrho)$ is the trivial $N(H)/H$-bundle over $X_1$ and every map $X_1 \to N(H)/H$ is homotopic to a constant map, so that

$$\pi_0\Gamma(\mathcal{M}ap^G(\varrho, \varrho)) \approx \pi_0\left(\frac{N(H)}{H}\right),$$

the group of arc components of $N(H)/H$ (that is, the quotient of $N(H)/H$ by its component of the identity). The action $\Phi_0$ is clearly induced, in the obvious way, by the right translation action of $N = N(H)/H$ on $S\backslash N = (N(H) \cap N(K))\backslash N(H)$. Thus we have the following corollary.

**6.2. Corollary** *In the situation of 6.1 with $X$ contractible, the set of equivalence classes of proper $G$-spaces over $X$ is in one–one correspondence with*

$$[B, (N(H) \cap N(K))\backslash N(H)]/\pi_0\left(\frac{N(H)}{H}\right). \quad ∎$$

**6.3. Example** Let $(G, K, H) = (\mathbf{O}(n), \mathbf{O}(n-1), \mathbf{O}(n-2))$ and $X = \mathbf{D}^2$. By 4.3 every such action over $\mathbf{D}^2$ is proper and it is also clearly locally smooth on a $(2n-1)$-manifold. Then $N = N(H)/H \approx \mathbf{O}(2)$ and $S = (N(H)$

$\cap N(K))/H \approx \mathbf{O}(1) \times \mathbf{O}(1) \approx \mathbf{Z}_2 \times \mathbf{Z}_2$. Thus $S\diagdown N = \mathbf{RP}^1$ ($\approx \mathbf{S}^1$) with the usual $\mathbf{O}(2)$-action. Note that $\mathbf{O}(1) \subset \mathbf{O}(2)$ reflects $\mathbf{S}^1$. Thus $\pi_0(N) \approx \mathbf{Z}_2$ acts on $[\mathbf{S}^1, S\diagdown N] \approx [\mathbf{S}^1, \mathbf{S}^1] \approx \mathbf{Z}$ by taking $m$ to $-m$. Thus $[\mathbf{S}^1, S\diagdown N]/\pi_0(N) \approx \mathbf{Z}^+$, the nonnegative integers. The actions of $\mathbf{O}(n)$ on $\Sigma_k^{2n-1}$ described in Chapter I, Section 7 are of this type, and we shall now show that the invariant in $\mathbf{Z}^+$ for $\Sigma_k^{2n-1}$ is just $|k|$. Another method of showing that the $\Sigma_k^{2n-1}$ exhaust these actions is given in Section 7. Of course, we could with sufficient care, simply compute the invariant for the explicitly given action, but it would seem to be preferable to give the following indirect argument. Let $Y_k^{2n-1}$ denote the $\mathbf{O}(n)$-space over $\mathbf{D}^2$ with invariant $k \geq 0$ in $\mathbf{Z}^+ = [\mathbf{S}^1, S\diagdown N]/\pi_0(N)$. Consider the fixed point set $F(\mathbf{O}(n-2), Y_k^{2n-1})$. Now $\mathbf{O}(2) \times \mathbf{O}(n-2) \subset \mathbf{O}(n)$ and the factor $\mathbf{O}(2)$ can be regarded as $N(H)/H = N$. Similarly $\mathbf{O}(1) \times \mathbf{O}(1) \times \mathbf{O}(n-2) \subset \mathbf{O}(n)$ and $\mathbf{O}(1) \times \mathbf{O}(1)$ can be regarded as $(N(H) \cap N(K))/H = S$. Clearly $\mathbf{O}(2)$ is transitive on both $F(\mathbf{O}(n-2), \mathbf{O}(n)/\mathbf{O}(n-2))$ and $F(\mathbf{O}(n-2), \mathbf{O}(n)/\mathbf{O}(n-1))$ and hence $F(\mathbf{O}(n-2), Y_k^{2n-1})/\mathbf{O}(2)$ is canonically identified with $Y_k^{2n-1}/\mathbf{O}(n)$. It is also clear that the invariant in $[\mathbf{S}^1, \mathbf{O}(1) \times \mathbf{O}(1)\diagdown\mathbf{O}(2)]/\pi_0(\mathbf{O}(2))$ for $\mathbf{O}(2)$ on $F(\mathbf{O}(n-2), Y_k^{2n-1})$ is just $k$. Thus it follows that

$$Y_k^3 \approx F(\mathbf{O}(n-2), Y_k^{2n-1})$$

as $\mathbf{O}(2)$-spaces over $\mathbf{D}^2$. Since we also had $\Sigma_k^3 \approx F(\mathbf{O}(n-2), \Sigma_k^{2n-1})$ it will suffice to treat the case $n = 2$.

For $k > 0$, consider the map $f_k$: $\mathbf{D}^2 \to \mathbf{D}^2$ given by $f_k(z) = z^k$. There is the pull-back diagram

$$\begin{array}{ccc} f_k^* Y_j^{2n-1} & \longrightarrow & Y_j^{2n-1} \\ \downarrow & & \downarrow \\ \mathbf{D}^2 & \xrightarrow{f_k} & \mathbf{D}^2 \end{array}$$

and the invariant in $[\mathbf{S}^1, S\diagdown N]/\pi_0(N)$ for the $\mathbf{O}(n)$-space $f_k^* Y_j^{2n-1}$ is clearly $jk$. Thus

$$f_k^* Y_j^{2n-1} \approx Y_{jk}^{2n-1}$$

as $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$. In particular, $f_k^* Y_1^{2n-1} \approx Y_k^{2n-1}$. Note that $f_k$ is the orbit map of a $\mathbf{Z}_k$-action on $\mathbf{D}^2$. Thus the pull-back $Y_k^{2n-1} \approx f_k^* Y_1^{2n-1}$ has an action of $\mathbf{Z}_k \times \mathbf{O}(n)$, and, moreover, the factor $\mathbf{Z}_k$ leaves stationary the orbit $\mathbf{O}(n)/\mathbf{O}(n-2)$ over $0 \in \mathbf{D}^2$ and acts freely outside this orbit. For $n = 2$, this fixed set of $\mathbf{Z}_k$ (and of any nontrivial subgroup) is $\mathbf{O}(2) \approx \mathbf{S}^1 + \mathbf{S}^1$. By the Smith Theorem III.7.11, $Y_k^3$ cannot be a sphere for $k \neq 1$, Thus the diagonal linear action of $\mathbf{O}(2)$ on $\mathbf{S}^3 \subset \mathbf{R}^2 \times \mathbf{R}^2$ must coincide with the $\mathbf{O}(2)$-space $Y_1{}^3$. (Recall that $\Sigma_1{}^3$ is also this linear $\mathbf{O}(2)$-space.)

Regarding $\mathbf{S}^3 \subset \mathbf{C} \times \mathbf{C}$, the given linear $\mathbf{O}(2)$-action is generated by the $\mathbf{S}^1 = \mathbf{SO}(2)$ action $z(s, t) = (zs, zt)$ and the $\mathbf{O}(1)$-action which is given by the involution $(s, t) \mapsto (\bar{s}, \bar{t})$. A change of coordinates

$$u = \frac{s - it}{\sqrt{2}} \quad \text{and} \quad v = \frac{s + it}{\sqrt{2}}$$

transforms this action into the same $\mathbf{S}^1$-action but with the $\mathbf{O}(1)$-action becoming $(u, v) \mapsto (\bar{v}, \bar{u})$. In these coordinates the circles $u = 0$ and $v = 0$ form (together) a principal orbit of $\mathbf{O}(2)$, and we may assume that this orbit corresponds to the origin $0 \in \mathbf{D}^2$.

Now the lens space $\mathbf{L}_k = \mathbf{L}(k, 1)$ is the quotient space $\mathbf{S}^3/\mathbf{Z}_k$, where $\mathbf{Z}_k \subset \mathbf{S}^1$ in the above action. Let $[u, v]$ denote the point of $\mathbf{L}_k$ which is the orbit of $\mathbf{Z}_k$ on $(u, v) \in \mathbf{S}^3$. Since $\mathbf{Z}_k$ is normal in $\mathbf{O}(2)$, there is an induced action of $\mathbf{O}(2)/\mathbf{Z}_k$ on $\mathbf{L}_k$. Now $z \mapsto z^k$ clearly induces an isomorphism

$$\mathbf{O}(2)/\mathbf{Z}_k \xrightarrow{\approx} \mathbf{O}(2)$$

and thus we may consider this action on $\mathbf{L}_k$ as an $\mathbf{O}(2)$-action. (Thus $\mathbf{S}^1$ acts on $\mathbf{L}_k$ by $z[u, v] = [z^{1/k}u, z^{1/k}v]$, with the same choice of $z^{1/k}$ for each factor; and $\mathbf{O}(1)$ acts via $[u, v] \mapsto [\bar{v}, \bar{u}]$.) Consider the map

$$\varphi\colon \mathbf{L}_k \to \mathbf{S}^3$$

$$\varphi[u, v] = \frac{(u^k, v^k)}{(|\, u\,|^{2k} + |\, v\,|^{2k})^{1/2}}.$$

This is clearly $\mathbf{O}(2)$-equivariant. If $\varphi[u, v] = \varphi[u', v']$, then there is a real number $t > 0$ with $(tu')^k = u^k$ and $(tv')^k = v^k$. Since

$$1 = |\, u\,|^2 + |\, v\,|^2 = |\, tu'|^2 + |\, tv'|^2 = t^2(|\, u'\,|^2 + |\, v'\,|^2) = t^2$$

we have $t = 1$. Thus $[u, v] = [u', \omega v']$ for some $k$th root of unity $\omega$.

Now $[u, v] \mapsto [u, \omega v]$ defines an action of $\mathbf{Z}_k$ on $\mathbf{L}_k$ and $\varphi$ can be regarded as the orbit map of this $\mathbf{Z}_k$-action. Moreover, this $\mathbf{Z}_k$-action commutes with the given $\mathbf{O}(2)$-action, so that together these give an action of $\mathbf{Z}_k \times \mathbf{O}(2)$ on $\mathbf{L}_k$. The diagram

$$\begin{array}{ccc} \mathbf{L}_k & \xrightarrow{\varphi} & \mathbf{S}^3 \\ \downarrow & & \downarrow \\ \mathbf{L}_k/\mathbf{O}(2) & \xrightarrow{\psi} & \mathbf{D}^2 = \mathbf{S}^3/\mathbf{O}(2) \end{array}$$

commutes and is clearly a pull-back diagram. The $\mathbf{O}(2)$-orbit $\{[u, v] \mid u = 0$ or $v = 0\}$ is fixed by the $\mathbf{Z}_k$-action on $\mathbf{L}_k$ and it is clear that no other $\mathbf{O}(2)$-orbit is taken into itself by *any* nontrivial element $\omega \in \mathbf{Z}_k$. That is, $\mathbf{Z}_k$ is free on $\mathbf{L}_k/\mathbf{O}(2)$ outside a single stationary point. Thus $\mathbf{L}_k/\mathbf{O}(2)$ is the cyclic $k$-fold covering of $\mathbf{D}^2$ branched at $\{0\}$.

It follows that

$$\mathbf{L}_k/\mathbf{O}(2) \approx \mathbf{D}^2$$

(which could be seen directly) and that the map $\psi$ in the above diagram can be taken to be $f_k$: $\mathbf{D}^2 \to \mathbf{D}^2$. Consequently

$$\mathbf{L}_k \approx Y_k{}^3$$

as $\mathbf{O}(2)$-spaces. Since the $\mathbf{L}_k$ are distinguished from one another as spaces by the order $k$, of $H_1(\mathbf{L}_k)$, it follows that $Y_k{}^3 \approx \Sigma_k{}^3$ for all $k$. (It is amusing to note that this shows that the definition of $\mathbf{L}(k, 1)$ used here is equivalent to that used to define $\Sigma_k{}^3$.) As remarked above, this implies in general that

$$Y_k^{2n-1} \approx \Sigma_k^{2n-1}$$

as $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$. We have proved the following result which gives the classification of $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ in a form which is convenient for some applications.

**6.4. Theorem** *Consider $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ with isotropy types $\mathbf{O}(n-1)$ on $\mathbf{S}^1$ and $\mathbf{O}(n-2)$ on* $\operatorname{int}(\mathbf{D}^2)$, $n \geq 2$. *Any such space $Y$ is equivalent over $\mathbf{D}^2$ to $\Sigma_k^{2n-1}$ for some $k \geq 0$. Moreover, for this $k$ we have $H_1(Y^H; \mathbf{Z}) \approx \mathbf{Z}_k$ ($\mathbf{Z}$ when $k = 0$), where $H = \mathbf{O}(n-2)$. Thus the $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ are classified by the first homology group of the fixed set of $\mathbf{O}(n-2)$.* ▌

Note that we have also shown that the $\mathbf{O}(n)$-action on $\Sigma_k^{2n-1}$ can be extended to an action of $\mathbf{Z}_k \times \mathbf{O}(n)$ with

$$F(\mathbf{Z}_k, \Sigma_k^{2n-1}) \approx \mathbf{O}(n)/\mathbf{O}(n-2).$$

In particular, for $k$ *odd*, $\mathbf{Z}_k$ acts on $\Sigma_k{}^5 \approx \mathbf{S}^5$ with fixed set $\mathbf{O}(3)/\mathbf{O}(1) \approx \mathbf{RP}^3$. We will see this $\mathbf{Z}_k$-action again in Section 9 from a different point of view. It was first noticed by Hirzebruch in the context of Section 9 (see Hirzebruch and Mayer [1]), and answered a question stated in Bredon [12]. Also Yang [7] independently found such examples by another method.

*Remark* In this section and in Section 5 we have been assuming that $X$ is a manifold with boundary $B$. However, we have not fully used this assumption (it is just the main case of interest). It clearly suffices for $X$ to be a metrizable space and for $B$ to be a closed subspace which is locally connected, has the homotopy type of a CW-complex, and is collared in $X$. For example, $X$ could be $\mathbf{D}^m$ and $B \subset \mathbf{S}^{m-1}$ a closed hemisphere.

## 7. CLASSIFICATION OF SELF-EQUIVALENCES

Let $G \supset K \supset H$ and $(X, B)$ be as in Section 6. Let $Y$ be a proper $G$-space over $X$. In this section we show how to classify the self-equivalences of $Y$ over $X$ up to equivariant homotopy over $X$. We may take $Y$ to be of the form $Y = Y(\varrho, (Q, f))$ in the notation of Section 6. Thus $P$ is a principal $N$-bundle over $X_1$ [with $N = N(H)/H$] with $\varrho$ the associated $G/H$-bundle $G/H \times_N P \to X_1$, $f\colon Q \to P_1$ is an $S$-reduction of the restriction $P_1$ of $P$ to $B_1$ [with $S = (N(H) \cap N(K))/H$], and

$$Y = (M_\pi \times_S Q) \cup_{f'} (G/H \times_N P),$$

where $f'\colon G/H \times_S Q \xrightarrow{\approx} G/H \times_N P_1$ is induced by $f$, and $\pi\colon G/H \to G/K$ is the projection.

Let $\xi$ be the bundle $G/H \times_S Q \to B_1$ and $\mathcal{M}ap^G(\xi, \xi)$ the bundle (defined in II.2.8) whose sections are the equivariant self-maps of $\xi$ over $B_1$. This bundle has fiber $N$ and may also be regarded as the bundle of $N$-equivalences of $\xi$ over $B_1$ (see Section 3). Then $f'$ clearly induces an isomorphism

$$\mathcal{M}ap^G(\xi, \xi) \xrightarrow{\approx} \mathcal{M}ap^G(\varrho, \varrho) \mid B_1$$

of bundles over $B_1$, which we may regard as an identification (with $f$ fixed). The $S$-equivalences of $\xi$ form a subbundle of $\mathcal{M}ap^G(\xi, \xi)$ which has fiber $S$. (As defined, this has structure group $S \times S$ made effective, acting by right and left translations, but this is easily seen to be reducible to $S$ acting by *conjugation*. Similarly for $\mathcal{M}ap^G(\varrho, \varrho)$. We do not need this fact, however.) Let us call this subbundle of $S$-equivalences

$$S\text{-}\mathcal{M}ap^G(\xi, \xi) \subset \mathcal{M}ap^G(\varrho, \varrho) \mid B_1.$$

Now suppose we have a self-equivalence $\varphi\colon Y \to Y$ over $X$. By 3.3 and 4.1, $\varphi \mid \mathbf{I} \times B$ is equivariantly homotopic to an $S$-equivalence. Now the part of $Y$ over $[1,2] \times B$ is equivalent to $[1,2] \times (G/H \times_N P_1) \approx [1,2] \times (G/H$

$\times_S Q)$. Let $\varphi_t$ be the $t$th stage of this homotopy on $M_\pi \times_S Q$ (where $\varphi_0 = \varphi$) and put $\psi_t = \varphi_t\varphi^{-1}$. Extend $\psi_t$ to $[1,2] \times (G/H \times_S Q)$ by putting

$$\psi_t(s, x) = \big(s, \psi_{t(2-s)}(x)\big).$$

Then extend $\varphi_t$ to $[1,2] \times (G/H \times_S Q)$ by $\varphi_t = \psi_t\varphi$. Since $\varphi_t = \varphi$ over $\{2\} \times B$ it extends over $X$ by $\varphi$ over $X_2$. Thus $\varphi$ is homotopic over $X$ through self-equivalences to $\varphi_1$ which is an $S$-equivalence over $\mathbf{I} \times B$. Similarly one sees that if two self-equivalences of $Y$ over $X$ which are $S$-equivalences over $\mathbf{I} \times B$ are equivariantly homotopic over $X$, then they are homotopic through self-equivalences which are $S$-equivalences over $\mathbf{I} \times B$. Let $\mathbf{Homeo}_X{}^G(Y)$ denote the group of self-equivalences of $Y$ over $X$. Let $\Gamma\big(\mathcal{M}ap^G(\varrho, \varrho), S\text{-}\mathcal{M}ap^G(\xi, \xi)\big)$ be the set of sections of the $N$-bundle $\mathcal{M}ap^G(\varrho, \varrho)$ over $X_1$ which, over $B_1$, take values in the $S$-subbundle $S\text{-}\mathcal{M}ap^G(\xi, \xi)$ over $B_1$. Then we have shown the following theorem.

**7.1. Theorem** *The natural map*

$$\pi_0\Gamma\big(\mathcal{M}ap^G(\varrho, \varrho), S\text{-}\mathcal{M}ap^G(\xi, \xi)\big) \to \pi_0\mathbf{Homeo}_X{}^G(Y)$$

*is a one–one correspondence.* ▮

Here $\pi_0\mathbf{Homeo}_X{}^G(Y)$ may be regarded as the set of equivariant homotopy classes over $X$ of self-equivalences of $Y$. The following fact is useful in determining how $S\text{-}\mathcal{M}ap^G(\xi, \xi)$ sits in $\mathcal{M}ap^G(\varrho, \varrho)$.

**7.2. Proposition** *Suppose that, in the above situation, $\varrho$ is trivial and that, for a given trivialization of $\varrho$, the $S$-reduction $f$: $Q \to P_1$ gives rise to the map $\tau$: $B_1 \to S\backslash N$ as in Section* 6 (*the induced cross section of the trivial $S\backslash N$-bundle $\eta \mid B_1$*). *Then, with respect to the induced trivialization of $\mathcal{M}ap^G(\varrho, \varrho)$ as $N \times X_1$, the fiber of $S\text{-}\mathcal{M}ap^G(\xi, \xi)$ over $b \in B_1$ is*

$$\tau(b)^{-1}S\tau(b).$$

***Proof*** Use a chart at $b$ to represent $S$ as the fiber of $Q$ over $b$. (This is determined up to right translation.) In terms of this choice $f$: $Q \to P_1 = N \times B_1$ induces

$$f_b\colon\ S \to N.$$

Let $n = f_b(e)$ so that $f_b(s) = sn$. Now an $S$-equivalence of $Q$ corresponds, in the fiber $S$ over $b$, to a right translation $R_s$ [where $R_s(s') = s's^{-1}$]. The $N$-equivalence of $\varrho \mid B_1$ induced by this is, over $b$, a right translation $R_m$ in $N$ for some $m \in N$. The diagram

$$\begin{array}{ccc} S & \xrightarrow{f_b} & N \\ {\scriptstyle R_s}\downarrow & & \downarrow{\scriptstyle R_m} \\ S & \xrightarrow{f_b} & N \end{array}$$

then commutes. Applied to $e \in S$ this means $nm^{-1} = s^{-1}n$; that is, $m = n^{-1}sn$. With $s$ running over $S$, $n^{-1}sn$ runs over

$$n^{-1}Sn = \tau(b)^{-1}S\tau(b)$$

since $\tau(b) = f_b(S) = Sn$, by definition. (Since $\tau(b)$ is a right coset of $S$, $\tau(b)^{-1}S\tau(b) = \tau(b)^{-1}\tau(b)$, but we include the $S$ in the middle to avoid misinterpretation.) ∎

**7.3. Corollary** *If $X$ is contractible and the invariant $[\tau]^* \in [B, S\diagdown N]/\pi_0(N)$ of 6.2 is trivial, then there is a one-one correspondence*

$$[X, B;\ N, S] \leftrightarrow \pi_0\mathbf{Homeo}_X{}^G(Y).$$

*Proof* Triviality of $[\tau]^*$ means that $\tau\colon B_1 \to S\diagdown N$ can be chosen to be the constant map $\tau(b) = Se$. With this choice, $S\text{-}\mathscr{M}ap^G(\xi, \xi) = S \times B_1 \subset N \times X_1$ by 7.2, and the result follows from 7.1. ∎

**7.4. Example** Let $(G, K, H) = (\mathbf{O}(n), \mathbf{O}(n-1), \mathbf{O}(n-2))$, $n \geq 2$, and $X = \mathbf{I}$, $B = \{0, 1\}$. By 6.2 there is just one such $G$-space over $\mathbf{I}$, namely the diagonal action of $\mathbf{O}(n)$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. By 7.3, the homotopy classes of self-equivalences of $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ over $\mathbf{I}$ correspond to

$$[\mathbf{I}, \partial\mathbf{I};\ N, S] = [\mathbf{I}, \partial\mathbf{I};\ \mathbf{O}(2), \mathbf{O}(1) \times \mathbf{O}(1)].$$

To any such self-equivalence $\varphi$ we may form the $\mathbf{O}(n)$-space

$$W(\varphi) = \mathbf{S}^{n-1} \times \mathbf{D}^n \cup_\varphi \mathbf{S}^{n-1} \times \mathbf{D}^n$$

over $\mathbf{D}^2$. Composition of $\varphi$ on either side by self-equivalences which extend to $\mathbf{S}^{n-1} \times \mathbf{D}^n$ alters this $\mathbf{O}(n)$-space only by an equivalence over $\mathbf{D}^2$. Now the self-equivalences of $\mathbf{S}^{n-1} \times \mathbf{D}^n$ over its orbit space (which can be identified with the upper half disk $\mathbf{D}^2_+$) correspond, up to homotopy, with $[\mathbf{D}^2_+, \mathbf{S}^1_+; N, S] = \pi_0(S)$. Two-sided composition clearly corresponds to the obvious action

$$\Psi\colon \pi_0(S) \times [\mathbf{I}, \partial\mathbf{I}; N, S] \times \pi_0(S) \to [\mathbf{I}, \partial\mathbf{I}; N,S].$$

It is clear, as in Sections 5 and 6 that $W(\varphi)$ and $W(\psi)$ are equivalent over $\mathbf{D}^2$ iff the corresponding homotopy classes $[\varphi]$ and $[\psi]$ in $[\mathbf{I}, \partial\mathbf{I}; N,S]$ are in the same orbit of $\Psi$. The reader may supply the details of this. If $f\colon (\mathbf{I}, \partial\mathbf{I}) \to (N,S)$, then by left, or right, multiplication by an element of $S$ we may change $f$ to $f'$ with $f'(0) = e$. Thus it is clear that the orbits of $\Psi$ on $[\mathbf{I}, \partial\mathbf{I}; N,S]$ are in one–one correspondence with the orbits of $\pi_0(S)$ acting by *conjugation* on $[\mathbf{I}, \{1\}, \{0\}; N,S,e] = \pi_1(N,S,e)$. Since, by 6.2, $\mathbf{S}^{n-1} \times \mathbf{D}^n$ is the only $\mathbf{O}(n)$-space over $\mathbf{D}^2_+$, to equivalence, we have set up a one–one correspondence between the equivalence classes of $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ and

$$\frac{\pi_1(\mathbf{O}(2), \mathbf{O}(1) \times \mathbf{O}(1), e)}{\pi_0(\mathbf{O}(1) \times \mathbf{O}(1))} \approx \frac{\pi_1(\mathbf{S}^1, \mathbf{Z}_2, e)}{\mathbf{Z}_2},$$

where $\mathbf{Z}_2$ acts on $\mathbf{S}^1$ by the nontrivial automorphism.

Now consider the self-equivalence $\varphi$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ defined in Chapter I, Section 7. That is,

$$\varphi(x, y) = (\theta_x(y), x),$$

where $\theta_x$ denotes the reflection through $\mathbf{R}x$ in $\mathbf{S}^{n-1}$. Let $H = \mathbf{O}(n-2)$ be regarded as acting on the *last* $n-2$ coordinates, so that $N = N(H)/H \approx \mathbf{O}(2)$ acting on the first two coordinates. Let $C \subset \mathbf{S}^{n-1}$ be the circle in the plane of the first two coordinates and $C_+$ the semicircle defined by requiring the second coordinate to be nonnegative. Let $x_0 = (1,0) \in C_+$. Then for $y$ varying over $C_+$, the points $(x_0, y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ describe a cross section for the $\mathbf{O}(n)$-action. Moreover, the isotropy group, for $y \neq \pm x_0$, is just $H = \mathbf{O}(n-2)$. Define

$$f(y) \in \mathbf{S}^1 = \mathbf{SO}(2)$$

to be the rotation which takes $y$ to $x_0$. Then for $y \in C_+$,

$$\varphi(x_0, y) = (f(y)(x_0), f(y)(y)) = f(y)(x_0, y).$$

This implies that $\varphi$ corresponds (via 7.3) to the map $f\colon C_+ \to \mathbf{S}^1$. (Here we think of $C_+$ as being parametrized by $\mathbf{I}$ with 0 corresponding to $x_0$ and 1 to $-x_0$.) This is a path starting at $e$ and going half way around $\mathbf{S}^1 = \mathbf{SO}(2)$ ending in the nontrivial element of $\mathbf{Z}_2 \subset \mathbf{S}^1$. By composition $\varphi^k$ corresponds to $f^k$, the path which goes $k/2$ times around $\mathbf{S}^1$ starting at $e$. Thus the $\varphi^k$ give all elements of $\pi_1(\mathbf{S}^1, \mathbf{Z}_2, e)/\mathbf{Z}_2$. This means that the examples $\Sigma_k^{2n-1}$ of $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ exhaust all $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ (with the given orbit structure) to equivalence over $\mathbf{D}^2$. Of course, this was also proved in 6.3 by another method. The present method, if done as directly as possible (see Bredon [11]), is probably the most efficient way of seeing this fact.

*Remark* The above method was essentially the original method used to classify $\mathbf{O}(n)$-spaces over $\mathbf{D}^2$ with the given orbit structure. This was done in Bredon [11], in both the topological and differentiable cases, although attention was restricted there to the situation in which the total space is a homology sphere (in particular, where $k$ is odd, which is equivalent to the nontriviality of the singular bundle). Actually the theorem proved there is considerably stronger than the present version since actions of general compact Lie groups were considered. (In fact, this accounts for most of the difficulties there.) The analogous cases treated in 5.2 can also be done easily by this method as the reader may verify.

*Remark* The discussion in 7.4 shows that the $\mathbf{O}(n)$-spaces $\Sigma_k^{2n-1}$ and $\Sigma_{-k}^{2n-1}$ over $\mathbf{D}^2$ are equivalent *over* $\mathbf{D}^2$, because of the action of $\mathbf{Z}_2$ on $\pi_1(\mathbf{S}^1, \mathbf{Z}_2, e)$. This also follows, of course, from 6.4. An equivalence of these was given in Chapter I, Section 7, namely the interchange of the copies of $\mathbf{S}^{n-1} \times \mathbf{D}^n$, but this is *not* an equivalence *over* $\mathbf{D}^2$. It will be useful to write down an explicit such equivalence. To do this let $\psi\colon \mathbf{S}^{n-1} \times \mathbf{D}^n \to \mathbf{S}^{n-1} \times \mathbf{D}^n$ be given by $\psi(x, y) = (x, \theta_x(y))$. On $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ we have $\varphi = T\psi$ where $T(x, y) = (y, x)$. Also note that $\psi^2 = 1$. Thus

$$\varphi^k\psi = (T\psi)^k\psi = (T\psi)^{k-1}T = T(\psi T)^{k-1} = \psi(\psi T)^k = \psi\varphi^{-k}$$

so that

$$\begin{array}{ccc} \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} & \xrightarrow{\varphi^{-k}} & \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} & \xrightarrow{\varphi^{k}} & \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \end{array}$$

commutes. Thus $\psi\colon \mathbf{S}^{n-1} \times \mathbf{D}^n \to \mathbf{S}^{n-1} \times \mathbf{D}^n$ on both copies defines an equivalence *over* $\mathbf{D}^2$ of $\Sigma_{-k}^{2n-1}$ to $\Sigma_k^{2n-1}$.

**7.5. Example** Let us consider the self-equivalences of $\Sigma_k^{2n-1}$ for $k \neq 0$ (the case $k = 0$ being trivial). First we treat those *over* $\mathbf{D}^2$. As we have shown, the invariant $[\tau]^* \in [\mathbf{S}^1, S\setminus N]/\pi_0(N) = \mathbf{Z}^+$ is $|k|$ so that any representative $\tau\colon \mathbf{S}^1 \to S\setminus N$ has degree $\pm k$. It follows easily that the part of the fiber $\tau(b)^{-1}S\tau(b) \approx S = \mathbf{O}(1) \times \mathbf{O}(1)$ which is in the component $\mathbf{O}(2) - \mathbf{SO}(2) \approx \mathbf{S}^1$ rotates with nonzero degree as $b$ traverses $\mathbf{S}^1 = \partial\mathbf{D}^2$. This means that there are no sections of $\mathcal{M}ap^G(\varrho, \varrho) = \mathbf{O}(2) \times \mathbf{D}^2$ in the component $(\mathbf{O}(2) - \mathbf{SO}(2)) \times \mathbf{D}^2$ which lie in $S\text{-}\mathcal{M}ap^G(\xi, \xi)$ over $\mathbf{S}^1$. In the component $\mathbf{SO}(2) \times \mathbf{D}^2$ we see that $S\text{-}\mathcal{M}ap^G(\xi, \xi)$ is just $\mathbf{Z}_2 \times \mathbf{S}^1$. Thus, since $\pi_2(\mathbf{S}^1) = 0$, there is exactly one *nontrivial* self-equivalence, up to homotopy, of $\Sigma_k^{2n-1}$, $k \neq 0$. Moreover, on a principal orbit $G/H = \mathbf{O}(n)/\mathbf{O}(n-2)$ this is equivalent to the right translation by

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbf{O}(2) = N(H)/H$$

and on singular orbits $G/K = \mathbf{O}(n)/\mathbf{O}(n-1) \approx \mathbf{S}^{n-1}$ it is right translation by $-1 \in \mathbf{O}(1) = N(K)/K$ (the antipodal map). The reader may verify that the explicit map

$$(x, y) \mapsto (-x, -y)$$

of $\mathbf{S}^{n-1} \times \mathbf{D}^n$ to itself (on both copies) induces such a self-equivalence of $\Sigma_k^{2n-1}$.

Since this equivalence preserves orientation on $\Sigma_k^{2n-1}$ we conclude that there are no orientation reversing self-equivalences of $\Sigma_k^{2n-1}$ *over* $\mathbf{D}^2$.

Now any homeomorphism of $\mathbf{D}^2$ to itself which preserves orientation is known to be isotopic to the identity. It follows from the Covering Homotopy Theorem of Palais, II.7.3, that any self-equivalence of $\Sigma_k^{2n-1}$ covering such a homeomorphism is isotopic (through self-equivalences) to a self-equivalence over the identity. Thus it suffices to discuss self-equivalences which induce a reflection on $\mathbf{D}^2$. Moreover, it suffices to consider one such equivalence since all others are obtained by composition with equivalences over the identity. However, from the discussion in the last remark, the map $\psi\colon \mathbf{S}^{n-1} \times \mathbf{D}^n \to \mathbf{S}^{n-1} \times \mathbf{D}^n$ given by $\psi(x, y) = (x, \theta_x(y))$ together with the *interchange* of the copies of $\mathbf{S}^{n-1} \times \mathbf{D}^n$, defines such a self-equivalence of $\Sigma_k^{2n-1}$. On the intersection $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$, $\psi$ clearly preserves orientation iff $n$ is odd. Thus this self-equivalence over a reflection of $\mathbf{D}^2$ preserves orientation iff $n$ is even. We conclude that there exists an orientation reversing self-equivalence of $\Sigma_k^{2n-1}$ iff $n$ is odd.

## 8. EQUIVARIANT PLUMBING

Let $\tau_n$ be the tangent disk bundle of $\mathbf{S}^n$ with total space $E(\tau_n)$ and projection $p$: $E(\tau_n) \to \mathbf{S}^n$. Let $\mathbf{D}^n$ be a tame $n$-disk in $\mathbf{S}^n$. Then there is a chart $\varphi$ over $\mathbf{D}^n$ of the form

$$\begin{array}{ccc} \mathbf{D}^n \times \mathbf{D}^n & \xrightarrow{\varphi} & p^{-1}(\mathbf{D}^n) \\ & \searrow \quad \swarrow & \\ & \mathbf{D}^n & \end{array}$$

with the projection $\mathbf{D}^n \times \mathbf{D}^n \to \mathbf{D}^n$ being $(x, y) \mapsto y$. We shall regard $\varphi$ as an inclusion. Let $\mu$: $\mathbf{D}^n \times \mathbf{D}^n \to \mathbf{D}^n \times \mathbf{D}^n$ be $\mu(x, y) = (y, x)$. Then the plumbing of two copies of $E(\tau_n)$ is defined to be

$$E(\tau_n) \cup_\mu E(\tau_n).$$

(See Figure V-1.)

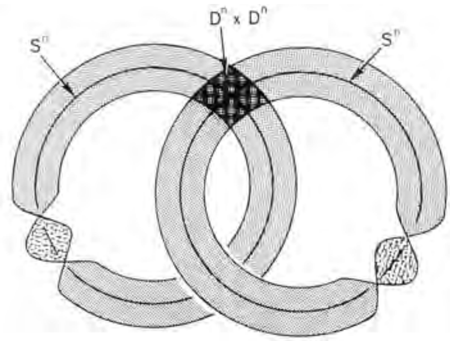
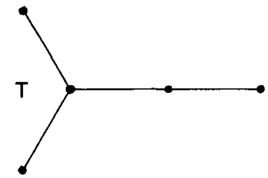



FIGURE V-1

More generally we may plumb several copies of $E(\tau_n)$ according to a graph. That is, if $T$ is a graph, then we take a copy of $E(\tau_n)$ for each vertex $v$ of the graph and we plumb the copies corresponding to $v$ and $w$ if there is an edge of the graph between $v$ and $w$. The disks used for the plumbing corresponding to the edge $(v, w)$ are taken to be disjoint from those used for any other edge, so that at most two points in the disjoint union of the $E(\tau_n)$ are identified to one point in the plumbing. We denote the result of this plumbing by $P^{2n}(T)$ which is a $2n$-manifold with boundary. (In the differentiable case this manifold has corners, which can be straightened by a standard process. We shall not concern ourselves with this here.) It is clear how to make this precise, and we shall not do it. Figure V-2 should suffice to make the meaning clear.



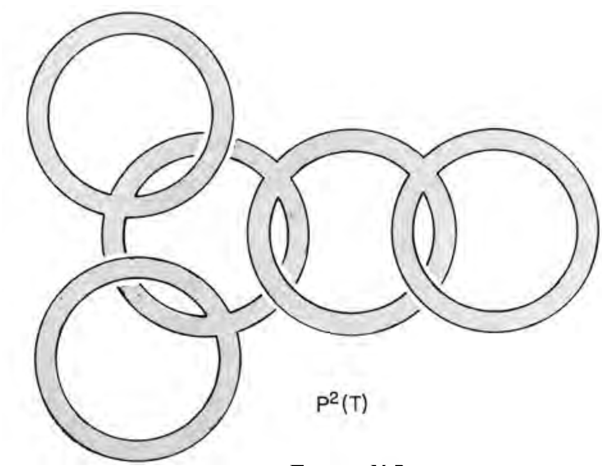




FIGURE V-2

The most important cases are those for which $T$ is a *tree* (a connected graph without circuits) and we shall restrict our attention to this case. There are certain classes of trees which are of primary importance (for reasons that we shall not enter into) and we illustrate these trees (with their standard designations) in Figure V-3, where $k$ stands for the number of vertices. Thus Figure V-1 illustrates $P^2(\mathbf{A}_2)$ and Figure V-2 illustrates $P^2(\mathbf{D}_5)$.

Now consider $\mathbf{O}(n) \subset \mathbf{O}(n+1)$ acting on $\mathbf{S}^n$, and hence on $E(\tau_n)$ in the standard way. Let $\pm x_0$ be the two fixed points of $\mathbf{O}(n)$ on $\mathbf{S}^n$. Then $\mathbf{O}(n)$ acts on the tangent disk at $x_0$ and at $-x_0$. The projection of the tangent disk at $x_0$ to $\mathbf{S}^n$ is obviously equivariant for this action. Thus, for an $\mathbf{O}(n)$-invariant disk $\mathbf{D}^n$ about $x_0$ in $\mathbf{S}^n$, the chart $\varphi$: $\mathbf{D}^n \times \mathbf{D}^n \to p^{-1}(\mathbf{D}^n)$ can be chosen to be $\mathbf{O}(n)$-equivariant, where $\mathbf{O}(n)$ acts *diagonally* on $\mathbf{D}^n \times \mathbf{D}^n$. Since $\mu$: $\mathbf{D}^n \times \mathbf{D}^n \to \mathbf{D}^n \times \mathbf{D}^n$ is equivariant for the diagonal action, there is an induced $\mathbf{O}(n)$-action on the plumbing

$$P^{2n}(\mathbf{A}_2) = E(\tau_n) \cup_\mu E(\tau_n).$$

Each of these copies of $E(\tau_n)$ has another fixed point $-x_0 \in \mathbf{S}^n$ about which we may perform further plumbing. Thus we clearly have a natural $\mathbf{O}(n)$-action on $P^{2n}(\mathbf{A}_k)$ for all $k$. This does not work for the other graphs since there are not enough fixed points in $\mathbf{S}^n$ at which to plumb. However, the subgroup $\mathbf{O}(n-1)$ has a *circle* of fixed points on $\mathbf{S}^n$ so that $P^{2n}(T)$ has an action of $\mathbf{O}(n-1)$ for any graph $T$. In general, the order of the plumbings around the circle of fixed points must be specified, but for the trees in Figure V-3 this is clearly irrelevant.

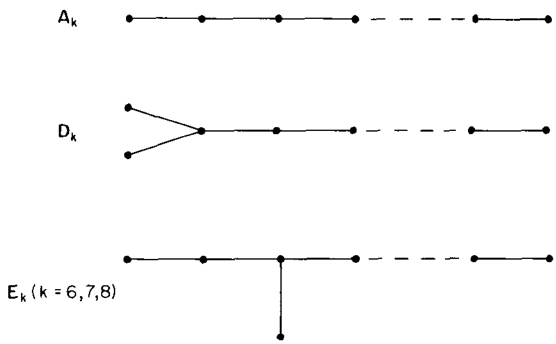



FIGURE V-3

In this section we shall be interested mainly in the $\mathbf{O}(n)$-spaces $\partial P^{2n}(\mathbf{A}_k)$. First let us show how to compute the homology of $\partial P^{2n}(T)$ for any tree $T$. It is clear that the union of the 0-sections $\mathbf{S}^n$ of the disk bundles $E(\tau_n)$ is a deformation retract of $P^{2n}(T)$ and this union has the homotopy type of the one-point union of $k$ copies of $\mathbf{S}^n$, where $k$ is the number of vertices of $T$. Thus $\tilde{H}_i(P^{2n}(T))$ is free abelian on $k$ generators (the 0-sections) for $i = n$ and is trivial for $i \neq n$. For $n \geq 2$, $P^{2n}(T)$ is simply connected and hence it is orientable, and $P^2(T)$ is also obviously orientable. (In fact it is easily seen that $P^{2n}(T)$ is parallelizable.)

More generally, let $M^{2n}$ be a compact oriented $2n$-manifold with boundary $B^{2n-1}$ and assume that $\tilde{H}_i(M)$ is free abelian for $i = n$ and is 0 for $i \neq n$. We wish to show how to compute $H_*(B)$. To do this consider the commutative Poincaré–Lefschetz duality diagram

$$\begin{array}{ccccccccc} \cdots \to & H_i(B) & \to & H_i(M) & \to & H_i(M, B) & \to & H_{i-1}(B) & \to \cdots \\ & \downarrow\approx & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx & \\ \cdots \to & H^{2n-i-1}(B) & \to & H^{2n-i}(M, B) & \to & H^{2n-i}(M) & \to & H^{2n-i}(B) & \to \cdots. \end{array}$$

This shows that $H_i(B)$ and $H^i(B)$ are 0 for $i \neq 0,\ n-1,\ n,\ 2n-1$.

Also we have the diagram

$$\begin{array}{ccccccccc}
0 \to & H_n(B) & \to & H_n(M) & \xrightarrow{j_*} & H_n(M, B) & \to & H_{n-1}(B) & \to 0 \\
& \downarrow \approx & & \Delta \downarrow \approx & & \Delta \downarrow \approx & & \downarrow \approx & \\
0 \to & H^{n-1}(B) & \to & H^n(M, B) & \xrightarrow[j^*]{} & H^n(M) & \to & H^n(B) & \to 0.
\end{array}$$

To compute $H_n(B)$ and $H_{n-1}(B)$ it clearly suffices to know the matrix of the homomorphism $j_*$ with respect to any bases of the free abelian groups $H_n(M)$ and $H_n(M, B)$ of rank $k$. Equivalently, it suffices to know the homomorphism

$$\Delta j_*\colon\ H_n(M) \to H^n(M).$$

Since $H_*(M)$ is free abelian, the evaluation homomorphism

$$H^n(M) \to \operatorname{Hom}(H_n(M), \mathbf{Z})$$

is an isomorphism. Recall that this is given by the cap product

$$\omega \mapsto \omega \cap (\cdot)$$

where we are identifying $H_0(M)$ with $\mathbf{Z}$. Thus it suffices to know the homomorphism

$$\psi\colon\ H_n(M) \to \operatorname{Hom}(H_n(M), \mathbf{Z})$$

given by

$$\psi(\alpha)(\beta) = \Delta j_*(\alpha) \cap \beta$$

where $\alpha, \beta \in H_n(M)$. Now let us recall that $\Delta^{-1}$ is given (in both cases) by the cap product

$$\Delta^{-1}(\omega) = \omega \cap [M]$$

with the fundamental (orientation) class $[M] \in H_{2n}(M, B)$ of $M$. Recall that the intersection number $\alpha \cdot \beta$ of $\alpha, \beta \in H_n(M)$ is defined to be

$$\alpha \cdot \beta = \Delta^{-1}(\Delta(\alpha) \cup \Delta(\beta)).$$

Thus

$$\begin{aligned}
\alpha \cdot \beta &= (\Delta(\alpha) \cup \Delta(\beta)) \cap [M] \\
&= \Delta(\alpha) \cap (\Delta(\beta) \cap [M]) \\
&= \Delta(\alpha) \cap \beta = j^*\Delta(\alpha) \cap \beta = \Delta j_*(\alpha) \cap \beta = \psi(\alpha)(\beta).
\end{aligned}$$

Therefore, given a basis of the free abelian group $H_n(M)$ and the dual basis of $\operatorname{Hom}(H_n(M), \mathbf{Z})$, the matrix of $\psi$ is just the *intersection matrix* of $M$ with respect to this basis of $H_n(M)$. Since $\alpha \cdot \beta = (-1)^n \beta \cdot \alpha$, this matrix is symmetric when $n$ is even and is skew-symmetric when $n$ is odd.

Returning to the manifold $M = P^{2n}(T)$, note that we may take the basis of $H_n(M)$ to be the fundamental classes of the 0-sections of the various copies of $E(\tau_n)$. Thus, denoting the vertices of $T$ by $v_1, \ldots, v_k$, we have a basis element $\alpha_i \in H_n(P^{2n}(T))$ corresponding to each $v_i$. Now $\alpha_i \cdot \alpha_i$ can be computed from $E(\tau_n)$ and this is well known to be $\pm 2$ [depending on the choice of orientation of $E(\tau_n)$] when $n$ is even, and (clearly) $\alpha_i \cdot \alpha_i = 0$ when $n$ is odd. Now when $n$ is even the interchange of factors $\mu\colon \mathbf{D}^n \times \mathbf{D}^n \to \mathbf{D}^n \times \mathbf{D}^n$ preserves orientation, so that an orientation may be chosen for $P^{2n}(T)$ such that $\alpha_i \cdot \alpha_i = 2$ for all $i$. For $i \neq j$ it is clear from the geometry that $\alpha_i \cdot \alpha_j = \pm 1$ if there is an edge of $T$ between $v_i$ and $v_j$ and $\alpha_i \cdot \alpha_j = 0$ otherwise. Now $\alpha_i \cdot (-\alpha_j) = -\alpha_i \cdot \alpha_j$. Thus when $n$ is even, and since $T$ is a tree, it is clear that the $\alpha_i$ may be chosen so that

$$\alpha_i \cdot \alpha_j = \begin{cases} 2 & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } (v_i, v_j) \text{ is an edge of } T, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the intersection matrix for the tree $\mathbf{A}_k$ is the $k \times k$ matrix

$$\underset{(n \text{ even})}{\begin{bmatrix} 2 & 1 & & & & 0 \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & \ddots & \ddots & \\ 0 & & & & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}}, \qquad \underset{(n \text{ odd})}{\begin{bmatrix} 0 & 1 & & & & 0 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & \ddots & \ddots & \\ 0 & & & & 0 & 1 \\ & & & & -1 & 0 \end{bmatrix}}.$$

The determinant of this matrix is easily computed to be $k + 1$ for *n even*. It follows immediately from the exact sequences above, that $H_{2m}(\partial P^{4m}(\mathbf{A}_k)) = 0$ and $H_{2m-1}(\partial P^{4m}(\mathbf{A}_k))$ has order $k + 1$. Of course one can see much more than this, but this information is all that we will need for the following result.

**8.1. Theorem** *The* $\mathbf{O}(n)$*-space* $\partial P^{2n}(\mathbf{A}_k)$ *is equivalent to the* $\mathbf{O}(n)$*-space* $\Sigma_{k+1}^{2n-1}$ *of Chapter* I, *Section* 7.

*Proof* Let $n \geq 2$. The action of $\mathbf{O}(n) \subset \mathbf{O}(n+1)$ on $E(\tau_n)$ is clearly equivalent to the action on orthogonal 2-frames in $\mathbf{R}^{n+1}$. Thus it is clear that each isotropy group of $\mathbf{O}(n)$ is either a conjugate of $\mathbf{O}(n-2)$ (when the frame and the fixed basis vector of $\mathbf{O}(n)$ on $\mathbf{R}^{n+1}$ span a 3-space) or a conjugate of $\mathbf{O}(n-1)$. From this we conclude that there is a canonical homeomorphism

$$\frac{F(\mathbf{O}(n-2),\ \partial P^{2n}(\mathbf{A}_k))}{\mathbf{O}(2)} \approx \frac{\partial P^{2n}(\mathbf{A}_k)}{\mathbf{O}(n)}$$

(compare 6.3) with orbits of type $\mathbf{O}(2)/\mathbf{O}(1)$ corresponding to those of type $\mathbf{O}(n)/\mathbf{O}(n-1)$. Also it is clear that there is a canonical equivalence of $\mathbf{O}(2)$-spaces

$$F(\mathbf{O}(n-2), \partial P^{2n}(\mathbf{A}_k)) \approx \partial P^4(\mathbf{A}_k).$$

For $n > 2$ we have $H_1(\partial P^{2n}(\mathbf{A}_k)) = 0$, as shown above, and it follows from IV.4.1, IV.8.3, and II.6.5 (for example), that $\partial P^{2n}(\mathbf{A}_k)/\mathbf{O}(n) \approx \mathbf{D}^2$ with the orbits of type $\mathbf{O}(n)/\mathbf{O}(n-1)$ corresponding to boundary points. This also follows for $n = 2$ from the above remarks. (Of course, it is not hard to verify these facts directly.) Thus $\partial P^{2n}(\mathbf{A}_k)$ fits the context of 6.4 and, since $H_1(\partial P^4(\mathbf{A}_k))$ has order $k + 1$, the result follows from 6.4. The case $n = 1$ also follows by noting that for any $n$, $F(\mathbf{O}(n-1), \partial P^{2n}(\mathbf{A}_k)) \approx \partial P^2(\mathbf{A}_k)$ as $\mathbf{O}(1)$-spaces. ▌

*Remarks* It follows from 8.1 that $H_{2m-1}(\partial P^{4m}(\mathbf{A}_k)) \approx \mathbf{Z}_{k+1}$, which is also easily seen directly. It also follows that $\partial P^{4m+2}(\mathbf{A}_k)$ is homeomorphic to $\mathbf{S}^{4m+1}$, for all $m$, when $k$ is *even*. For $k = 2$ this is the famous Kervaire sphere and has an exotic differentiable structure for suitable values of $m$. From the intersection form of the tree $\mathbf{E}_8$ one sees that $\partial P^{4m}(\mathbf{E}_8)$ is homeomorphic to $\mathbf{S}^{4m-1}$ for $m \geq 2$. (For $m = 1$ it is a homology 3-sphere, but it is not simply connected. In fact it is the Poincaré dodecahedral space $\mathbf{SO}(3)/I$.) This is an exotic sphere for all $m \geq 2$ and is called the "Milnor sphere". In fact, the main use of 8.1 is to identify the differentiable structures on the $\Sigma_k^{2n-1}$. (Of course, one must prove the Classification Theorem in the smooth case.) We shall discuss this further in Chapter VI. The present use of plumbing to study transformation groups is due to Hirzebruch and a more extensive treatment of it may be found in Hirzebruch and Mayer [1].

## 9. ACTIONS ON BRIESKORN VARIETIES

Let $k$ be a nonzero integer, let $n \geq 2$ and let $W_k^{2n-1}$ be the space of all points $(z_0, z_1, \ldots, z_n)$ in complex $(n + 1)$-space which lie on the variety

$V_k^{2n}$ defined by the equation

$$z_0^k + z_1^2 + z_2^2 + \cdots + z_n^2 = 0$$

and which also lie on the sphere

$$\| z \|^2 = | z_0 |^2 + | z_1 |^2 + \cdots + | z_n |^2 = 1.$$

(The case $k = 0$, where $z_0^k = 1$, can also be treated if we set $\| z \|^2 = 2$, for example. For $k \neq 0$ the radius has no importance. However, the case $k = 0$ requires special arguments and, since it is not of primary interest, we shall relegate it to Exercise 4.)

It is clear that the origin is the only singularity of $V_k^{2n}$. Also, if $z = (z_0, \ldots, z_n) \in V_k^{2n} - \{0\}$, then for $t > 0$,

$$\lambda(t, z) = (t^2 z_0, t^k z_1, \ldots, t^k z_n)$$

is a curve on $V_k^{2n}$ with

$$d\| \lambda(t, z) \|^2/dt > 0.$$

Thus this curve is not tangent to the unit sphere $\| z \|^2 = 1$, and this implies that $V_k^{2n}$ is transverse to the sphere. Thus $W_k^{2n-1}$ is a $(2n - 1)$-manifold.

Let $\mathbf{O}(n)$ act on the coordinates $z_1, \ldots, z_n$ as a subgroup of $\mathrm{U}(n)$. Then $W_k^{2n-1}$ is invariant under this action. If we put $z_j = x_j + iy_j$, then this $\mathbf{O}(n)$-action is just the standard action on $(x_1, \ldots, x_n)$ and on $(y_1, \ldots, y_n)$. Thus the $\mathbf{O}(n)$-action on $\mathbf{C}^{n+1}$ can be thought of as the action on $\mathbf{R}^2 \times \mathbf{R}^n \times \mathbf{R}^n$ which is trivial on $\mathbf{R}^2$ and is the standard diagonal action on $\mathbf{R}^n \times \mathbf{R}^n$; that is, it is the sum of a trivial 2-dimensional (real) representation and twice the standard representation. It follows that the isotropy groups of $\mathbf{O}(n)$ on $\mathbf{C}^{n+1}$ are the conjugates of $\mathbf{O}(n)$, $\mathbf{O}(n - 1)$, and $\mathbf{O}(n - 2)$. Also $(z_0, \ldots, z_n)$ is a fixed point iff $z_1 = z_2 = \cdots = z_n = 0$ and hence fixed points do not occur in $W_k^{2n-1}$.

We shall need the following characterization of the orbits of the linear $\mathbf{O}(n)$-action on $\mathbf{C}^n = \mathbf{R}^n \times \mathbf{R}^n$.

**9.1. Lemma** *For the action of* $\mathbf{O}(n)$ *on* $\mathbf{C}^n$ *as a subgroup of* $\mathrm{U}(n)$, $n \geq 2$, *the orbit map can be identified with the map*

$$\mathbf{C}^n \to V = \{(r, \mu) \in \mathbf{R} \times \mathbf{C} \mid | \mu | \leq r\}$$

*given by*

$$(z_1, \ldots, z_n) \mapsto (| z_1 |^2 + \cdots + | z_n |^2,\ z_1^2 + \cdots + z_n^2).$$

***Proof*** Put $z_j = x_j + iy_j$, $z = (z_1, \ldots, z_n)$, $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$. Then $r = \| z \|^2 = \| x \|^2 + \| y \|^2$ and

$$\mu = z_1^2 + \cdots + z_n^2 = \| x \|^2 - \| y \|^2 + 2i\langle x, y\rangle$$

are invariants for the action. Moreover, $r$ and $\mu$ determine $\| x \|$, $\| y \|$, and $\langle x, y\rangle$. If $x'$ and $y'$ are two other vectors in $\mathbf{R}^n$ with $\| x' \| = \| x \|$, $\| y' \| = \| y \|$, and $\langle x', y'\rangle = \langle x, y\rangle$, then it is clear from elementary linear algebra that there is an orthogonal transformation of $\mathbf{R}^n$ taking $(x', y')$ into $(x, y)$. The inequality $| \mu | \leq r$ is clear and it is also easy to see how to manipulate the first two coordinates, with the others zero, to achieve any such inequality. ▌

Consider the map

$$\varphi\colon W_k^{2n-1} \to \mathbf{S}^{2n-1}$$

given by

$$\varphi(z_0, \ldots, z_n) = \frac{(z_1, \ldots, z_n)}{(| z_1 |^2 + \cdots + | z_n |^2)^{1/2}}.$$

This is $\mathbf{O}(n)$-equivariant. We claim that it may be regarded as the orbit map of the $\mathbf{Z}_k$-action on $W_k^{2n-1}$ defined by

$$\big(\omega, (z_0, \ldots, z_n)\big) \mapsto (\omega z_0, z_1, \ldots, z_n),$$

where $\omega^k = 1$. To see this let us first show that $\varphi$ is onto. If $(u_1, \ldots, u_n) \in \mathbf{S}^{2n-1}$ and if we let $u_0 \in \mathbf{C}$ be such that

$$u_0^k = -(u_1^2 + \cdots + u_n^2),$$

then the distance from the origin to the point

$$\lambda(t, u) = (t^2 u_0, t^k u_1, \ldots, t^k u_n) \in \mathbf{C}^{n+1}$$

increases from 0 to $\infty$ with $t$. Thus $\lambda(t, u) \in W_k^{2n-1}$ for some $t > 0$, and $\varphi\big(\lambda(t, u)\big) = (u_1, \ldots, u_n)$. Suppose now that $\varphi(z_0, \ldots, z_n) = \varphi(v_0, \ldots, v_n)$ for two points $z$ and $v$ of $W_k^{2n-1}$. Then for some $t > 0$ we clearly have $v_i = t^k z_i$ for $1 \leq i \leq n$. Then

$$v_0^k = -(v_1^2 + \cdots + v_n^2) = -t^{2k}(z_1^2 + \cdots + z_n^2) = t^{2k} z_0^k$$

so that $t^2 z_0 = \omega v_0$ for some $k$th root of unity $\omega$. Since $(z_0, z_1, \ldots, z_n)$and $(\omega v_0, v_1, \ldots, v_n) = (t^2 z_0, t^k z_1, \ldots, t^k z_n) = \lambda(t, z)$ are both in $W_k^{2n-1}$ and since $\| \lambda(t, z) \|$ increases strictly with $t$, we must have $t = 1$. Thus $(z_0,$

$z_1, \ldots, z_n) = (\omega v_0, v_1, \ldots, v_n)$ and this proves our contention that $\varphi$ may be regarded as the orbit map of the $\mathbf{Z}_k$-action on $W_k^{2n-1}$. Now this $\mathbf{Z}_k$-action commutes with the $\mathbf{O}(n)$-action and the only $\mathbf{O}(n)$-orbit taken into itself by any nontrivial $\omega \in \mathbf{Z}_k$ is the orbit $\{(0, z_1, \ldots, z_n) \in W_k^{2n-1}\}$. (This *is* an orbit by 9.1.)

Thus $\mathbf{Z}_k$ has exactly one fixed point on $W_k^{2n-1}/\mathbf{O}(n)$ and acts freely outside this point. The diagram

$$\begin{array}{ccc} W_k^{2n-1} & \xrightarrow{\varphi} & \mathbf{S}^{2n-1} \\ \downarrow & & \downarrow \\ W_k^{2n-1}/\mathbf{O}(n) & \xrightarrow{\psi} & \mathbf{S}^{2n-1}/\mathbf{O}(n) \approx \mathbf{D}^2 \end{array}$$

is a pull-back diagram since $\varphi$ preserves $\mathbf{O}(n)$-orbit types. By the above remarks, $\psi$ may be regarded as the projection for the $k$-fold cyclic covering of $\mathbf{D}^2$ branched at 0 (corresponding to the orbit $W_2^{2n-3} \subset \mathbf{S}^{2n-1}$). That is, $W_k^{2n-1}/\mathbf{O}(n) \approx \mathbf{D}^2$ and $\psi$ becomes $z \mapsto z^k$ of $\mathbf{D}^2 \to \mathbf{D}^2$. By 6.3 we have the equivalences

$$W_k^{2n-1} \approx Y_k^{2n-1} \approx \Sigma_k^{2n-1}$$

of $\mathbf{O}(n)$-spaces. Thus we have proved the following theorem.

**9.2. Theorem** *The $\mathbf{O}(n)$-space $\Sigma_k^{2n-1}$ of Chapter* I, *Section 7 is equivalent to the $\mathbf{O}(n)$-space $W_k^{2n-1}$ for $k \neq 0$.* ∎

**9.3. Corollary** *For $n$ and $k$ both odd, $W_k^{2n-1}$ is homeomorphic to $\mathbf{S}^{2n-1}$. For $n$ even, $H_{n-1}(W_k^{2n-1}) \approx \mathbf{Z}_k$ and $H_i(W_k^{2n-1}) = 0$ for $i \neq 0, n-1, 2n-1$. Also $W_k^3 \approx \mathbf{L}(k, 1)$.* ∎

*Remark* The consideration of the $W_k^{2n-1}$ is due to Brieskorn [1] and was used by him to investigate the topological nature of an isolated singularity of an algebraic variety. Corollary 9.3 is due to him. The application of transformation groups to this subject, and vice versa, is due to Hirzebruch [2] (see Hirzebruch and Mayer [1]). A study of a much more general, and useful, class of varieties with isolated singularities may be found in Milnor [7].

Note that $W_k^{2n-1}$ has further symmetry besides the given $\mathbf{O}(n)$- and $\mathbf{Z}_k$-actions. Indeed

$$(z, (z_0, z_1, \ldots, z_n)) \mapsto (z^2 z_0, z^k z_1, \ldots, z^k z_n)$$

defines an action of $\mathbf{S}^1$ on $W_k^{2n-1}$ which commutes with the given $\mathbf{O}(n)$-action. Thus we have an action of $\mathbf{S}^1 \times \mathbf{O}(n)$ on $W_k^{2n-1}$. The kernel of this action is just the subgroup of order two generated by $(-1, -I)$. Note that in the most interesting case, for which $n$ is *odd*, then $(-1, -I)$ is not in the identity component of $\mathbf{S}^1 \times \mathbf{O}(n)$. Thus the inclusion induces an *isomorphism*

$$\mathbf{S}^1 \times \mathbf{SO}(n) \xrightarrow{\approx} \frac{\mathbf{S}^1 \times \mathbf{O}(n)}{\mathbf{Z}_2} \qquad (n \quad \text{odd}),$$

so that the effective part of this action is an action of the connected group $\mathbf{S}^1 \times \mathbf{SO}(n)$.

These actions on $W_k^{2n-1}$ yield some interesting examples via standard constructions and we shall briefly discuss some of these. For the sake of simplicity we shall take $k = 3$ and $n = 3$. Let $\omega$ be a primitive cube root of unity and let

$$\eta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathbf{O}(3).$$

Then $\omega$ and $\eta$ commute and $\omega\eta$ has order 6 on $W_3{}^5 \approx \mathbf{S}^5$. Now

$$\omega\eta(z_0, z_1, z_2, z_3) = (\omega z_0, z_1, z_2, -z_3)$$

so that the fixed set of $\omega\eta$ is

$$W_2{}^1 \approx \mathbf{O}(2) \approx \mathbf{S}^1 + \mathbf{S}^1.$$

If we remove an invariant disk about some fixed point of this $\mathbf{Z}_6$-action on $\mathbf{S}^5$, we obtain a $\mathbf{Z}_6$-action on $\mathbf{D}^5$ with fixed set $\mathbf{D}^1 + \mathbf{S}^1$. Multiplying this by the trivial action on $\mathbf{I}$ gives a $\mathbf{Z}_6$-action on $\mathbf{D}^6$ with fixed point set $\mathbf{D}^2 + (\mathbf{D}^1 \times \mathbf{S}^1)$; the part in the boundary $\mathbf{S}^5$ being $\mathbf{S}^1 + (\mathbf{S}^0 \times \mathbf{S}^1) \approx \mathbf{S}^1 + \mathbf{S}^1 + \mathbf{S}^1$.

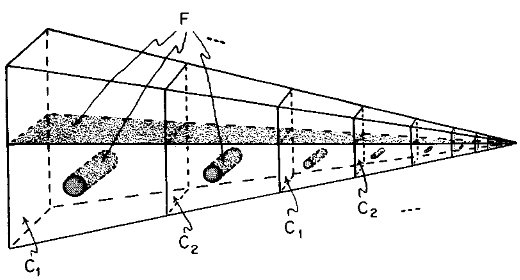



FIGURE V-4

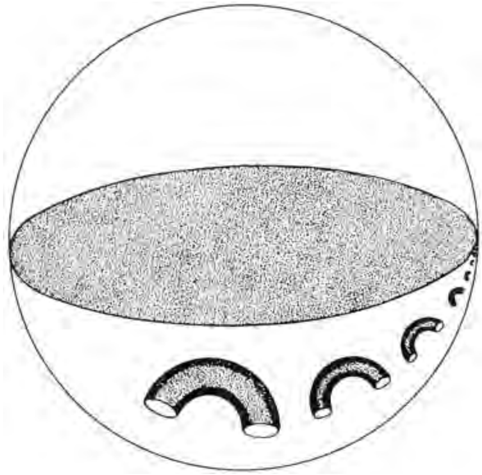


FIGURE V-5

Now let $C_1$ and $C_2$ be disjoint 5-cell neighborhoods in $\mathbf{S}^5$ of two points in the $\mathbf{D}^2$ component of the fixed set of this $\mathbf{Z}_6$-action on $\mathbf{D}^6$. We may regard $\mathbf{D}^6$ as a cube with $C_1$ and $C_2$ as opposite faces. An infinite number of copies of this action on $\mathbf{D}^6$ may then be pasted together (every other $\mathbf{D}^6$ being reflected in the parallel to $C_1$ and $C_2$) converging to an ideal point, as indicated in Figure V-4. This yields a $\mathbf{Z}_6$-action on $\mathbf{D}^6$ with fixed set $F$ being $\mathbf{D}^2$ together with a sequence of cylinders $\mathbf{D}^1 \times \mathbf{S}^1$ converging to a point of $\mathbf{D}^2$ as indicated in Figure V-5. On the boundary $\mathbf{S}^5$ the fixed set is a circle to-

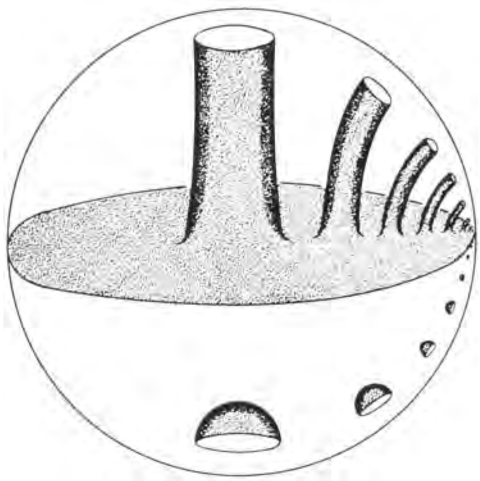


FIGURE V-6

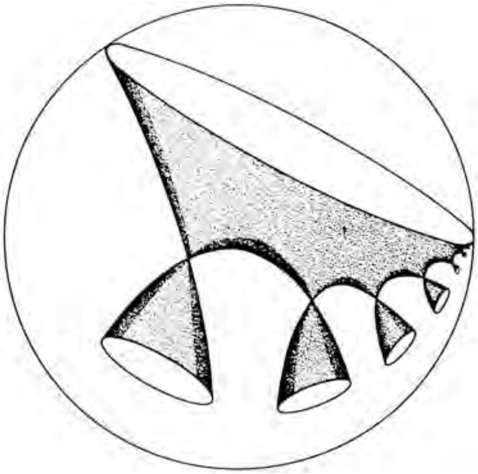


FIGURE V-7

gether with a sequence of circles converging to a point of this circle. If $C_1$ and $C_2$ were taken about points of $\mathbf{D}^1 \times \mathbf{S}^1$, then one would obtain an action of $\mathbf{Z}_6$ on $\mathbf{D}^6$ with fixed set $F$ as indicated in Figure V-6. Other possibilities are evident. These actions are locally smooth except at one point. Wilder actions can be constructed in a similar manner by pasting together the $\mathbf{Z}_6$-actions on $\mathbf{D}^6$ which are the *cones* on the original $\mathbf{Z}_6$-action on $\mathbf{S}^5$ (with $\mathbf{S}^1 + \mathbf{S}^1$ as fixed point set). By doing this one obtains actions of $\mathbf{Z}_6$ on $\mathbf{D}^6$ with fixed sets as illustrated in Figure V-7 or as in Figure V-8.

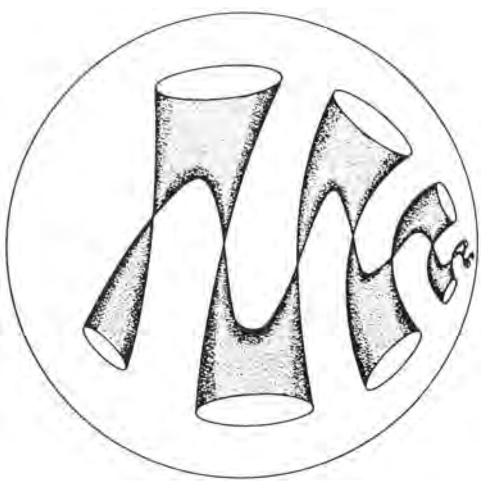


FIGURE V-8

We have given these examples to illustrate how complicated topological actions (not locally smooth) can be. Also note the actions on the boundary 5-spheres and the actions on $\mathbf{S}^6$ obtained by doubling the actions on $\mathbf{D}^6$.

Let us now briefly consider the general Brieskorn varieties. Thus let $\alpha = (a_0, \ldots, a_n)$ with the $a_i$ positive integers, and let $W_\alpha^{2n-1}$ be the intersection of the sphere $\| z \| = 1$ with the variety in $\mathbf{C}^{n+1}$ defined by the equation

$$z_0^{a_0} + \cdots + z_n^{a_n} = 0.$$

We shall restrict our attention to the proof of the following result.

**9.4. Theorem** *If $a_0$ is relatively prime to each $a_j$ for $j \neq 0$, then $a_0\beta = 0$ for every element $\beta \in H_i(W_\alpha^{2n-1}; \mathbf{Z})$ with $0 < i < 2n - 1$.*

*Proof* Let $a = a_0 a_1 \cdots a_n$. For $z = (z_0, \ldots, z_n)$ in $W_\alpha^{2n-1}$ and $t > 0$ consider the point

$$(t^{a/a_1} z_1, \ldots, t^{a/a_n} z_n) \in \mathbf{C}^n.$$

The norm of this is nonzero and is strictly increasing with $t$. Thus there is a unique value of $t$ such that this point is in the unit sphere $\mathbf{S}^{2n-1} \subset \mathbf{C}^n$. Let

$$\varphi\colon W_\alpha^{2n-1} \to \mathbf{S}^{2n-1}$$

be the map taking $z$ to this point. As above, some easy calculations show that $\varphi$ may be identified with the orbit map of the $\mathbf{Z}_{a_0}$-action

$$(\omega, (z_0, \ldots, z_n)) \mapsto (\omega z_0, z_1, \ldots, z_n)$$

on $W_\alpha^{2n-1}$. Now note that there is an $\mathbf{S}^1$-action

$$(z, (z_0, \ldots, z_n)) \mapsto (z^{a/a_0} z_0, \ldots, z^{a/a_n} z_n)$$

on $W_\alpha^{2n-1}$. If $\omega$ is a primitive $a_0$th root of unity, then $\omega^{a/a_0}$ is also a primitive $a_0$th root of unity and $\omega^{a/a_i} = 1$ for $i > 0$, since $a_0$ is relatively prime to each $a_i$ for $i \neq 0$. Thus this $\mathbf{S}^1$-action contains the $\mathbf{Z}_{a_0}$-action (up to automorphism of $\mathbf{Z}_{a_0}$) and this implies that $\mathbf{Z}_{a_0}$ acts *trivially* on $H_i(W_\alpha^{2n-1}; \mathbf{Z})$. Thus, by III.(7.1), the composition

$$H_i(W_\alpha^{2n-1}) \to H_i(\mathbf{S}^{2n-1}) \to H_i(W_\alpha^{2n-1})$$

of the orbit map for the $\mathbf{Z}_{a_0}$-action on $W_\alpha^{2n-1}$ with the transfer is just multiplication by $a_0$, so that this must vanish for $i \neq 0, 2n - 1$. ∎

**9.5. Corollary** *If there are at least two values of i for which $a_i$ is relatively prime to all $a_j$ for $j \neq i$, then $W_\alpha^{2n-1}$ is an integral homology $(2n-1)$-sphere.* ∎

*Remark* This result is a special case of a sharper theorem of Milnor [7]. (Also see Brieskorn [1] and Hirzebruch and Mayer [1]). It can be shown (see these references) that $W_\alpha^{2n-1}$ is simply connected (in fact, $(n-2)$-connected) for $n \geq 3$, and hence in the case of 9.5 it is homeomorphic to $S^{2n-1}$. Determination of the differentiable structures can also be found in the above references. In the 3-dimensional case, $n = 2$, $W_\alpha^3$ is not generally simply connected, even in the case of 9.5. For example, it can be shown that $W^3_{(2,3,5)}$ is the Poincaré dodecahedral space $\mathbf{SO}(3)/I$.

## 10. ACTIONS WITH THREE ORBIT TYPES

To motivate the present section and the next one, let us discuss the Brieskorn manifold $W_{p,q}^{2n+1}$ defined to be the space of all points $(u, v, z_1, z_2, \ldots, z_n) \in \mathbf{C}^{n+2}$ on the intersection of the sphere

$$|u|^2 + |v|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1$$

and the variety

$$u^p + v^q + z_1^2 + \cdots + z_n^2 = 0,$$

where $p$ and $q$ are positive integers. (Note that the case $q = 2$, studied in Section 9, is included.)

Now $\mathbf{O}(n)$ acts on $W_{p,q}^{2n+1}$ by its action on $(z_1, \ldots, z_n)$ and, as seen before, its isotropy types are $\mathbf{O}(n)$, $\mathbf{O}(n-1)$, and $\mathbf{O}(n-2)$. Given the coordinates $u, v$ of a point we see that $|z_1|^2 + \cdots + |z_n|^2 = 1 - (|u|^2 + |v|^2)$ and $z_1^2 + \cdots + z_n^2 = -(u^p + v^q)$ are determined, and it follows from 9.1 that the map

$$\psi\colon W_{p,q}^{2n+1} \to \mathbf{C}^2,$$

$$(u, v, z_1, \ldots, z_n) \mapsto (u, v)$$

can be identified with the orbit map of this $\mathbf{O}(n)$-action (once we find its image).

Since

$$|u^p + v^q| = |z_1^2 + \cdots + z_n^2| \leq |z_1|^2 + \cdots + |z_n|^2 = 1 - (|u|^2 + |v|^2),$$

the image of $\psi$ consists of points $(u, v)$ satisfying the inequality

$$| u^p + v^q | + | u |^2 + | v |^2 \leq 1.$$

It is easy to see that any such $(u, v)$ comes from a point of $W_{p,q}^{2n+1}$ if $n \geq 2$. Moreover, it is easily seen that this inequality is an equality iff the real and imaginary parts of $(z_1, \ldots, z_n)$ are dependent as vectors in $\mathbf{R}^n$; that is, iff the corresponding $\mathbf{O}(n)$-orbit is nonprincipal [i.e., of isotropy type $\mathbf{O}(n)$ or $\mathbf{O}(n-1)$].

Now it can be seen that the inequality $| u^p + v^q | + | u |^2 + | v |^2 \leq 1$ defines a 4-disk in $\mathbf{C}^2$. One way to see this is to consider the action of the multiplicative group $\mathbf{R}^+$ of positive reals on $\mathbf{C}^2$ given by

$$t(u, v) = (t^q u, t^p v).$$

Since, for $(u, v) \neq (0, 0)$ fixed,

$$t^{pq} | u^p + v^q | + t^{2q} | u |^2 + t^{2p} | v |^2$$

increases with $t$, it is equal to 1 for exactly one value of $t$. Similarly

$$|t^q u |^2 + | t^p v |^2 = t^{2q} | u |^2 + t^{2p} | v |^2 = 1$$

for exactly one value of $t$. Thus each trajectory of this flow (except for the origin) cuts each of the surfaces $| u^p + v^q | + | u |^2 + | v |^2 = 1$ and $| u |^2 + | v |^2 = 1$ in a single point. Using this, it is clear how to set up a homeomorphism of the region defined by

$$| u^p + v^q | + | u |^2 + | v |^2 \leq 1$$

(i.e., the image of $\psi$) with the unit disk $| u |^2 + | v |^2 \leq 1$.

Now the space of nonprincipal orbits can be identified with

$$F(\mathbf{O}(n-1), W_{p,q}^{2n+1})/\mathbf{O}(1)$$

which is just the orbit space of

$$W_{p,q}^3 = \{(u, v, z) \in \mathbf{C}^3 \mid | u |^2 + | v |^2 + | z |^2 = 1, \quad u^p + v^q + z^2 = 0\}$$

under the involution $(u, v, z) \mapsto (u, v, -z)$. Also $F(\mathbf{O}(n), W_{p,q}^{2n+1})$ is identified with $F(\mathbf{O}(1), W_{p,q}^3)$. As in Section 9 we see that the map

$$\varphi\colon W_{p,q}^3 \to \mathbf{S}^3,$$

which takes the point $(u, v, z) \in W^3_{p,q}$ to $(t^{2q}u, t^{2p}v)$, for that unique value of $t > 0$ such that this has norm 1, can be identified with the orbit map of this involution. Then $F = F(\mathbf{O}(1), W^3_{p,q})$, regarded as a subspace of the orbit space, becomes

$$\{(u, v) \in \mathbf{S}^3 \mid u^p + v^q = 0\}.$$

If we change coordinates in $\mathbf{S}^3$ by multiplying $v$ by a $q$th root of $-1$, then we change this representation of $F$ to

$$\{(u, v) \in \mathbf{S}^3 \mid u^p = v^q\}.$$

Letting $a, b$ be the unique positive real numbers such that $a^2 + b^2 = 1$ and $a^p = b^q$, this is

$$\{(ae^{2\pi ix}, be^{2\pi iy}) \in \mathbf{S}^3 \mid px = qy + m, \quad m \in \mathbf{Z}\},$$

which is the image under the exponential map $\mathbf{R}^2 \to \mathbf{S}^1 \times \mathbf{S}^1 \subset \mathbf{S}^3$ of the union of all lines of slope $p/q$ passing through the points $(m/p, 0)$, $m \in \mathbf{Z}$, If $p$ and $q$ are relatively prime, then this consists of exactly one curve which, in $\mathbf{S}^3$, is the "torus knot" of type $(p, q)$.

Thus the orbit structure of $W^{2n+1}_{p,q}$ is that of a 4-disk $\mathbf{D}^4$, with interior points of type $\mathbf{O}(n)/\mathbf{O}(n-2)$, and a 1-manifold in its boundary $\mathbf{S}^3$ consisting of the fixed points [in this case, a torus link of type $(p, q)$], with the remainder of $\mathbf{S}^3$ consisting of points of type $\mathbf{O}(n)/\mathbf{O}(n-1)$. If one generalizes this situation and introduces further coordinates on which $\mathbf{O}(n)$ does not act, then it can be seen that one obtains orbit spaces which are higher-dimensional disks and with the fixed point set consisting of a submanifold of codimension 2 of the boundary.

Before considering this situation in general, let us prove a general fact which allows us to discard the fixed point set in classifying actions over a given orbit structure.

**10.1. Proposition** *Let $G$ be a compact Lie group. Let $X$ be a locally compact space with a given orbit structure $X \to \mathscr{E}_G$ and let $F$ be the closed subspace of $X$ consisting of fixed points (points having type $G/G$). Then deletion of fixed point sets gives a one–one correspondence between equivalence classes of $G$-spaces over $X$ and those over $X - F$.*

***Proof*** If $Y$ is a $G$-space over $X - F$, then the orbit map $Y \to X - F$ is proper and hence extends to the one-point compactifications $Y_+$

$\to (X - F)_+$. There is also an orbit structure preserving map $X \to (X - F)_+$ taking $F$ to $\infty$. We let $W$ be the $G$-space over $X$ which is the pull-back

$$\begin{array}{ccc} W & \longrightarrow & Y_+ \\ \downarrow & & \downarrow \\ X & \longrightarrow & (X - F)_+ . \end{array}$$

Conversely, if $W'$ is any $G$-space over $X$, then we may regard $F$ as a subspace of $W'$ and put $Y = W' - F$, a $G$-space over $X - F$. It is clear that $W'$ is equivalent over $X$ to the above pull-back $W$ (see Chapter I, Exercise 10). ▌

*Remark* In the differentiable case, 10.1 is generally not true and one must be careful to consider how a tubular neighborhood of the fixed point set is attached. Because of this, some of the theorems stated in Hsiang and Hsiang [4] are false, as was pointed out to us by W.-C. Hsiang.

For the remainder of this section we shall let $X^m$ be a compact contractible $m$-manifold, $m \geq 4$, with oriented boundary $B^{m-1}$. Since $X$ is compact, $B$ is automatically an integral homology sphere. Also let $A^{m-3} \subset B^{m-1}$ be an oriented submanifold of dimension $m - 3$ (possibly wild). Let $\{A_\alpha\}$ be the components of $A$. We endow $X^m$ with the $\mathbf{O}(n)$-orbit structure ($n \geq 2$) which assigns type$(\mathbf{O}(n)/\mathbf{O}(n-2))$ to int$(X)$, type$(\mathbf{O}(n)/\mathbf{O}(n-1))$ to $B - A$ and type$(\mathbf{O}(n)/\mathbf{O}(n))$ to $A$. (We are primarily interested in the case in which $X^m = \mathbf{D}^m$ and in which $A$ is connected.)

Now $B$ is collared in $X$ and it follows that $X - A$ is a contractible (open) manifold with boundary $B - A$. Thus by 6.2, the equivalence classes of $\mathbf{O}(n)$-spaces over $X - A$ (and hence those over $X$, by 10.1) are in one–one correspondence with

$$\begin{aligned}[B - A, \mathbf{O}(1) \times \mathbf{O}(1)\backslash\mathbf{O}(2)]/\pi_0(\mathbf{O}(2)) &\approx [B - A, \mathbf{S}^1]/\mathbf{Z}_2 \\ &\approx H^1(B - A; \mathbf{Z})/\mathbf{Z}_2,\end{aligned}$$

where $\mathbf{Z}_2$ acts on $H^1(B - A; \mathbf{Z})$ via the automorphism $k \mapsto -k$ of the coefficients $\mathbf{Z}$, and hence takes $\gamma \in H^1(B - A; \mathbf{Z})$ to $-\gamma$.

Recall that by Poincaré–Lefschetz duality and the homology sequence of $(B^{m-1}, A^{m-3})$ we have the Alexander duality

$$H^1(B - A) \approx H_{m-2}(B, A) \approx H_{m-3}(A) \approx H^0(A) \approx \prod_\alpha H^0(A_\alpha) \approx \prod_\alpha \mathbf{Z}.$$

Via this isomorphism, let $\gamma \in H^1(B - A)$ correspond to $\{\gamma_\alpha\}$, where $\gamma_\alpha \in \mathbf{Z} \approx H^0(A_\alpha)$. Note that a change of the orientation of $A_\alpha$ changes the sign of $\gamma_\alpha$; that is, the particular isomorphism $H^1(B-A) \approx \prod_\alpha \mathbf{Z}$ depends on the choice of orientation of the $A_\alpha$, but otherwise this choice is unimportant. Also note that the $\mathbf{Z}_2$-action on $H^1(B-A)$ corresponds to the *simultaneous* change of sign of the $\gamma_\alpha$.

Now suppose that $a \in A_\alpha$ is a point at which $A_\alpha$ is tame; that is, there is a half disk $\mathbf{D}^m{}_+$ in $X^m$ giving a local coordinate system about $a$; viz.

$$\mathbf{D}^m{}_+ = \{(x_1, \ldots, x_m) \mid \sum x_i{}^2 \leq 1, \quad x_m \geq 0\}$$

with $a$ at the origin and with

$$B \cap \mathbf{D}^m{}_+ = \mathbf{D}^{m-1} = \{x \in \mathbf{D}^m{}_+ \mid x_m = 0\},$$
$$A \cap \mathbf{D}^m{}_+ = \mathbf{D}^{m-3} = \{x \in \mathbf{D}^m{}_+ \mid x_{m-2} = x_{m-1} = x_m = 0\}.$$

For $\gamma \in H^1(B-A)$ let $M_\gamma$ be the corresponding $\mathbf{O}(n)$-space over $X$, which depends only on $\pm\gamma$. By naturality, it is clear that the restriction

$$H^0(A_\alpha) \xrightarrow{\approx} H^0(\mathbf{D}^{m-3})$$

takes $\gamma_\alpha$ into the corresponding invariant (up to sign) for the $\mathbf{O}(n)$-action on the part of $M_\gamma$ over $\mathbf{D}^m{}_+$. Now consider the 2-disk

$$C = \{x \in \mathbf{D}^m{}_+ \mid x_1 = \cdots = x_{m-3} = 0, \quad x_{m-2}^2 + x_{m-1}^2 + x_m{}^2 = 1\},$$

and let $\Sigma$ be the part of $M_\gamma$ lying over $C$. It is clear that the part of $M_\gamma$ over $\mathbf{D}^m{}_+$ is equivalent as an $\mathbf{O}(n)$-space to the product of $\mathbf{D}^{m-3}$ with the cone over $\Sigma$. Also the restriction

$$H^1(\mathbf{D}^{m-1} - \mathbf{D}^{m-3}) \xrightarrow{\approx} H^1(\partial C)$$

clearly preserves the invariants for these actions. Thus $\Sigma \approx \Sigma_k^{2n-1}$ by 6.3, where $k = |\gamma_\alpha|$ (regarding $\gamma_\alpha \in \mathbf{Z}$). If $k \neq 1$ and $n$ is even, then $\Sigma$ is not a homology sphere, so that $M_\gamma$ is not a manifold near the given point $a \in A_\alpha$. Similarly if $k \neq 1$ and $n$ is odd, then $\Sigma^{\mathbf{O}(1)} \approx \Sigma_k^{2n-3}$ is not a homology sphere, so that $M_\gamma^{\mathbf{O}(1)}$ is not a manifold near $a$ (and thus $\mathbf{O}(n)$ does not act locally smoothly near $a$). On the other hand, if $k = 1$ then $\Sigma$ is an *orthogonal* $\mathbf{O}(n)$-space and thus $\mathbf{O}(n)$-acts *locally smoothly* on $M_\gamma$ near $a$.

It is clear that $A$ is locally flat if $\mathbf{O}(n)$ acts locally smoothly. Thus for locally smooth actions we must have that each $\gamma_\alpha = \pm 1$. Moreover, the signs are invariants up to a *simultaneous* change of sign. (Changing orientations of the $A_\alpha$ merely changes the correspondence between actions and collections of signs.)

Let us call an action over $X^m$ **amenable** if each of the $\gamma_\alpha$ is $\pm 1$. This makes sense even if $A$ is wild everywhere. We have shown the following theorem.

**10.2. Theorem** *Let $X^m$ be a compact contractible m-manifold with boundary $B^{m-1}$, $m \geq 4$, and let $A^{m-3}$ be a closed submanifold of $B^{m-1}$ with k components. Let $X^m$ have the $\mathbf{O}(n)$-orbit structure defined above. An $\mathbf{O}(n)$-space M over $X^m$ is locally smooth iff A is locally flat and the action is amenable. In general, if the action on M is amenable, then it is locally smooth everywhere except at those points of A near which A is wild (not locally flat). Moreover, there are exactly $2^{k-1}$ equivalence classes of amenable actions over $(X, B, A)$. Thus, if A is locally flat, then there are exactly $2^{k-1}$ equivalence classes of locally smooth actions over $(X, B, A)$.* ∎

*Remark* In the smooth case, and with $A$ connected, these results are due to Hsiang and Hsiang [4] and, independently, to Jänich [1]. See Chapter VI, Section 7 for that case.

Let us now briefly consider the case $m = 3$ which was excluded above. Here $A$ is a finite set of $k$ points in $B^2 = \partial X^3$ (and hence is locally flat). The invariant $\gamma$ is now in

$$H^1(B - A) \approx H_1(B, A) \approx \tilde{H}_0(A)$$

and corresponds to a collection of $k$ integers $\gamma_\alpha \in H_0(A_\alpha)$ with $\sum \gamma_\alpha = 0$ (determined only up to simultaneous change of sign). Again the action is locally smooth iff each $\gamma_\alpha = \pm 1$ and $k$ must be even since $\sum \gamma_\alpha = 0$. Thus a locally smooth action is classified *over* $X$ by the partition of $A$ (given by the signs) into two subsets of $k/2$ elements each. Since any two such partitions can be taken into one another by a homeomorphism of $(X^3, B^2, A)$ it follows that (given $k$) all these $\mathbf{O}(n)$-spaces are equivalent (*not over* $X$). Thus we have the following theorem.

**10.3. Theorem** *Let $X^3$ be a compact contractible 3-manifold with boundary $B^2$ and let A be a set of $k > 0$ points in $B^2$. Let $X^3$ have the $\mathbf{O}(n)$-orbit structure defined above. There are exactly $\binom{k-1}{k/2}$ equivalence classes over X of locally smooth $\mathbf{O}(n)$-spaces over X if k is even, and all of them are equivalent as $\mathbf{O}(n)$-spaces (not over X). For k odd, there are no locally smooth $\mathbf{O}(n)$-spaces over X.* ∎

It is of interest to consider a similar situation in which $X^3$ is now $\mathbf{D}^3{}_+ = \{(x, y, z) \in \mathbf{D}^3 \mid z \geq 0\}$, $B^2 = \mathbf{D}^3{}_+ \cap \mathbf{S}^2$, and $A$ consists of $k$ points in the interior of the 2-disk $B^2$. Let $\mathbf{D}^2 = \{(x, y, 0) \in \mathbf{D}^3{}_+\}$. We endow $\mathbf{D}^3{}_+$ with the usual $\mathbf{O}(n)$-orbit structure; that is, isotropy types $\mathbf{O}(n)$, $\mathbf{O}(n-1)$, $\mathbf{O}(n-2)$ on $A$, $B - A$, $\mathbf{D}^3{}_+ - B$, respectively. Then a locally smooth $\mathbf{O}(n)$-space over $\mathbf{D}^3{}_+$ will be a manifold *with boundary*, which is the part over $\mathbf{D}^2$. As has been remarked, the general discussion applies to this case. Now an $\mathbf{O}(n)$-space $W$ over this orbit structure is determined by an invariant (up to sign)

$$\gamma \in H^1(B - A) \approx H_1(B, A \cup \mathbf{S}^1) \approx \tilde{H}_0(A \cup \mathbf{S}^1),$$

where $\mathbf{S}^1 = \partial B = \partial \mathbf{D}^2$. Thus $\gamma$ corresponds to a set of integers $\gamma_\alpha \in H_0(A_\alpha)$ and $\delta \in H_0(\mathbf{S}^1)$ with

$$\delta = -\sum \gamma_\alpha$$

and which is determined only up to simultaneous change of sign of the $\gamma_\alpha$ *and* $\delta$. Again $\gamma_\alpha = \pm 1$ for locally smooth actions. Also $|\,\delta\,|$ is clearly the invariant for the $\mathbf{O}(n)$-space $\partial W$ (the part of $W$ over $\mathbf{D}^2$); that is,

$$\partial W \approx \Sigma^{2n-1}_{|\delta|}$$

as an $\mathbf{O}(n)$-space. If we are given $\delta = -\sum \gamma_\alpha$ and $k$ then $\delta = k_- - k_+$, where $k_+$ and $k_-$ are the numbers of $+$ and $-$ signs among the $\gamma_\alpha$. If we *allow* arbitrary equivalences [i.e., over homeomorphisms of $(\mathbf{D}^3{}_+, B^2, A)$], then $|\,\delta\,|$ and $k$ are clearly the only invariants. Moreover $k \geq |\,\delta\,|$ and $k - \delta$ is even. Thus we have the following theorem.

**10.4. Theorem** *Consider $\mathbf{O}(n)$-spaces over $(\mathbf{D}^3{}_+, B^2, A)$ as above where $A$ consists of $k \geq 0$ points in* int $B^2$. *Let $\delta \geq 0$ be an integer. If $k \geq \delta$, with $k - \delta$ even, then there is a unique (not over $\mathbf{D}^3{}_+$) locally smooth $\mathbf{O}(n)$-space over $(\mathbf{D}^3{}_+, B^2, A)$ with boundary equivalent to the $\mathbf{O}(n)$-space $\Sigma^{2n-1}_\delta$. If $k < \delta$, or if $k - \delta$ is odd, then there is no such locally smooth $\mathbf{O}(n)$-space.* ▌

***Remarks*** Note that 10.4 gives another characterization of the $\mathbf{O}(n)$-spaces $\Sigma^{2n-1}_\delta$ over $\mathbf{D}^2$. The reader can verify that an example of the case $k = \delta \geq 0$ of 10.4 is given by the $\mathbf{O}(n)$-action on the space in $\mathbf{C}^{n+1}$ defined by

$$z_0{}^k + z_1{}^2 + \cdots + z_n{}^2 = 1,$$
$$|\,z_0\,|^2 + \cdots + |\,z_n\,|^2 \leq 2.$$

Also, for $k = \delta \geq 2$ it is given by the $\mathbf{O}(n)$-action on the plumbing $P^{2n}(\mathbf{A}_{k-1})$. It is not hard to construct explicit examples for all $k \geq \delta$ with $k - \delta$ even.

## 11. KNOT MANIFOLDS

Let $\Sigma^k \subset \mathbf{S}^{k+2}$ be a *connected* orientable submanifold (possibly wild) of the $(k+2)$-sphere and let $n \geq 2$. Then by 10.2 there is a unique (to equivalence over $\mathbf{D}^{k+3}$) amenable $\mathbf{O}(n)$-space $M^{2n+k}(\Sigma^k)$ over $\mathbf{D}^{k+3}$ [with the orbit structure as in Section 10; that is, the isotropy types are $\mathbf{O}(n)$ on $\Sigma^k$, $\mathbf{O}(n-1)$ on $\mathbf{S}^{k+2} - \Sigma^k$, and $\mathbf{O}(n-2)$ on $\operatorname{int}(\mathbf{D}^{k+3})$]. Also recall from 10.2 that if $\Sigma^k$ is locally flat, then $M^{2n+k}(\Sigma^k)$ is the unique locally smooth $\mathbf{O}(n)$-manifold over $(\mathbf{D}^{k+3}, \mathbf{S}^{k+2}, \Sigma^k)$. In this section we address ourselves to the study of this space $M^{2n+k}(\Sigma^k)$ with emphasis on finding conditions under which it is a sphere.

Although we are primarily interested in the locally smooth case it will be mildly interesting (and will not present much more difficulty) to also consider some matters in the wild case. Thus let $W \subset \Sigma^k$ be the (closed) set of wild points; that is, the set of points at which $\Sigma^k$ is not locally flat. For convenience in the homology calculations we shall assume that $W \neq \Sigma^k$, although this is not really necessary.

When the data are understood we put $M = M^{2n+k}(\Sigma^k)$ and we also regard $\Sigma^k$ as the subspace $M^{\mathbf{O}(n)}$ of $M$. We let $C$ denote the part of $M$ over $\mathbf{S}^{k+2}$.

**11.1. Theorem** *If $W \neq \Sigma^k$, then $M - W$ is simply connected.*

***Proof*** For future use, we remark that we shall prove this without the assumption that $\Sigma^k$ is connected. Now $M - W$ is a *manifold* of dimension $2n + k \geq 4 + k$ and $\Sigma^k - W$ is a $k$-dimensional locally flat submanifold. It follows that $\pi_1(M - \Sigma) \to \pi_1(M - W)$ is onto. (In fact it is an isomorphism.) Now $C - \Sigma$ (being an $\mathbf{S}^{n-1}$-bundle over $\mathbf{S}^{k+2} - \Sigma$) is a locally flat $(n + k + 1)$-manifold in $M - \Sigma$. If $n \geq 3$, then $(2n + k) - (n + k + 1) = n - 1 \geq 2$ so that $\pi_1(M - C) \to \pi_1(M - \Sigma)$ is onto. However, $M - C$ is an $\mathbf{O}(n)/\mathbf{O}(n-2)$-bundle over $\mathbf{R}^{k+3}$ and thus $\pi_1(\mathbf{O}(n)/\mathbf{O}(n-2)) \to \pi_1(M - W)$ is onto (via inclusion of any principal orbit). However, there is a principal orbit in a linear disk in $M$ about any point of $\Sigma - W$ so that this inclusion clearly is trivial on the fundamental group.

Now consider the case $n = 2$. Note that

$$M - C \approx \mathbf{R}^{k+3} \times \mathbf{O}(2) \approx (\mathbf{R}^{k+3} \times \mathbf{S}^1) + (\mathbf{R}^{k+3} \times \mathbf{S}^1).$$

Also $C - \Sigma$ is an $\mathbf{O}(2)/\mathbf{O}(1)$-bundle over $\mathbf{S}^{k+2} - \Sigma^k$. The projection $\mathbf{O}(2) \to \mathbf{O}(2)/\mathbf{O}(1) \approx \mathbf{S}^1$ is a homeomorphism on each of the two components of $\mathbf{O}(2)$. It follows that the closures $X_1$ and $X_2$ in $M - \Sigma$ of the two components of $M - C$ are manifolds, each of which has interior homeomorphic to $\mathbf{R}^{k+3} \times \mathbf{S}^1$ and with common boundary $C - \Sigma$. Since $C - \Sigma$, being an $\mathbf{S}^1$-bundle over the connected set $\mathbf{S}^{k+2} - \Sigma^k$, is connected, it follows from the Van Kampen Theorem that $\pi_1(M - \Sigma)$ is generated by the images of the two copies of $\pi_1(\mathbf{R}^{k+3} \times \mathbf{S}^1) \approx \pi_1(\mathbf{S}^1)$. Since these circles are the components of a principal orbit $\mathbf{O}(2)$ it follows as before that this fundamental group goes trivially into $\pi_1(M - W)$. ∎

**11.2. Theorem** *Assume that $W \neq \Sigma^k$. If $\Sigma^k$ is a* mod 2 *(Čech) cohomology sphere, then $M^{2n+k}(\Sigma^k)$ is a* mod 2 *cohomology sphere. If $\Sigma^k$ is an integral cohomology sphere and $n$ is even, then $M^{2n+k}(\Sigma^k)$ is an integral cohomology sphere.*

*Proof* To simplify notation we shall use cohomology with compact supports. Thus, for a locally compact space $U$, $\check{H}_c^*(U) = \check{H}^*(U_+, \infty)$, and this may also be defined directly by using Alexander–Spanier cohomology or sheaf cohomology; see Spanier [1] and Bredon [13]. For a compact pair $(X, A)$ there is a canonical isomorphism $\check{H}^*(X, A) \approx \check{H}_c^*(X - A)$. The coefficients will be understood to be in $\mathbf{Z}_2$ for the first case of the theorem and in $\mathbf{Z}$ for the second case.

Let $x \in \Sigma^k - W$ and let $U$ be an open $(2n + k)$-disk neighborhood in $M$ of $x$ on which $\mathbf{O}(n)$ acts orthogonally. Let $U^* \subset \mathbf{D}^{k+3}$ be the orbit space of $U$. Then $U^*$ is a open half disk neighborhood of $x^* \in \Sigma^k - W \subset \mathbf{S}^{k+2} = \partial\mathbf{D}^{k+3}$ in $\mathbf{D}^{k+3}$.

Now $U - C$ is an $\mathbf{O}(n)/\mathbf{O}(n - 2)$-bundle over $U^* - \mathbf{S}^{k+2} \approx \mathbf{R}^{k+3}$ contained in the $\mathbf{O}(n)/\mathbf{O}(n - 2)$-bundle $M - C$ over $\mathbf{D}^{k+3} - \mathbf{S}^{k+2} \approx \mathbf{R}^{k+3}$. Since these are trivial bundles it is clear that the inclusion induces an isomorphism

$$\check{H}_c^*(U - C) \xrightarrow{\approx} \check{H}_c^*(M - C)$$

since, by Poincaré duality, this is equivalent to the homomorphism

$$H_*(U - C) \to H_*(M - C)$$

induced by the inclusion $U - C \subset M - C$ (which is a homotopy equivalence). Let $y \in \Sigma$ be another point, and assume, as we may, that $y \notin U$. Since $U \cap \Sigma^k$ is an open $k$-cell and since $\Sigma^k - \{y\}$ is acyclic, we have that

$$\check{H}_c^*(U \cap \Sigma) \xrightarrow{\approx} \check{H}_c^*(\Sigma - \{y\}).$$

Now $C - \Sigma$ is an $\mathbf{S}^{n-1} = \mathbf{O}(n)/\mathbf{O}(n-1)$-bundle over $\mathbf{S}^{k+2} - \Sigma$ with structure group $\mathbf{O}(1) \approx N(\mathbf{O}(n-1))/\mathbf{O}(n-1)$ which acts on $\mathbf{S}^{n-1}$ via the antipodal map. If $n$ is even, then the antipodal map preserves orientation on $\mathbf{S}^{n-1}$ so that this is an orientable bundle, and when coefficients are in $\mathbf{Z}_2$ it is orientable in the sense of cohomology for arbitrary $n$. (We remark for later reference that this is the only use of the assumption that $n$ is even when coefficients are in $\mathbf{Z}$.) In the neighborhood of $x$ we have the subbundle

$$U \cap C - \Sigma \to U^* \cap \mathbf{S}^{k+2} - \Sigma \approx \mathbf{R}^{k+2} - \mathbf{R}^k$$

of this, and the inclusion gives a map of Gysin sequences

$$\begin{array}{ccccccc} \cdots \to \check{H}_c^i(\mathbf{R}^{k+2} - \mathbf{R}^k) & \to & \check{H}_c^i(U \cap C - \Sigma) & \to & \check{H}_c^{i-n+1}(\mathbf{R}^{k+2} - \mathbf{R}^k) \to \cdots \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \to \check{H}_c^i(\mathbf{S}^{k+2} - \Sigma) & \longrightarrow & \check{H}_c^i(C - \Sigma) & \longrightarrow & \check{H}_c^{i-n+1}(\mathbf{S}^{k+2} - \Sigma) \to \cdots \end{array}$$

The homomorphism $\check{H}_c^*(\mathbf{R}^{k+2} - \mathbf{R}^k) \to \check{H}_c^*(\mathbf{S}^{k+2} - \Sigma)$ is an isomorphism by Poincaré–Alexander duality, and hence

$$\check{H}_c^*(U \cap C - \Sigma) \xrightarrow{\approx} \check{H}_c^*(C - \Sigma)$$

by the 5-lemma.

Putting these facts together, we see from the diagram

$$\begin{array}{ccccccc} \cdots \to \check{H}_c^i(U \cap C - \Sigma) & \to & \check{H}_c^i(U \cap C) & \to & \check{H}_c^i(U \cap \Sigma) \to \cdots \\ \downarrow \approx & & \downarrow & & \downarrow \approx \\ \cdots \longrightarrow \check{H}_c^i(C - \Sigma) & \longrightarrow & \check{H}_c^i(C - \{y\}) & \to & \check{H}_c^i(\Sigma - \{y\}) \to \cdots \end{array}$$

that

$$\check{H}_c^*(U \cap C) \xrightarrow{\approx} \check{H}_c^*(C - \{y\}).$$

Similarly, the diagram

$$\begin{array}{ccccccc} \cdots \to \check{H}_c^i(U - C) & \longrightarrow & \check{H}_c^i(U) & \longrightarrow & \check{H}_c^i(U \cap C) \to \cdots \\ \downarrow \approx & & \downarrow & & \downarrow \approx \\ \cdots \to \check{H}_c^i(M - C) & \to & \check{H}_c^i(M - \{y\}) & \to & \check{H}_c^i(C - \{y\}) \to \cdots \end{array}$$

shows that

$$\check{H}_c^*(U) \xrightarrow{\approx} \check{H}_c^*(M - \{y\}).$$

However, $U \approx \mathbf{R}^{2n+k}$ and it follows that $\check{H}^*(M) \approx H^*(\mathbf{S}^{2n+k})$. ∎

As a converse to this result we note the following fact.

**11.3. Theorem** *If $M^{2n+k}(\Sigma^k)$ is a* mod 2 *cohomology sphere then $\Sigma^k$ is a* mod 2 *cohomology sphere. If n is even and $M^{2n+k}(\Sigma^k)$ is an integral cohomology sphere, then $\Sigma^k$ is an integral cohomology sphere.*

***Proof*** Let $T$ be a maximal torus of $\mathbf{O}(n)$. If $n$ is even, then $T$ is not conjugate to a subgroup of $\mathbf{O}(n-1)$, so that $M^{\mathbf{O}(n)} = M^T$. For arbitrary $n$, let $L = \mathbf{O}(1) \times \mathbf{O}(1) \times \cdots \times \mathbf{O}(1) \subset \mathbf{O}(n)$ be the subgroup of diagonal matrices (the $\mathbf{Z}_2$-maximal torus). Then $L$ is not conjugate to a subgroup of $\mathbf{O}(n-1)$, so that $M^{\mathbf{O}(n)} = M^L$. The result follows from III.10.2 and III.7.11. ▌

**11.4. Theorem** *Let n be even and assume that $\Sigma^k$ is an integral cohomology k-sphere with at most one wild point in $\mathbf{S}^{k+2}$ (or with finitely many wild points when $k = 1$). Then $M^{2n+k}(\Sigma^k) \approx \mathbf{S}^{2n+k}$.*

***Proof*** If there are no wild points, then this follows from 11.1, 11.2, and the generalized Poincaré conjecture (see Connell [1]) since $2n + k \geq 5$. Thus assume that $w \in \Sigma^k$ is the unique wild point. By 11.1 we have $\pi_1(M - \{w\}) = 1$. Moreover, the proof of 11.1 shows that the part of $M - \{w\}$ over $U - \{w^*\}$ is simply connected, where $U$ is any disk neighborhood of $w^*$ in $\mathbf{D}^{k+3}$ (since connectivity of $\Sigma^k$ was not used in the proof of 11.1). Thus by 11.1, and 11.2, $M - \{w\}$ is a contractible open manifold which is simply connected at infinity. Thus $M - \{w\} \approx \mathbf{R}^{2n+k}$ by a theorem of Siebenmann [1], and it follows that $M \approx \mathbf{S}^{2n+k}$. If $k = 1$ and there are finitely many wild points of $\Sigma$ it clearly suffices to show that $M$ is a manifold about each of these points $w$. However, $\Sigma$ can obviously be altered, without changing it near a given point $w$, so that it is wild only at $w$. This changes $M$ only away from $w$, so that the result follows. ▌

For example, there is an $\mathbf{O}(2)$-action on $\mathbf{S}^5$ corresponding to the wild knot illustrated in Figure V-9. We remark that for wild knots which are infinite sums of tame knots, such as that in Figure V-10, the corresponding $\mathbf{O}(n)$-spaces are infinite connected sums of locally smooth $\mathbf{O}(n)$-spaces. For example, the cloverleaf knot [which is the torus knot (3,2)] corresponds by Section 10 to the $\mathbf{O}(n)$-manifold $\Sigma_3^{2n+1} \approx W_3^{2n+1}$ (see 9.2), where $\mathbf{O}(n)$ acts as the standard subgroup of $\mathbf{O}(n+1)$. Clearly the wild knot in Figure V-10 corresponds to an infinite equivariant connected sum, about fixed points, of $\Sigma_3^{2n+1}$. If $n = 2$, then it is of interest to note that the subgroup $\mathbf{SO}(2)$ acts *linearly* on $\Sigma_3^5$ (see Exercise 2) and has the same fixed set as

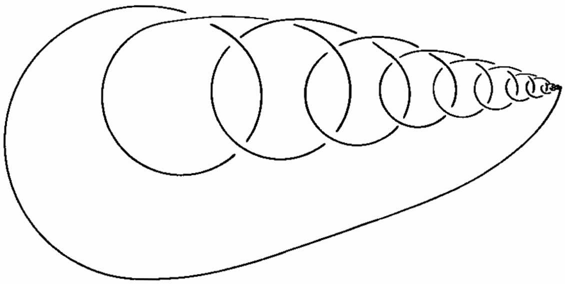


FIGURE V-9

does $\mathbf{O}(2)$. The infinite connected sum is still a linear $\mathbf{SO}(2)$-action on $\mathbf{S}^5$. It is easily seen that for this $\mathbf{O}(2)$-action on $\mathbf{S}^5$ (corresponding to Figure V-10), we have that $\mathbf{S}^5/\mathbf{SO}(2) \approx \mathbf{S}^4$ and is just the double of $\mathbf{S}^5/\mathbf{O}(2) \approx \mathbf{D}^4$. Thus the wild knot $\Sigma^1$ of Figure V-10 is tame in $\mathbf{S}^4$, since it corresponds to the fixed set, in the orbit space, of a *linear* $\mathbf{SO}(2)$-action on $\mathbf{S}^5$. (Of course, the tameness of $\Sigma^1$ in $\mathbf{S}^4$ could easily be seen directly.)

Now we turn our attention to the computation of the cohomology of $M^{2n+1}(\Sigma^1)$ when $n$ is *odd* and $\Sigma = \Sigma^1$ is a tame knot in $\mathbf{S}^3 \subset \mathbf{D}^4$. Let $M^3(\Sigma)$ denote the double covering of $\mathbf{S}^3$ branched at $\Sigma$. (This is an $\mathbf{O}(1)$-space over $\mathbf{S}^3$ and not over $\mathbf{D}^4$.)

**11.5. Theorem** *Let $n$ be odd and $k = 1$. Assume that $\Sigma$ is tame. Then $H^i(M^{2n+1}(\Sigma);\ \mathbf{Z}) = 0$ for $i \neq 0,\ n+1$, and $2n+1$. Also*

$$H^{n+1}(M^{2n+1}(\Sigma); \mathbf{Z}) \approx H^2(M^3(\Sigma); \mathbf{Z}).$$

*Proof* We first remark that, by 11.2, $H^{n+1}(M^{2n+1}(\Sigma);\ \mathbf{Z})$ is finite, so that the result will imply that

$$H_n(M^{2n+1}(\Sigma); \mathbf{Z}) \approx H^{n+1}(M^{2n+1}(\Sigma); \mathbf{Z}) \approx H^2(M^3(\Sigma); \mathbf{Z}) \approx H_1(M^3(\Sigma); \mathbf{Z}).$$

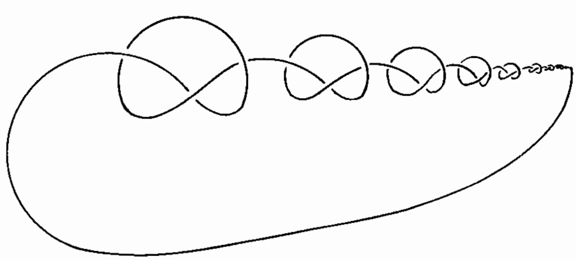


FIGURE V-10

Also, the order of this group is known to be $\Delta(-1)$, where $\Delta(t)$ is the Alexander polynomial of the knot $\Sigma$.

To do the computation we first follow the lines of the proof of 11.2. The only place at which we used the hypothesis that $n$ was even was the fact that the $\mathbf{S}^{n-1}$-bundle $C - \Sigma$ over $\mathbf{S}^3 - \Sigma$ is orientable for $n$ even. Thus the Gysin sequence with ordinary coefficients could be used in that case. For *odd n* this bundle is nonorientable, but we still have a Gysin sequence diagram

$$\begin{array}{ccccccccc}
\to \check{H}_c^{i}(\mathbf{R}^3 - \mathbf{R}^1; \mathbf{Z}^\tau) & \to & \check{H}_c^{i+n}(\mathbf{R}^3 - \mathbf{R}^1) & \to & \check{H}_c^{i+n}(U \cap C - \Sigma) & \to & \check{H}_c^{i+1}(\mathbf{R}^3 - \mathbf{R}^1; \mathbf{Z}^\tau) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to \check{H}_c^{i}(\mathbf{S}^3 - \Sigma; \mathbf{Z}^\tau) & \to & \check{H}_c^{i+n}(\mathbf{S}^3 - \Sigma) & \to & \check{H}_c^{i+n}(C - \Sigma) & \to & \check{H}_c^{i+1}(\mathbf{S}^3 - \Sigma; \mathbf{Z}^\tau) \to,
\end{array}$$

where $\mathbf{Z}^\tau$ denotes twisted integer coefficients and elsewhere the coefficients are in $\mathbf{Z}$. (The twisting is clearly via the homomorphism $\pi_1(\mathbf{S}^3 - \Sigma) \to H_1(\mathbf{S}^3 - \Sigma; \mathbf{Z}) \xrightarrow{\approx} \mathbf{Z} \to \mathbf{Z}_2 \approx \operatorname{Aut}(\mathbf{Z})$.)

Now $\check{H}_c^0(\mathbf{R}^3 - \mathbf{R}^1; \mathbf{Z}^\tau) \approx 0 \approx \check{H}_c^0(\mathbf{S}^3 - \Sigma; \mathbf{Z}^\tau)$ since these are the sections with compact support of the bundle $\mathbf{Z}^\tau$ of coefficients. Also, by Poincaré duality (see Bredon [13]),

$$\check{H}_c^1(\mathbf{R}^3 - \mathbf{R}^1; \mathbf{Z}^\tau) \approx H_2(\mathbf{R}^3 - \mathbf{R}^1; \mathbf{Z}^\tau) \approx H_2(\mathbf{S}^1; \mathbf{Z}^\tau) = 0.$$

Thus $\check{H}_c^1(\mathbf{R}^3 - \mathbf{R}^1; \mathbf{Z}^\tau) \to \check{H}_c^1(\mathbf{S}^3 - \Sigma; \mathbf{Z}^\tau)$ is at least a monomorphism.

Using the 5-lemma on the above Gysin diagram, we obtain that $\check{H}_c^j(U \cap C - \Sigma) \to \check{H}_c^j(C - \Sigma)$ is an isomorphism for $j < n$ and is a monomorphism for $j = n$.

From the diagrams used in the proof of 11.2 we obtain that $\check{H}_c^j(U \cap C) \to \check{H}_c^j(C - \{y\})$ is an isomorphism for $j < n$ and is a monomorphism for $j = n$. Similarly we conclude that $\check{H}_c^j(U) \to \check{H}_c^j(M - \{y\})$ is an isomorphism for $j < n$, and hence that $\check{H}^j(M) = 0$ for $0 < j < n$. By Poincaré duality, and the Universal Coefficient Theorem, it follows that $\check{H}^j(M) = 0$ for $j \neq 0, n, n+1, 2n+1$. By 11.2, $M$ is a mod 2 cohomology sphere, so that this group must be finite of odd order for $j = n, n+1$. Again the Universal Coefficient Theorem and Poincaré duality imply that it must vanish for $j = n$. Let $x$, $y$ be two distinct points in $\Sigma$. Then we have that

$$\check{H}^j(M - \{y\}, \{x\}) = 0 \qquad \text{for} \quad j \neq n + 1,$$

and it remains to find this group for $j = n + 1$.

Now let $T$ be the circle subgroup of $\mathbf{O}(n)$ which is the diagonal of $1 \times \mathbf{SO}(2) \times \mathbf{SO}(2) \times \cdots \times \mathbf{SO}(2) \subset \mathbf{O}(1) \times \mathbf{O}(2) \times \cdots \times \mathbf{O}(2)$

$\subset \mathbf{O}(n)$ [$(n-1)/2$ copies of $\mathbf{SO}(2)$]. By considering the action of this on the Stiefel manifold $\mathbf{O}(n)/\mathbf{O}(n-2)$ of 2-frames in $\mathbf{R}^n$ and on the $(n-1)$-sphere $\mathbf{O}(n)/\mathbf{O}(n-1)$, we see that $T$ acts *semifreely* on each $\mathbf{O}(n)$-orbit in $M$ and hence semifreely on $M$. Also, $T$ has no fixed points on principal orbits and exactly two fixed points (antipodal) on $\mathbf{S}^{n-1}$ which are interchanged by the action of $\mathbf{O}(1)$ on $M^T$. It follows that $M^T \approx M^3(\Sigma)$. Also, since $T$ acts semifreely, there is the Smith-Gysin sequence with integral coefficients

$$\cdots \to \check{H}^i(X^*, X^T) \to \check{H}^i(X, \{x\}) \to \check{H}^{i-1}(X^*, X^T) \oplus \check{H}^i(X^T, \{x\}) \to \check{H}^{i+1}(X^*, X^T) \to \cdots$$

(see III.(10.6)), where $X = M - \{y\}$ and $X^* = X/T$. As we have seen, $\check{H}^i(X, \{x\})$ vanishes except for $i = n+1$, and similarly $\check{H}^i(X^T, \{x\}) = \check{H}^i(M^3(\Sigma) - \{y\}, \{x\}) = 0$ for $i \neq 2$. Thus, an obvious induction, using the Smith–Gysin sequence, provides the isomorphisms

$$\check{H}^{n+1}(X, \{x\}) \xrightarrow{\approx} \check{H}^n(X^*, X^T) \xleftarrow{\approx} \check{H}^{n-2}(X^*, X^T) \xleftarrow{\approx} \cdots \xleftarrow{\approx} \check{H}^3(X^*, X^T) \xleftarrow{\approx} \check{H}^2(X^T, \{x\})$$

and this gives the desired isomorphism $\check{H}^{n+1}(M^{2n+1}(\Sigma)) \approx \check{H}^2(M^3(\Sigma))$.

*Remarks* This relationship between knots and $\mathbf{O}(n)$-manifolds was first studied by Jänich [1] and by Hsiang and Hsiang [4] in the differentiable case. The homology computations for $k = 1$ were done by Hirzebruch [2] by a different method. Hirzebruch [2] also determined the differentiable structure of $M^{2n+1}(\Sigma^1)$ when $n$ is even and $\Sigma$ is smooth. For the case $k > 1$ see Bredon [26].

## EXERCISES FOR CHAPTER V

**1.** Consider $\mathbf{O}(n)$-spaces over $\mathbf{D}^3 \times \mathbf{S}^1$ with isotropy type $\mathbf{O}(n-1)$ on $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{O}(n-2)$ on the interior. Show that there are precisely four such actions, up to equivalence over $\mathbf{D}^3 \times \mathbf{S}^1$, and that they are distinguished from one another by the triviality or nontriviality of the regular and singular bundles $\varrho$ and $\sigma$. Also show, however, that the set $\pi_0\Gamma(\eta \mid B_1)$, as in 6.1, of homotopy classes of reductions of the structure group of $\varrho \mid B_1$ to $N(\mathbf{O}(n-2)) \cap N(\mathbf{O}(n-1))/\mathbf{O}(n-2)$ is infinite.

**2.** Consider the $\mathbf{O}(3)$-spaces $\Sigma_k^5$ of Chapter I, Section 7 (recall that $\Sigma_k^5 \approx W_k^5$ of 9.2) for $k$ *odd*. Show that the subgroup $\mathbf{SO}(3)$ is transitive on each $\mathbf{O}(3)$-orbit and hence that its orbit space is the same as that of $\mathbf{O}(3)$ (which is $\mathbf{D}^2$). Show, however, that all of these $\mathbf{SO}(3)$-actions are equivalent to the linear action of $\mathbf{SO}(3)$ on $\Sigma_1^5 \approx \mathbf{S}^5$. Readers familiar with the exceptional Lie group $\mathbf{G}_2$ should similarly show that the action of $\mathbf{G}_2 \subset \mathbf{O}(7)$ on $\Sigma_k^{13}$ for $k$ *odd* has the same orbits as does $\mathbf{O}(7)$, but is equivalent to the linear action on $\Sigma_1^{13} \approx \mathbf{S}^{13}$. (In particular, this shows that $\Sigma_k^5 \approx \mathbf{S}^5$ and $\Sigma_k^{13} \approx \mathbf{S}^{13}$ for $k$ odd without use of the generalized Poincaré "conjecture.")

**3.** Let $A^{m-3}$ be a submanifold of $\mathbf{S}^{m-1}$. Consider $\mathbf{O}(n)$-spaces $W$ over $\mathbf{D}^m$ with isotropy types $\mathbf{O}(n)$ on $A^{m-3}$, $\mathbf{O}(n-1)$ on $\mathbf{S}^{m-1} - A^{m-3}$, and $\mathbf{O}(n-2)$ on $\mathbf{D}^m - \mathbf{S}^{m-1}$. If $A^{m-3}$ is nonorientable, show that no such $\mathbf{O}(n)$-space is locally smooth.

**4.** Consider the $\mathbf{O}(n)$-space $W_0^{2n-1}$ defined to be the space of all points $(z_0, z_1, \ldots, z_n) \in \mathbf{C}^{n+1}$ on the variety

$$1 + z_1^2 + \cdots + z_n^2 = 0$$

and on the sphere

$$|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 2.$$

Put $z_j = x_j + iy_j$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. Show that the map $(z_0, x, y) \mapsto (1, 0, y/\|y\|)$ defines an equivariant retraction of $W_0^{2n-1}$ onto an orbit. Conclude that

$$W_0^{2n-1} \approx \mathbf{O}(n) \times_{\mathbf{O}(n-1)} \mathbf{S}^n$$

as $\mathbf{O}(n)$-spaces, where $\mathbf{O}(n-1)$ acts on $\mathbf{S}^n$ through the inclusion in $\mathbf{O}(n+1)$. Also show that $W_0^{2n-1} \approx \mathbf{S}^{n-1} \times \mathbf{S}^n \approx \Sigma_0^{2n-1}$ as $\mathbf{O}(n)$-spaces, where $\mathbf{O}(n)$ acts diagonally on $\mathbf{S}^{n-1} \times \mathbf{S}^n$.

**5.** Verify the statements following Theorem 10.4.

**6.** Let $X^m$ be a contractible compact manifold with boundary $B^{m-1}$. Let $\Sigma^{m-4}$ be a locally flat submanifold of $B^{m-1}$ which is an integral homology sphere. Show that there is a unique locally smooth $\mathbf{U}(n)$-space over $X^m$ with isotropy types $\mathbf{U}(n)$ on $\Sigma^{m-4}$, $\mathbf{U}(n-1)$ on $B^{m-1} - \Sigma^{m-4}$, and $\mathbf{U}(n-2)$ on $X^m - B^{m-1}$. Also prove the corresponding result for $\mathbf{Sp}(n)$-spaces over $X^m$, where now the fixed point set is an integral homology sphere $\Sigma^{m-6}$ in $B^{m-1}$. (These results would be false in the smooth case.)

**7.** Consider the $\mathbf{O}(n+1)$-spaces $X_k = \mathbf{O}(n+1) \times_{\mathbf{O}(n)} \Sigma_k^{2n-1}$, where $\Sigma_k^{2n-1}$ is the $\mathbf{O}(n)$-space defined in Chapter I, Section 7. Show that the $X_k$ are mutually equivalent for all odd $k$ (respectively, for all even $k$). In particular, show that $X_k \approx \mathbf{S}^n \times \mathbf{S}^{2n-1}$ for $k$ odd.

**8.** Let $G = \mathbf{O}(n+1)$, $K = \mathbf{O}(n-1)$, and $H = \mathbf{O}(n-2)$, $n \geq 2$. Consider $G$-spaces over $\mathbf{D}^{k+1}$ with orbit type $G/K$ over $\mathbf{S}^k$ and $G/H$ over $\operatorname{int} \mathbf{D}^{k+1}$. For $k \geq 3$ show that such $G$-spaces over $\mathbf{D}^{k+1}$ are classified by the elements of $\pi_k(\mathbf{S}^2)$, while for $k = 2$ they are classified by $\mathbf{Z}^+$. If $X$ is any such $G$-space for $k \geq 2$, show that the $\mathbf{O}(n+2)$-space $\mathbf{O}(n+2) \times_{\mathbf{O}(n+1)} X$ is equivalent to $\mathbf{O}(n+2) \times_{\mathbf{O}(n-1)} \mathbf{S}^{n+k-1}$, where $\mathbf{O}(n-1)$ acts on $\mathbf{S}^{n+k-1} \subset \mathbf{R}^{n+k} = \mathbf{R}^{n-1} \times \mathbf{R}^{k+1}$ via the standard representation plus a trivial one.

**9.** For $n \geq 2$ consider the diagonal action of $\mathbf{U}(n)$ on $\mathbf{S}^{4n-1} \subset \mathbf{C}^n \times \mathbf{C}^n$, which has $\mathbf{D}^3$ as orbit space. Show that the equivariant homotopy classes over $\mathbf{D}^3$ of self-equivalences of this $\mathbf{U}(n)$-space are in one–one correspondence with $\pi_3(\mathbf{U}(2)) \approx \mathbf{Z}$.

# CHAPTER VI

## SMOOTH ACTIONS

In this chapter we assume a knowledge of elementary differential geometry. Some of the needed material is reviewed in Section 1 in order to set up terminology and notation, but this material is not intended as a self-contained introduction to differential geometry.

In the first four sections we prove some basic facts concerning the differential topology of smooth, compact, Lie group actions. For example, we prove an existence and uniqueness theorem for invariant tubular neighborhoods in Section 2. A general theorem is proved in Section 3 which allows certain types of isotopies to be replaced by equivariant ones. In Section 4 we prove a smooth equivariant embedding theorem, due to Palais and Mostow, and we apply it to prove a smooth equivariant approximation theorem for continuous equivariant maps and homotopies.

In Section 5 the induced functional structure on a very simple type of orbit space is studied in detail. This is used in Section 6 to prove a smooth version of the classification theorem for actions over a manifold with boundary, studied in Chapter V, Sections 5–7 in the topological case. This is applied in Section 7 to obtain the results of Jänich on knot manifolds, the topological case of which was discussed in Chapter V, Sections 10 and 11.

In the last four sections we assume considerably more background in differential topology. Involutions on spheres are studied in Section 8 using the Eells–Kuiper invariant, the results of Section 7, and an invariant of Browder and Petrie (of which we construct a modified version). Some results on semifree circle actions on spheres and disks are proved in Section 9. Equivariant $K$-theory is used in Sections 10 and 11 to compare representations at two fixed points in a smooth action.

### 1. FUNCTIONAL STRUCTURES AND SMOOTH ACTIONS

In this section we shall review some elementary definitions and a few facts from differential geometry. We assume that the reader is already familiar with this material and it is given here only to maintain continuity, to establish notation, and to single out some special items of importance to us. Many proofs and explanatory material will be omitted, so that the reader should understand that this section is *not* intended as an introduction to differential geometry or differential topology.

Following Hochschild [1] we define a **functional structure** on a topological space $X$ to be a function $\mathscr{F}_X$ which assigns to each open set $U \subset X$ a subalgebra $\mathscr{F}_X(U)$ of the algebra of all continuous real-valued functions on $U$ and which satisfies the following conditions:

(1) The subalgebra $\mathscr{F}_X(U)$ contains the constant functions on $U$.

(2) If $V \subset U$, then the restriction of a function $f \in \mathscr{F}_X(U)$ to $V$ is in $\mathscr{F}_X(V)$.

(3) If $U = \bigcup U_\alpha$ is an arbitrary union of open sets, then a function $f\colon U \to \mathbf{R}$ is in $\mathscr{F}_X(U)$ iff the restriction of $f$ to each $U_\alpha$ is in $\mathscr{F}_X(U_\alpha)$.

In other words, $\mathscr{F}_X$ is a subsheaf of algebras of the sheaf of germs of continuous real-valued functions on $X$ containing the constant subsheaf.

A **functionally structured space** $(X, \mathscr{F}_X)$ is a space $X$ together with a functional structure $\mathscr{F}_X$ on $X$. A **morphism** $\varphi\colon (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ between functionally structured spaces is a map $\varphi\colon X \to Y$ such that composition

$$f \mapsto f \circ \varphi$$

with $\varphi$ carries $\mathscr{F}_Y(U)$ into $\mathscr{F}_X(\varphi^{-1}U)$ for each open $U \subset Y$. An **isomorphism** is, of course, a morphism having an inverse which is a morphism.

If $\varphi\colon X \to Y$ is a map and $\mathscr{F}$ is a functional structure on $X$, then we may define a functional structure $\varphi_*\mathscr{F}$ on $Y$ by letting $(\varphi_*\mathscr{F})(U)$ consist of all those continuous functions $f\colon U \to \mathbf{R}$ such that $f \circ \varphi \in \mathscr{F}(\varphi^{-1}U)$. This structure $\varphi_*\mathscr{F}$ is called the "functional structure on $Y$ **induced** by $\varphi$ from $\mathscr{F}$." Clearly $\varphi\colon (X, \mathscr{F}) \to (Y, \varphi_*\mathscr{F})$ is a morphism.

If $\varphi\colon X \to Y$ is a map and $\mathscr{S}$ is a functional structure on $Y$, then we can define a functional structure $\varphi^*\mathscr{S}$ on $X$ by letting $(\varphi^*\mathscr{S})(U)$ consist of those functions $f\colon U \to \mathbf{R}$ such that $f$ is *locally* the composition of $\varphi$ with a member of $\mathscr{S}$. (That is, for each $x \in U$ there is a neighborhood $U_x \subset U$ of $x$, an open set $V \subset Y$ containing $\varphi(U_x)$, and a member $g \in \mathscr{S}(V)$ such that $f = g \circ \varphi$ on $U_x$.) Then $\varphi\colon (X, \varphi^*\mathscr{S}) \to (Y, \mathscr{S})$ is a morphism. We call $\varphi^*\mathscr{S}$ the "functional structure on $X$ **induced** by $\varphi$ from $\mathscr{S}$." Of main interest to us will be the case of a subspace. If $A \subset X$ is a subspace and $\mathscr{F}$ is a functional structure on $X$, then we let $\mathscr{F}_A$ denote $i^*\mathscr{F}$ where $i\colon A \to X$ is the inclusion. Thus for $U \subset X$ open, $\mathscr{F}_A(A \cap U)$ consists of those functions $f\colon A \cap U \to \mathbf{R}$ such that, for each $a \in A \cap U$, there is a neighborhood

$V$ of $a$ in $X$ and an extension of $f|A \cap V$ to a member of $\mathscr{F}(V)$. If $A$ is *open*, then $\mathscr{F}_A(A \cap U) = \mathscr{F}(A \cap U)$, so that $\mathscr{F}_A$ is the restriction of $\mathscr{F}$ to open sets of $U$.

The $C^\infty$ functional structure on $\mathbf{R}^n$ assigns to $U \subset \mathbf{R}^n$ the algebra of $C^\infty$ real-valued functions on $U$. A **$C^\infty$ n-manifold** (or a **smooth $n$-manifold**) is a paracompact Hausdorff space $M^n$ together with a functional structure $\mathscr{F}$ such that $(M^n, \mathscr{F})$ is locally isomorphic to $(\mathbf{R}^n, C^\infty)$; that is, each point $x \in M$ has an open neighborhood $U$ such that $(U, \mathscr{F}_U)$ is isomorphic to $(V, C_V^\infty)$ where $V \subset \mathbf{R}^n$ is some open set. A map $\varphi\colon U \to V$ realizing such an isomorphism is called a **chart**, and $\mathscr{F}$ is called a **$C^\infty$ structure** on $M$. For smooth manifolds $M^m$ and $N^n$ a morphism $\varphi\colon M \to N$ of their given $C^\infty$ structures is called simply a **smooth map** (or a $C^\infty$ map; or a differentiable map). An isomorphism $\varphi$ is called a **diffeomorphism**.

Consider the half space $\mathbf{R}^n{}_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 \geq 0\}$. As a subspace of $\mathbf{R}^n$, $\mathbf{R}^n{}_+$ inherits a functional structure from $C^\infty$, which will also be denoted by $C^\infty$. Then a **$C^\infty$ manifold with boundary** is a functionally structured paracompact Hausdorff space $(M, \mathscr{F})$ which is locally isomorphic to $(\mathbf{R}^n{}_+, C^\infty)$. The boundary (possibly empty) consists of those points that are taken by charts to points on the hyperplane $x_1 = 0$, and this is a smooth $(n-1)$-manifold with the induced structure. Unless otherwise specified in this chapter, the word "manifold" will refer to a $C^\infty$ manifold without boundary.

If $M^m$ and $N^n$ are $C^\infty$ manifolds (without boundary), then there is a unique $C^\infty$ structure on $M \times N$ such that $\varphi \times \psi\colon U \times V \to \mathbf{R}^{m+n}$ is a chart of $M \times N$ whenever $\varphi\colon U \to \mathbf{R}^m$ and $\psi\colon V \to \mathbf{R}^n$ are charts of $M$ and $N$, respectively. This structure is understood when products of smooth manifolds are considered.

Recall that a Lie group $G$ has a *unique* $C^\infty$ structure for which the map $G \times G \to G$ taking $(g, h) \mapsto gh^{-1}$ is smooth. By a **smooth action** of a Lie group $G$ on a smooth manifold $M$ we mean an action

$$\Theta\colon G \times M \to M$$

which is a smooth map. (Actually it suffices that each $\theta_g\colon M \to M$ is smooth, but this is difficult to prove and we shall not consider this question; see Montgomery and Zippin [4].)

Of course, the basic example of a smooth action is the canonical action of the general linear group $\mathbf{Gl}(n)$ on $\mathbf{R}^n$. Now $\mathbf{Gl}(n)$ is an open subset of $\mathbf{R}^{n^2}$ (all $n \times n$ matrices over $\mathbf{R}$) and inherits its $C^\infty$ structure therefrom. Smoothness of $\Theta\colon \mathbf{Gl}(n) \times \mathbf{R}^n \to \mathbf{R}^n$ is clear. A homomorphism of Lie groups is automatically smooth (Chapter 0, Section 5) and it follows that the action $G \times \mathbf{R}^n \to \mathbf{R}^n$ given by any representation $G \to \mathbf{Gl}(n)$ is smooth. Other examples are found in earlier chapters. For instance, the actions of $\mathbf{O}(n)$ given in Chapter I, Section 7 and those on the Brieskorn varieties in Chapter V, Section 9 are smooth.

A $\mathbf{C}^\infty$ **fiber bundle** is defined as in Chapter II, Section 1, where now the base $B$ and fiber $F$ are $C^\infty$ manifolds, the structure group $K$ is a Lie group (this may sometimes be weakened) acting smoothly on $F$, and the transition functions $U \to K$ are required to be smooth. In this case, the total space $X$ of the bundle has a canonical $C^\infty$ structure defined by requiring the charts $\varphi\colon F \times U \to p^{-1}(U)$ to be isomorphisms of functional structures.

If $\gamma\colon \mathbf{R} \to M$ is a smooth curve in the manifold $M$ with $\gamma(0) = p$, and if $f$ is a $C^\infty$ function defined on an open neighborhood of $p$, then we can define the **directional derivative** of $f$ at $p$ along $\gamma$ to be

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t))_{t=0},$$

where $D_\gamma$ is also called the **tangent vector** to $\gamma$ at $p$. Then $D = D_\gamma$ is a derivation of the algebra of germs of smooth real-valued functions at $p$; that is, $D$ satisfies

(1) $D(af + bg) = aD(f) + bD(g)$,
(2) $D(fg) = f(p)D(g) + D(f)g(p)$,

where $a, b \in \mathbf{R}$ and $f, g$ are smooth real-valued functions defined near $p$.

In terms of a local coordinate system $x_1, \ldots, x_n$ about $p$ we can consider $f$ as a function of $x_1, \ldots, x_n$ and we can write the curve $\gamma$ as $t \mapsto (x_1(t), \ldots, x_n(t))$. Then

$$D_\gamma(f) = \sum \frac{\partial f}{\partial x_i}\bigg|_p \frac{dx_i}{dt}\bigg|_{t=0}$$

so that

$$D_\gamma = \sum a_i \frac{\partial}{\partial x_i}\bigg|_p,$$

where $a_i = dx_i/dt\,|_{t=0}$. Also $\partial/\partial x_i\,|_p = D_{\mu_i}$, where $\mu_i$ is the curve given in coordinate form by $x_j(t) = \delta_{i,j}t$. It follows that the set of tangent vectors $D$ to curves at $p$ forms a vector space with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$. This vector space is called the **tangent space** $T_p(M)$ to $M$ at $p$. In fact, it is not hard to show that this coincides with the vector space of all derivations of the algebra of germs of smooth real-valued functions at $p$. (This depends on the fact that we work with the $C^\infty$ case. In the $C^k$ case, for $k$ finite, the latter vector space is much larger than the tangent space.) The union $T(M) = \bigcup_{p \in M} T_p(M)$ can be made into a vector bundle over $M$, called the **tangent bundle** of $M$ [a smooth bundle with fiber $\mathbf{R}^n$ and structure group $\mathbf{Gl}(n)$].

If $\varphi: M \to N$ is a smooth map between manifolds, then there is an induced map, the **differential**, $\varphi_*: T(M) \to T(N)$ taking $T_p(M)$ to $T_{\varphi(p)}(N)$ defined as follows: If $\gamma: \mathbf{R} \to M$ is a smooth curve with $\gamma(0) = p$, then we let

$$\varphi_*(D_\gamma) = D_{\varphi \circ \gamma}.$$

Thus if $g: U \to \mathbf{R}$ is smooth with $U$ a neighborhood of $\varphi(p)$, we have

$$\varphi_*(D_\gamma)(g) = D_{\varphi \circ \gamma}(g) = \frac{d}{dt} g(\varphi(\gamma(t)))\big)_{t=0} = D_\gamma(g \circ \varphi).$$

This equation

$$(\varphi_* D)(g) = D(g \circ \varphi)$$

shows that the differential $\varphi_*$ is well defined. If $\gamma: \mathbf{R} \to M$ is a smooth curve, then it is clear that

$$\gamma_*\left(\frac{d}{dt}\right) = D_\gamma.$$

If $\varphi: M \to N$ is smooth and if each $\varphi_*: T_p(M) \to T_{\varphi(p)}(N)$ is a monomorphism, then $\varphi$ is called an **immersion.** If $\varphi$ is *also* injective, then $\varphi$ is called an **embedding.** If $\varphi$ is *also* a homeomorphism onto its image $\varphi(M)$, then $\varphi(M)$ is called a **submanifold** of $N$ (in the sense of differential topology) and in this case it follows from the Implicit Function Theorem that $\varphi: M \to \varphi(M)$ is a diffeomorphism, where $\varphi(M)$ has the functional structure induced from that of $N$.

**1.1. Lemma** *If $\Theta: \mathbf{R} \times M \to M$ is a smooth action of the additive group of reals on the smooth manifold $M$, then the fixed point set $M^{\mathbf{R}}$ coincides with the set of points $p \in M$ for which the tangent vector to the curve $\tau_p: \mathbf{R} \to M$ is zero, where $\tau_p(t) = \Theta(t, p) = \theta_t(p) = t(p)$.*

***Proof*** If $p \in M^{\mathbf{R}}$, then $\tau_p$ is the constant curve at $p$ and $D_{\tau_p} = 0$. Conversely, suppose that $D_{\tau_p} = 0$. For $s \in \mathbf{R}$ we have

$$\tau_{s(p)}(t) = \theta_t\theta_s(p) = \theta_s(\tau_p(t))$$

so that

$$\tau_{s(p)} = \theta_s \circ \tau_p.$$

Thus

$$D_{\tau_{s(p)}} = (\theta_s)_*(D_{\tau_p}) = (\theta_s)_*(0) = 0$$

for all $s \in \mathbf{R}$. Now if $f\colon U \to \mathbf{R}$ is $C^\infty$ and defined near $p$, then for sufficiently small $s$ we have

$$\frac{d}{dt} f(\theta_t(p))_{t=s} = \frac{d}{dt} f(\theta_{s+t}(p))_{t=0} = D_{\tau_{s(p)}}(f) = 0$$

which implies that $t \mapsto f(\theta_t(p))$ is constant (in a suitable neighborhood oι 0). Applying this to the coordinate functions in a coordinate neighborhood of $p$ shows that $t \mapsto \theta_t(p)$ is constant (near $t = 0$ and hence for all $t$ since $\theta_{s+t} = \theta_s\theta_t$). Thus $p$ is a fixed point. ▌

Suppose that $\Theta\colon G \times M \to M$ is a smooth action of a Lie group $G$ on a smooth manifold $M$. Then the orbit map $\pi\colon M \to M/G$ induces a functional structure on $M/G$ which we call the **induced smooth structure**. Thus a function $f$ on an open set $U \subset M/G$ to $\mathbf{R}$ is called **smooth** iff $f \circ \pi\colon \pi^{-1}(U) \to \mathbf{R}$ is smooth. It should be emphasized that the use of this terminology does not imply that $M/G$ is a manifold (as it usually is not.) For example, if $M = \mathbf{R}^2$ and $G = \mathbf{Z}_2$ acting by $(x, y) \mapsto (-x, -y)$, then $M/G$ is *homeomorphic* to $\mathbf{R}^2$ but the *induced* smooth structure on $M/G$ from that of $M$ is *not* the structure of a smooth *manifold*. On the other hand, if $M = \mathbf{R}^2$ and if $G = \mathbf{Z}_2$ acts by $(x, y) \mapsto (x, -y)$, then $M/G$ is homeomorphic to a half space $\mathbf{R}^2{}_+$ and, in fact, can be shown to be *diffeomorphic* to $\mathbf{R}^2{}_+$. (This is a significant fact which is not easy to see. It is a special case of results proved in Section 4, but the reader may find it a good exercise to attempt to prove this for himself.)

Let $G$ be a Lie group and recall that each tangent vector $X$ at $e$ in $G$ is the tangent vector $X = D_{\gamma_X}$ to a unique one-parameter subgroup $\gamma_X\colon \mathbf{R} \to G$ [i.e., $\gamma_X(s + t) = \gamma_X(s)\gamma_X(t)$]. The map $T_e(G) \to G$ taking $X$ to $\gamma_X(1)$ is called the "exponential map"; thus $\exp(X) = \gamma_X(1)$. Since

$$\frac{d}{dt} f(\gamma_X(st))_{t=0} = s\frac{d}{dt} f(\gamma_X(t))_{t=0} = sX$$

we see that

$$\gamma_{sX}(t) = \gamma_X(st)$$

and

$$\exp(sX) = \gamma_X(s).$$

Thus the differential $\exp_*$ takes the tangent vector to the curve $s \mapsto sX$

to $D_{\gamma_X} = X$. However, this is the canonical identification of the tangent space at 0 of a real vector space with the vector space itself. Thus, under this identification, the differential $\exp_*\colon T_e(G) \to T_e(G)$ is the identity. Consequently exp maps some neighborhood of 0 in $T_e(G)$ *diffeomorphically* to a neighborhood of $e$ in $G$.

Now let $H$ be a closed (hence Lie) subgroup of $G$. Then $T_e(H)$ can be identified with a subspace of $T_e(G)$; the tangent vectors to one-parameter subgroups of $G$ lying in $H$. Let $V$ be a complementary subspace and consider the map

$$\varphi\colon T_e(G) = V \times T_e(H) \to G$$

given by $\varphi(X, Y) = \exp(X) \cdot \exp(Y)$. By restricting attention to each factor one sees that the differential $\varphi_*$ at the origin is the same as that for exp; that is, $\varphi_* = 1$ on $T_e(G)$. Thus $\varphi$ is a diffeomorphism on some neighborhood of 0 to neighborhood of $e$ in $G$. Taking a coordinate system in $T_e(H)$ and one in $V$, $\varphi$ induces a coordinate system $x_1, \ldots, x_p, y_1, \ldots, y_h$ in $G$ with $e$ at the origin and such that the left cosets of $H$ are given by the coordinate slices $x_1 = c_1, \ldots, x_k = c_k$ ($c_i$ constant). By possibly restricting the domain of these coordinates it is easy to see that the coordinate slice $y_1 = 0, \ldots, y_h = 0$ defines a local cross section $C$ at $e$ for the left cosets of $H$ (i.e., for the canonical map $\pi\colon G \to G/H$). Now a function defined near $e$ on $G$ is constant on the left cosets of $H$ iff it is independent of the coordinates $y_1, \ldots, y_h$. It follows that the map $\pi$ takes $C$ *diffeomorphically* to a neighborhood of $eH$ in $G/H$, where $G/H$ has the functional structure induced by $\pi$ from that of $G$. It follows that $G/H$ is a smooth manifold. From the commutativity of the diagram

$$\begin{array}{ccccc} G \times G & \longrightarrow & G & & \\ \downarrow & & \downarrow & & \\ G \times (G/H) & \longrightarrow & G/H & \xrightarrow{f} & \mathbf{R} \end{array}$$

(for all smooth $f$) and the definition of the functional structure on $G/H$, one deduces that the natural action of $G$ on $G/H$ is smooth.

**1.2. Theorem** *If $G \times M \to M$ is a smooth action and $x \in M$, then the canonical map $\alpha_x\colon G/G_x \to M$ given by $\alpha_x(gG_x) = g(x)$ is an embedding.*

***Proof*** Since left translations in $G/G_x$ and the operations by elements of $G$ on $M$ are diffeomorphisms it clearly suffices to show that the differential of $\alpha_x$ at $eG_x$ is a monomorphism. If $X$ is a tangent vector at $eG_x$ of $G/G_x$, then $X$ comes (via the differential of $G \to G/G_x$) from a tangent vector $Y$ at $e$ in $G$. Moreover, $Y$ is the tangent vector $Y = D_\gamma$ to some one-parameter subgroup $\gamma$: $\mathbf{R} \to G$ of $G$. If $X$ goes to zero in $T_x(M)$, then so does $D_\gamma$. Then by 1.1, $x$ must be left fixed by the action of $\mathbf{R}$ via $\gamma$: $\mathbf{R} \to G$. Thus the image of $\gamma$ is in $G_x$ and $\gamma$ is taken to a constant curve by the projection $\pi$: $G \to G/G_x$. Thus

$$X = \pi_*(D_\gamma) = D_{\pi \circ \gamma} = 0$$

as desired. ∎

**1.3. Corollary** *If the compact Lie group $G$ acts smoothly on the smooth manifold $M$, then each orbit $G(x)$ is a submanifold of $M$ and the map $\alpha_x$: $G/G_x \to G(x)$ is a diffeomorphism.* ∎

## 2. TUBULAR NEIGHBORHOODS

As in the last section we assume that the reader is already familiar with elementary differential geometry, and in particular with the notion of riemannian manifold and of a geodesic; or at least that he is willing to accept those few properties of them that we will need.

Let $M$ be a smooth manifold and suppose that $G$ acts smoothly on $M$. By a (smooth) $G$-**vector bundle** $\xi$ on $M$ we mean a smooth vector bundle [i.e., fiber $\mathbf{R}^n$ and structure group $\mathbf{Gl}(n)$] together with a smooth action of $G$ on the total space $E(\xi)$ by bundle maps (i.e., linear on the fibers) and such that the projection $\pi$: $E(\xi) \to M$ is equivariant. The main example is, of course, the tangent bundle $T(M)$ on which $G$ acts via the differential oı the given action on $M$.

An **inner product** on $\xi$ is a function which assigns to each point $p \in M$ an inner product (positive definite, symmetric, bilinear form) $\langle \cdot, \cdot \rangle_p$ on the fiber $F_p = \pi^{-1}(p)$ in a "smooth way." In terms of a chart

$$\varphi\colon\ U \times \mathbf{R}^n \xrightarrow{\approx} \pi^{-1}(U),$$

$\langle \cdot, \cdot \rangle_p$ is expressed as a positive definite symmetric matrix $[a_{i,j}(p)]$ for each $p \in U$; that is

$$\langle \varphi(p, (x_1, \ldots, x_n)), \varphi(p, (y_1, \ldots, y_n)) \rangle_p = \sum_{i,j} x_i a_{i,j}(p) y_j.$$

Smoothness of the inner product means that the matrix coefficients $a_{i,j}(p)$

are smooth functions of $p$ in $U$. An inner product $\langle\cdot,\cdot\rangle$ on $\xi$ is said to be **invariant** (with respect to the given $G$-action) if

$$\langle g(v), g(w)\rangle_{g(p)} = \langle v, w\rangle_p$$

for all $v, w \in F_p$ and all $p \in M$.

It is easy to see that inner products on $\xi$ correspond to *smooth* reductions of the structure group $\mathbf{Gl}(n)$ of $\xi$ to $\mathbf{O}(n)$. We will not need this fact.

A smooth $G$-vector bundle $\xi$ together with an invariant inner product on $\xi$ will be called a **euclidean $G$-bundle** on $M$.

**2.1. Theorem** *Let $G$ be a compact Lie group acting smoothly on $M$ and let $\xi$ be a $G$-vector bundle on $M$. Then there exists an invariant inner product on $\xi$.*

***Proof*** First we show that there is a (noninvariant) inner product. To see this let $\{U_\alpha\}$ be a locally finite covering of $M$ by open sets over which there are charts

$$\varphi_\alpha\colon\ U_\alpha \times \mathbf{R}^n \to \pi^{-1}(U_\alpha).$$

Over each $U_\alpha$ we can define an inner product $\langle\cdot,\cdot\rangle_\alpha$ by means of the chart $\varphi_\alpha$; e.g., given by the identity matrix. Then if $\{f_\alpha\}$ is a smooth partition of unity subordinate to $\{U_\alpha\}$ we put

$$\langle v, w\rangle_p = \sum_\alpha f_\alpha(p)\langle v, w\rangle_{\alpha,p}$$

for $v$, $w \in F_p$, which clearly defines an inner product on $\xi$.

Now define a new inner product on $\xi$ by putting

$$\{v, w\}_p = \int_G \langle gv, gw\rangle_{g(p)}\, dg.$$

For $h \in G$ we have

$$\begin{aligned}\{hv, hw\}_{h(p)} &= \int_G \langle ghv, ghw\rangle_{gh(p)}\, dg\\ &= \int_G \langle gv, gw\rangle_{g(p)}\, dg = \{v, w\}_p\end{aligned}$$

so that $\{\cdot,\cdot\}$ is invariant. Since the integrand is smooth in $p$ and $g$ it follows from 0.3.3 that $\{\cdot,\cdot\}$ is smooth. Moreover $\{\cdot,\cdot\}$ is clearly symmetric bilinear and is positive definite because

$$\{v, v\}_p = \int_G \langle gv, gv \rangle_{g(p)}\, dg > 0$$

for $v \neq 0$ since the integrand is positive. ∎

A **riemannian metric** on $M$ is a smooth inner product on the tangent bundle $T(M)$ of $M$. Thus we have shown, in particular, that a smooth $G$-manifold $M$ has an *invariant* riemannian metric if $G$ is compact. For such a metric, $G$ is said to act by **isometries**.

If $X \in T_p(M)$, where $M$ is riemannian, then there is a unique geodesic $\gamma_X\colon U \to M$ with $\gamma_X(0) = p$ and tangent vector $D_{\gamma_X} = X$ at $p$. Here the open set $U \subset \mathbf{R}$ is taken as the natural domain of definition of the geodesic. Clearly $\gamma_{sX}(t) = \gamma_X(st)$ and the exponential map is defined by

$$\exp(X) = \gamma_X(1).$$

This map to $M$ is defined on some open neighborhood $W \subset T(M)$ of the 0-section. Moreover exp is a smooth map.

Note that the tangent space to $T(M)$ at a point $p$ of the 0-section (which we identify with $M$) decomposes into

$$T_p(T(M)) = T_{\text{v}} \oplus T_{\text{h}}$$

(vertical and horizontal vectors), where $T_{\text{v}}$ consists of tangent vectors to $T_p(M) \subset T(M)$ and hence is canonically isomorphic to $T_p(M)$, and $T_{\text{h}}$ consists of tangent vectors to the 0-section $M$ and hence is also canonically isomorphic to $T_p(M)$. The differential of exp restricted to $T_{\text{v}}$ is clearly just this identification of $T_{\text{v}}$ with $T_p(M)$ since

$$\frac{d}{dt} f(\exp(tX))_{t=0} = \frac{d}{dt} f(\gamma_X(t))_{t=0} = X(f).$$

Also exp is the identity map on the 0-section and hence $\exp_*|T_{\text{h}}$ is the canonical identification of $T_{\text{h}}$ with $T_p(M)$. Since the differential is a linear map on the tangent space at a point, it follows that at a point $p \in M$ and under the above decomposition $T_p(T(M)) = T_{\text{v}} \oplus T_{\text{h}} = T_p(M) \oplus T_p(M)$, the differential

$$\exp_*\colon T_p(M) \oplus T_p(M) \to T_p(M)$$

is just vector addition.

Suppose now that $G$ acts smoothly on $M$ and that the given riemannian metric on $M$ is $G$-invariant. Then, since geodesics and the exponential map are defined *canonically* in terms of the riemannian metric, the map exp (and its natural domain of definition) is *equivariant*, where $G$ acts on $T(M)$ via the differential of its action on $M$. The above facts about the exponential map are all that we shall use.

Let $G$ act smoothly on $M$ and let $A \subset M$ be a smooth, invariant, closed submanifold. The restriction $T(M)|A$ has $T(A)$ as a subbundle. The quotient bundle

$$N(A) = (T(M) \mid A)/T(A)$$

is called the **normal bundle** to $A$ in $M$ and is clearly a smooth $G$-vector bundle on $A$. If $M$ is riemannian and $G$ acts by isometries, then $T(A)$ has an orthogonal complement $T(A)^\perp$ in $T(M) \mid A$ and $T(A)^\perp$ is canonically isomorphic to $N(A)$, and is a euclidean $G$-bundle on $A$. By an **open invariant tubular neighborhood** of $A$ in $M$ we mean a smooth $G$-vector bundle $\xi$ on $A$ and an equivariant diffeomorphism

$$\varphi\colon\ E(\xi) \to M$$

onto some open neighborhood of $A$ in $M$, such that the restriction of $\varphi$ to the 0-section $A$ of $\xi$ is the inclusion of $A$ in $M$. If $\xi$ is a euclidean $G$-bundle on $A$, then the restriction of such a diffeomorphism $\varphi$ to the unit disk bundle $D(\xi) \to M$ is called a **closed invariant tubular neighborhood** of $A$. By 2.1 every open invariant tubular neighborhood "contains" a closed one when $G$ is compact.

**2.2. Theorem** *If the compact Lie group $G$ acts smoothly on $M$ and if $A$ is a closed invariant submanifold, then $A$ has an open invariant tubular neighborhood in $M$.*

*Proof* We may assume that $G$ acts by isometries in some riemannian metric on $M$. Thus we may also regard the normal bundle $N(A)$ as the perpendicular complement of $T(A)$ in $T(M) \mid A$. Then the exponential map is defined on some open invariant neighborhood $U$ of $A$ in $N(A)$ and $\exp\colon U \to M$ is equivariant. For $a \in A$, the differential

$$\exp_*\colon T_a(N(A)) = N_a(A) \oplus T_a(A) = T_a(M) \to T_a(M)$$

is the identity. Thus there is a smaller invariant open neighborhood $V$ of $A$ in $N(A)$ on which $\exp_*$ is an isomorphism on the tangent space at each point of $V$. That is, exp: $V \to M$ is an immersion. Since $A$ is closed it is clear that $V$ may be taken to be so small that $\exp^{-1}(A) = A$. Since exp is the identity on $A$, it will follow from Lemma 2.3 (to be proved later) that there is a smaller invariant neighborhood $W$ of $A$ in $N(A)$ on which exp is an embedding. Define a function $f$: $A \to \mathbf{R}$ by letting $f(a)$ be the supremum of the set of real numbers $r$ for which the open ball of radius $r$ in $N_a(A)$ is in $W$. Then $f(ga) = f(a)$ for all $a \in A$ and $g \in G$, and $f$ is a lower semicontinuous positive function on $A$. By a theorem of Dowker (see Dugundji [1, p. 170]) there is a continuous function $h$ on $A$ with $0 < h(a) < f(a)$. By the Smooth Approximation Theorem, we may assume $h$ to be smooth. Using the normalized integral on $G$ we define a smooth function $k$: $A \to \mathbf{R}$ by

$$k(a) = \int_G h(ga)\, dg.$$

Then

$$0 < k(a) < \int_G f(ga)\, dg = f(a) \int_G dg = f(a)$$

and

$$k(ga) = k(a) \qquad \text{for all } g \in G.$$

Now define $\psi$: $N(A) \to N(A)$ by

$$\psi(v) = \frac{k(\pi(v))}{(1 + \langle v, v\rangle)^{1/2}}\, v,$$

where $\pi$ is the projection $N(A) \to A$. Then $\psi$ is an equivariant diffeomorphism onto its image, which is the open set

$$\{v \in N(A) \mid \|v\| < k(\pi(v))\}.$$

Thus $\varphi = \exp \circ \psi$: $N(A) \to M$ is an open invariant tubular neighborhood of $A$. ∎

As mentioned, we still must prove the following lemma.

**2.3. Lemma** *Let $X$ and $Y$ be metric spaces and let $f$: $X \to Y$ be a local homeomorphism* (i.e., *each $x \in X$ has an open neighborhood mapped homeomorphically onto an open set in $Y$*). *Suppose that $f$ is one–one on a subspace $A \subset X$, and that $f^{-1}f(A) = A$. Then $A$ has an open neighborhood $U$ on which $f$ is a homeomorphism to $f(U)$.*

***Proof*** Put $B = f(A) \subset Y$. We may as well assume that $f$ is onto. Then for $y \in Y$ there is an $x \in X$ with $f(x) = y$, and $f$ maps some neighborhood of $x$ homeomorphically to a neighborhood of $y$, so that an inverse to $f$ can be defined near $y$. Since $Y$ is paracompact we can find a locally finite covering $\{U_\alpha\}$ of $Y$ and open maps $g_\alpha\colon U_\alpha \to X$ such that $fg_\alpha$ is the identity on $U_\alpha$. Let $\{V_\alpha\}$ be a covering of $Y$ with $\bar{V}_\alpha \subset U_\alpha$. Let $W$ be the set of points $y \in Y$ such that if $y \in \bar{V}_\alpha \cap \bar{V}_\beta$, then $g_\alpha(y) = g_\beta(y)$. Then the maps $g_\alpha \mid \bar{V}_\alpha \cap W$ patch together consistently to define a map $g\colon W \to X$ with $fg = 1_W$. Since $f^{-1}(B) = A$ and $f \mid A$ is one–one, we see that $W \supset B$.

We claim that $W$ is open. To see this let $y \in W$ and put $g(y) = x$, so that $f(x) = y$. Let $N$ be an open neighborhood of $x$ on which $f$ is one–one. Suppose that $y \in \bar{V}_{\alpha_1} \cap \cdots \cap \bar{V}_{\alpha_k}$ but that $y \notin \bar{V}_\beta$ for $\beta \neq \alpha_1, \ldots, \alpha_k$. There is an open neighborhood $M$ of $y$ not touching any of the $\bar{V}_\beta$. Moreover, since $g_{\alpha_i}(y) = x$, we may take $M$ to be so small that $g_{\alpha_i}(M) \subset N$. Now if $z \in M$ and if $g_{\alpha_i}(z) \neq g_{\alpha_j}(z)$, then, since $f$ is one–one on $N$, $z = fg_{\alpha_i}(z) \neq fg_{\alpha_j}(z) = z$; a contradiction. Thus $M \subset W$ and $W$ is open, as was to be shown.

Now $g$ is open since $g = g_\alpha$ on the open set $V_\alpha \cap W$, and thus $g(W)$ is an open set containing $A$. Also $f$ is one–one on $g(W)$ since $fg = 1_W$. Since $f$ is open it is a homeomorphism on $g(W)$ to $W$, and its inverse is $g$. ∎

**2.4. Corollary** *A smooth action of a compact Lie group is locally smooth.*

***Proof*** Let $G$ act by left translation on $G/H$ and let $\xi$ be a (smooth) euclidean $G$-bundle on $G/H$. Let $V$ be the fiber over the point $eH/H$ and note that $H$ acts orthogonally on $V$. The canonical map

$$\psi\colon G \times_H V \to E(\xi)$$

defined by $\psi[g, v] = g(v)$ is an equivalence of $G$-spaces by II.4.4. Thus if $G_x = H$ and if $\varphi\colon E(\xi) \to M$ is an invariant tubular neighborhood of $G(x)$ (which exists for some such $\xi$ by 1.3, 2.1, and 2.2), then $\varphi \circ \psi$ is a linear tube about $G(x)$ in the sense of Chapter IV, Section 1.) ∎

We recall for future reference that $G \times_H V$ is the bundle over $G/H$ associated with the smooth principal $H$-bundle $G \to G/H$ and that, since $H$ acts smoothly on $V$, there is a canonical smooth structure on $G \times_H V$. Since the map $\psi\colon G \times_H V \to E(\xi)$ covers the identity on $G/H$ and is a diffeomorphism on the fiber $V$, it is clearly a *diffeomorphism*. Thus the tube $\varphi \circ \psi\colon G \times_H V \to M$ is a diffeomorphism onto an open neighborhood of the given orbit $G(x)$; a fact that we shall use below.

We remark that this shows that a locally smooth $G$-space ($G$ compact Lie) is just a $G$-space such that each orbit has an open invariant neighborhood on which there exists a $C^\infty$-structure in which $G$ acts smoothly. This accounts for the terminology "locally smooth" (although "locally smoothable" would be more accurate). Such results as IV.*1.1*, that the restriction of a locally smooth action to a subgroup is still locally smooth, now follow trivially from these remarks and 2.4. It is definitely false that locally smooth actions are (globally) smooth in some differentiable structure. In fact there are topological manifolds admitting locally smooth actions but not admitting any differentiable structure. There are also smooth manifolds admitting no smooth $S^1$-actions, but whose underlying topological manifolds admit locally smooth $S^1$-actions (see 9.6).

Note that if $p \in M^G$, then 2.1 and 2.2 applied to $A = \{p\}$ mean that there are local coordinates about $p$ in which the $G$-action is expressed as an orthogonal representation. This result is due to Bochner [1].

From the fact that the tube $\varphi \circ \psi$ in the proof of 2.4 is a diffeomorphism, and from the fact that $(G \times_H V)_{(H)} \approx (G/H) \times V^H$ we deduce the following result (see IV.3.3).

**2.5. Corollary** *Let $G$ be a compact Lie group acting smoothly on $M$. Then the subspace $M_{(H)}$ of points on orbits of* type *$G/H$ is a smooth submanifold (locally closed) of $M$ and is a smooth $G/H$-bundle over its orbit space $M^*_{(H)}$.* ▮

We now take up the question of the uniqueness of invariant tubular neighborhoods. Two invariant tubular neighborhoods $\varphi\colon E(\xi) \to M$ and $\psi\colon E(\eta) \to M$ of a closed invariant submanifold $A \subset M$ are said to be **equivariantly isotopic** if there are invariant tubular neighborhoods $\varphi_t$: $E(\xi) \to M$ of $A$, $t \in [0,1]$, and a smooth vector bundle equivalence

$$\theta\colon E(\xi) \to E(\eta)$$

such that $\varphi_1 = \varphi$, $\varphi_0 = \psi \circ \theta$, and the map $[0,1] \times E(\xi) \to M$ taking $(t, v) \mapsto \varphi_t(v)$ is smooth. By a standard argument, that we omit, it may be assumed that $\varphi_t$ is constant for $t$ near 0 and near 1, and it follows easily that isotopy is an equivalence relation between invariant tubular neighborhoods of $A$. (The definition of isotopy of *closed* invariant tubular neighborhoods is the same, except of course that $\theta$ is required to be an equivalence of $\mathbf{O}(n)$-bundles.) Note that $\theta$ is necessarily $G$-equivariant.

**2.6. Theorem** *If $G$ is a compact Lie group acting smoothly on $M$ and if $A \subset M$ is a closed invariant submanifold, then any two (open or closed) invariant tubular neighborhoods of $A$ are equivariantly isotopic.*

***Proof*** Note that by the proof of 2.2 there exists an invariant tubular neighborhood of $A$ whose image is contained in that of both $\varphi$ and $\psi$. Since isotopy is an equivalence relation, it thus suffices to treat the case in which the image of $\varphi$ is contained in the image of $\psi$. (In particular $\psi^{-1}\varphi\colon E(\xi) \to E(\eta)$ is defined.) Thus it makes sense to define

$$\varphi_t(v) = \psi\left(\frac{1}{t}\, \psi^{-1}\varphi(tv)\right).$$

We need only investigate this as $t$ approaches 0.

Let $p \in A$ and let $x_1, \ldots, x_m$ be local coordinates on an open neighborhood $U \subset A$ of $p$. Let $y_1, \ldots, y_n$ be coordinates in $\mathbf{R}^n$ (the fiber of $\xi$ and $\eta$). Using a chart for $\eta$ we may regard $E(\eta \mid U) = U \times \mathbf{R}^n$. Then $x_1, \ldots, x_m, y_1, \ldots, y_n$ can be regarded as local coordinates on $M$ about $p$ by using $\psi\colon U \times \mathbf{R}^n \to M$ as a chart. (Thus $\psi$ is the identity in these coordinates.) Using a chart $E(\xi \mid U) \approx U \times \mathbf{R}^n$ for $\xi$, we can represent $\varphi$ in the neighborhood of $p \in A \subset E(\xi)$ by

$$\varphi(x, y) = \big(\lambda(x, y), \mu(x, y)\big),$$

where

$$\lambda(x, y) = \big(\lambda_1(x, y), \ldots, \lambda_m(x, y)\big) \in \mathbf{R}^m,$$
$$\mu(x, y) = \big(\mu_1(x, y), \ldots, \mu_n(x, y)\big) \in \mathbf{R}^n.$$

Since $\varphi(x, 0) = (x, 0)$ we have $\lambda(x, 0) = x$ and $\mu(x, 0) = 0$. Thus

$$\varphi_t(x, y) = \psi\left(\frac{1}{t}\, \psi^{-1}\varphi(x, ty)\right) = \left(\lambda(x, ty), \frac{1}{t}\, \mu(x, ty)\right)$$

(which is defined for $t \neq 0$ sufficiently small).

Now $\lambda(x, ty)$ is defined and smooth in $x, y, t$ even for $t = 0$. Since $\mu(x, 0) = 0$ we can write

$$\mu_i(x, y) = \sum_j a_{i,j}(x) y_j + \sum_{j,k} \omega_{i,j,k}(x, y) y_j y_k$$

by Taylor's Theorem, where the $\omega_{i,j,k}$ are smooth and

$$a_{i,j}(x) = (\partial \mu_i / \partial y_j)\,(x, 0).$$

Thus

$$\frac{1}{t}\mu_i(x, ty) = \sum_j a_{i,j}(x)y_j + t\sum_{j,k} \omega_{i,j,k}(x, ty)y_jy_k$$

is defined and smooth in $x, y, t$ even at $t = 0$. Moreover

$$\varphi_0(x, y) = (x, L_x(y)),$$

where $L_x$: $\mathbf{R}^n \to \mathbf{R}^n$ is the linear map given by the matrix $[a_{i,j}(x)]$. Now $L_x$ is just the differential of the map $y \mapsto \mu(x, y)$ at $y = 0$ which is nonsingular since the differential of $\varphi$ at $(x, 0)$ has the matrix form

$$\begin{bmatrix} I & 0 \\ * & L_x \end{bmatrix}$$

which must be nonsingular. Thus $\theta = \psi^{-1}\varphi_0$: $E(\xi) \to E(\eta)$ is a vector bundle equivalence given in terms of these coordinates by $\theta(x, y) = (x, L_x(y))$ (since $\psi$ is the identity in the coordinates used).

This finishes the proof of the case of *open* tubular neighborhoods. To conclude the case of *closed* tubular neighborhoods (where $\xi$ and $\eta$ are now euclidean $G$-bundles) it suffices to show that an equivariant $\mathbf{Gl}(n)$-bundle equivalence $\theta$: $E(\xi) \to E(\eta)$ is (equivariantly) isotopic to an $\mathbf{O}(n)$-bundle equivalence. In fact the isotopy shall be through $\mathbf{Gl}(n)$-bundle equivalences and will be canonical.

Let us recall some linear algebra. If $Q$ is a positive definite symmetric (real) matrix, then $Q$ can be diagonalized by an orthogonal change of basis and has a unique positive diagonal square root in this basis, which we denote by $P = Q^{1/2}$. Moreover, $Q \mapsto Q^{1/2}$ is smooth in the matrix coefficients. It is also clear that if $P_1$ and $P_2$ are positive definite symmetric, then so is any convex linear combination $tP_1 + (1 - t)P_2$, $0 \le t \le 1$. Now if $B$ is an arbitrary nonsingular matrix, put $P = (B'B)^{1/2}$ (positive definite symmetric) and put $O = BP^{-1}$. Then

$$OO' = BP^{-1}P^{-1}B' = B(P^2)^{-1}B' = B(B'B)^{-1}B' = BB^{-1}B'^{-1}B' = I,$$

so that $O$ is orthogonal. Clearly $P$ and $O$ depend smoothly on $B$. The decomposition $B = OP$ ($O$ orthogonal and $P$ positive definite symmetric) is unique since, for any such decomposition, $B'B = (OP)'(OP) = P'O'OP = PO'OP = P^2$, whence $P = (B'B)^{1/2}$. In particular the decomposition $B = OP$ is natural with respect to *orthogonal* change of basis.

Since $\xi$ and $\eta$ are $\mathbf{O}(n)$-bundles, this means that $\theta\colon E(\xi) \to E(\eta)$ can be factored uniquely into

$$\theta\colon E(\xi) \xrightarrow{\theta'} E(\xi) \xrightarrow{\theta''} E(\eta),$$

where $\theta'$ is represented on each fiber by a positive definite symmetric matrix and $\theta''$ is an orthogonal equivalence. Also $\theta'$ and $\theta''$ are smooth (since $O$ and $P$ depend smoothly on $B$) and $G$-equivariant (since the decomposition $B = OP$ is preserved by *orthogonal* transformations). Thus putting

$$\theta_t = \theta'' \circ \bigl(t\theta' + (1 - t)I\bigr)$$

provides the desired isotopy between $\theta$ and the orthogonal equivalence $\theta''$. ∎

*Remark* Our treatment of the existence and uniqueness of tubular neighborhoods essentially follows that of Milnor [3]; also see Lang [1]. It follows from a theorem of Thom (see Milnor [3]) that if $A$ is compact, then any two *closed* tubular neighborhoods of $A$ are ambient isotopic by an isotopy $\mathbf{I} \times M \to M$ which is constant outside a compact neighborhood of $A$. It is not hard to prove the equivariant version of this result, but we shall omit it for two reasons: First, the proof is similar to that of a theorem to be proved in the next section; and second, the equivariant result follows from the nonequivariant result upon application of the theorem of the next section.

## 3. INTEGRATION OF ISOTOPIES

If $M$ is a smooth manifold, then an **isotopy** of $M$ is a smooth map $\varphi\colon \mathbf{I} \times M \to M$ such that each $\varphi_t\colon M \to M$ is a diffeomorphism, where $\varphi_t(x) = \varphi(t, x)$, and such that $\varphi_t$ is independent of $t$ in some neighborhood of 0 and in some neighborhood of 1. (With little loss of generality one could assume that $\varphi_0$ is the identity.) If $G$ acts smoothly on $M$, then the set

$$\varphi^G = \{x \in M \mid \varphi_t(gx) = g\varphi_t(x) \text{ for all } t \in \mathbf{I} \text{ and all } g \in G\}$$

is a closed invariant subspace of $M$ called the **subspace of equivariance** of $\varphi$ on $M$. If $\varphi^G = M$, then $\varphi$ is called "equivariant." The following result allows us to replace certain types of isotopies by equivariant ones.

**3.1. Theorem** *Let $\varphi$ be an isotopy of $M$ and let $G$ be a compact Lie group acting smoothly on $M$. Assume that $\varphi_0$: $M \to M$ is equivariant and that $M - \varphi^G$ has compact closure in $M$. Then there exists an equivariant isotopy $\psi$ of $M$ with $\psi_0 = \varphi_0$ and which coincides with $\varphi$ on $\mathbf{I} \times \varphi^G$.*

*Proof* By composing with $\varphi_0^{-1}$ we may as well assume that $\varphi_0$ is the identity. Since $\varphi$ is constant in $t$ near 0 and near 1 it can be extended in the obvious way to $\varphi$: $\mathbf{R} \times M \to M$ (a minor convenience). Let $F$: $\mathbf{R} \times M \to \mathbf{R} \times M$ be the diffeomorphism defined by

$$F(t, x) = \big(t, \varphi(t, x)\big).$$

Apply the differential $F_*$ of $F$ to the vector field $d/dt$ on $\mathbf{R} \times M$, obtaining a vector field $X = F_*(d/dt)$ on $\mathbf{R} \times M$. The $\mathbf{R}$ component of $X$ is clearly $d/dt$. The integral curves of $X$ are the transforms under $F$ of those of $d/dt$; that is, they are $t \mapsto \big(t, \varphi(t, x)\big)$.

Using the normalized integral on $G$ we define a new vector field $Y$ on $\mathbf{R} \times M$ by

$$Y_{(t,x)} = \int_G g_*(X_{(t,g^{-1}x)})\, dg.$$

(This is the integral of a vector-valued function on $G$ to $T_{(t,x)}(\mathbf{R} \times M)$.) Then

$$\begin{aligned} Y_{(t,hx)} &= \int_G g_*(X_{(t,g^{-1}hx)})\, dg \\ &= \int_G (hg)_*(X_{(t,g^{-1}x)})\, dg \\ &= h_*(Y_{(t,x)}) \end{aligned}$$

which means that $Y$ is invariant under the canonical $G$-action on $T(\mathbf{R} \times M)$. Since the $\mathbf{R}$-component of $X$ is constant $= d/dt$ and since the action of $G$ on $T(\mathbf{R} \times M)$ preserves this component, the $\mathbf{R}$-component of $Y$ is also $d/dt$.

Suppose that $x \in \varphi^G$. Then $F(t, x) = gF(t, g^{-1}x)$ for all $t$ so that $X_{(t,x)} = g_*(X_{(t,g^{-1}x)})$, and it follows that $Y_{(t,x)} = X_{(t,x)}$.

Now integral curves to the vector field $Y$ exist locally by the Existence Theorem for Ordinary Differential Equations (see e.g., Sternberg [1, p. 90]). Since $Y$ has $\mathbf{R}$-component $d/dt$, and hence has no singularities, each integral curve eventually gets outside the *compact* set $\mathbf{I} \times \big(M - \text{int}(\varphi^G)\big)$. Since $Y = X$ outside this compact set, the integral curves of $Y$ are globally defined. That is, there is a smooth action $\Theta$: $\mathbf{R} \times (\mathbf{R} \times M) \to (\mathbf{R} \times M)$ of $\mathbf{R}$ on $\mathbf{R} \times M$ such that the tangent vector field to the orbits (from $d/dt$ on the group) is $Y$. Since the $\mathbf{R}$-component of $Y$ is $d/dt$ the diffeomorphism $(t, x) \mapsto \Theta(t, (0, x))$ has the form $\Theta(t, (0, x)) = (t, \psi(t, x))$ and $\psi_t$: $x \mapsto \psi(t, x)$ must be a diffeomorphism of $M$. Now $t \mapsto \Theta(t, (0, x)) = (t, \psi_t(x))$ is the integral curve of $Y$ with initial point $(0, x)$, and $t \mapsto F(t, x) = (t, \varphi_t(x))$ is the integral curve of $X$ with initial point $(0, \varphi_0(x)) = (0, x)$. It follows that $\psi_t(x) = \varphi_t(x)$ for all $t$ when $x \in \varphi^G$. Since $Y$ is $G$-invariant it follows that $\psi$ is an equivariant isotopy. ∎

To illustrate the use of 3.1 let us prove the following fact.

**3.2. Corollary** *Let $G$ be a compact Lie group acting smoothly on $M$ and let $x$ and $y$ be points in the same component of the fixed point set $M^G$. Then there exists an equivariant isotopy $\psi$: $\mathbf{I} \times M \to M$ with $\psi_0$ the identity and with $\psi_1(x) = y$. Moreover, $\psi$ can be taken to be constant in $t$ outside some compact set.*

*Proof* We assume the standard fact that such a (nonequivariant) isotopy $\varphi$ exists which is constant (hence equivariant) outside some compact set and which moves $x$ along some path in $M^G$ to $y$. Since $M - \varphi^G$ then has compact closure and since $x \in \varphi^G$ (the path $\varphi_t(x)$ being in $M^G$), the result follows from 3.1. ∎

*Remark* Recall the fact (mentioned in Section 2) that two closed tubular neighborhoods of $y$ are ambient isotopic by an isotopy constant outside a compact set. Then 3.1 applies to show that this is also true equivariantly. Thus we deduce from 3.2 that $\psi_1$ may be assumed to take a given disk neighborhood of $x$ orthogonally onto any *given* disk neighborhood of $y$ (where $G$ is assumed to act orthogonally on both of these disks).

## 4. EQUIVARIANT SMOOTH EMBEDDINGS AND APPROXIMATIONS

In this section we shall prove a theorem, due independently to Mostow [1] and to Palais [2], that a compact manifold with a smooth action of a compact Lie group $G$ can be smoothly embedded in an orthogonal $G$-action. Sharper results, analogous to the Whitney Embedding Theorem, have been obtained by Wasserman [2], and the compactness condition can be weakened to finiteness of the number of orbit types, but we shall not discuss these improvements here. The Embedding Theorem is then applied to show that an equivariant map can be approximated by an equivariantly homotopic smooth map; see Bredon [12] and Wasserman [2].

The main case of the following theorem is, of course, that for which $M = K = N$ is compact.

**4.1. Theorem** *Let $G$ be a compact Lie group acting smoothly on a manifold $M$. Let $K$ be a compact invariant subspace of $M$ and let $N \supset K$ be an open invariant neighborhood of $K$. Then there exists an orthogonal action of $G$ on some euclidean space $\mathbf{R}^n$ and a smooth equivariant map $\theta\colon M \to \mathbf{R}^n$ which is an embedding (in the smooth sense) on $K$ and is zero outside $N$.*

*Proof* Let $H$ be a closed subgroup of $G$. By 0.5.2 there exists an orthogonal representation of $G$ on some euclidean space $V_0$ and a point $v_0 \in V_0$ with $G_{v_0} = H$. Suppose we are given an orthogonal representation of $H$ on a euclidean space $V$. By 0.4.2 there is an orthogonal representation of $G$ on some euclidean space $V' \supset V$ extending the $H$-action on $V$. Then $G$ acts orthogonally on $W = V_0 \oplus V'$ via the sum of these two representations (i.e., diagonally). Consider the map

$$\varphi\colon G \times_H V \to V_0 \oplus V' = W$$

defined by $\varphi[g, v] = g(v_0 + v)$. If $\varphi[g, v] = \varphi[g', v']$, then $g(v_0 + v) = g'(v_0 + v')$ so that $g^{-1}g'(v_0) = v_0$ and $g^{-1}g'(v') = v$. Thus $h = g^{-1}g' \in H$ and $h(v') = v$ which shows that $[g, v] = [gh, h^{-1}v] = [g', v']$ and hence that $\varphi$ is one–one. Since $G \times_H V$ has the differentiable structure induced from that of $G \times V$ and since the action map $G \times V \to W$ is smooth, it follows that $\varphi$ is smooth.

The isotropy group at $[e, v]$ is $H_v$ and this is also the isotropy group at $v_0 + v \in W$. Thus $\varphi$ takes the orbit of $[e, v]$ diffeomorphically onto $G(v_0 + v)$ (by 1.3). The differential of $\varphi$ is thus one–one on the tangent space to the orbit at $[e, v]$. However, the normal space to the orbit of $[e, v]$ is a subspace of $V$ and $\varphi$ maps this one–one affinely into $W$, whence $\varphi_*$ is one–one on the whole tangent space to $G \times_H V$ at $[e, v]$. By equivariance, $\varphi_*$ is everywhere one–one, so that $\varphi$ is a one–one immersion. Since $\varphi$ is obviously proper, it is an embedding. (It is only important that $\varphi$ be an embedding near the 0-section $G/H$.)

Let $s > r > 0$ and let $f$: $\mathbf{R} \to \mathbf{R}$ be a smooth function with

$$\begin{aligned} f(t) &= 1 \quad \text{for} \quad t \leq r, \\ f(t) &\neq 0 \quad \text{for} \quad t < s, \\ f(t) &= 0 \quad \text{for} \quad t \geq s. \end{aligned}$$

Define $\psi$: $G \times_H V \to W$ by

$$\psi[g, v] = f(\| v \|) \cdot \varphi[g, v].$$

Then $\psi$ is equivariant, is an embedding for $\| v \| < r$, and is the 0-map for $\| v \| \geq s$.

Now if $x \in M$ with $G_x = H$, then $G(x)$ has an invariant tubular neighborhood of this form $G \times_H V$. Since $\psi[g, v] = 0$ for $\| v \| \geq s$, $\psi$ extends by zero to all of $M$. Similarly the map $G \times_H V \to \mathbf{R}$ taking $[g, v] \mapsto f(\| v \| s/r)$ extends to an equivariant map $\tau$: $M \to \mathbf{R}$ which is nonzero exactly when $\| v \| < r$ (where $G$ acts trivially on $\mathbf{R}$).

That is, for any $x \in M$ we have found an orthogonal representation of $G$ on a euclidean space $W_x$ and a smooth equivariant map $\psi_x$: $M \to W_x$ which is an embedding on some open neighborhood $U_x$ of $G(x)$, and a smooth invariant function $\tau_x$: $M \to \mathbf{R}$ which is nonzero exactly on $U_x$. Moreover, we may assume that $\psi_x(y) = 0$ for $y$ outside any preassigned neighborhood of $\bar{U}_x$.

If $K \subset M$ is any compact invariant set, then $K$ can be covered by a finite number

$$K \subset U_{x_1} \cup \cdots \cup U_{x_k}$$

of such open sets and it can be assumed that the $\psi_{x_i}$ vanish outside any given neighborhood $N$ of $K$. The map

$$\theta: M \to W_{x_1} \oplus \cdots \oplus W_{x_k} \oplus \mathbf{R}^k$$
$$\theta(x) = (\psi_{x_1}(x), \ldots, \psi_{x_k}(x), \tau_{x_1}(x), \ldots, \tau_{x_k}(x))$$

is smooth and equivariant. If $x, y \in \bigcup U_{x_i}$ and if $\theta(x) = \theta(y)$, then for some $i$, $\tau_{x_i}(x) = \tau_{x_i}(y) \neq 0$, which implies that $x, y \in U_{x_i}$ and hence that $x = y$ since $\psi_{x_i}$ is one–one on $U_{x_i}$. Since the differential of $\psi_{x_i}$ is one–one on $U_{x_i}$ it follows that the equivariant smooth map $\theta$ is an embedding on a neighborhood of $K$ and vanishes outside $N$. ▌

By far the most important application of the Smooth Embedding Theorem is the following Smooth Approximation Theorem. When we say that a map $\varphi\colon M \to N$ can be "approximated" by a map $\psi\colon M \to N$ (with certain properties) we mean that for a given metric on $N$ and for a given positive function $\varepsilon\colon M \to \mathbf{R}^+$, $\psi$ can be found such that $\operatorname{dist}(\psi(x), \varphi(x)) < \varepsilon(x)$ for all $x \in M$. This notion is independent of the particular metric chosen for $N$.

**4.2. Theorem** *Let $G$ be a compact Lie group acting smoothly on the manifolds $M$ and $N$. Let $\varphi\colon M \to N$ be an equivariant (continuous) map. Then $\varphi$ can be approximated by a smooth equivariant map $\psi\colon M \to N$ which is equivariantly homotopic to $\varphi$ by a homotopy approximating the constant homotopy. Moreover, if $\varphi$ is already smooth on the closed invariant set $A \subset M$, then $\psi$ can be chosen to coincide with $\varphi$ on $A$, and the homotopy between $\varphi$ and $\psi$ to be constant there.*

*Proof* The idea is to embed $N$ smoothly in an orthogonal representation on $\mathbf{R}^n$, to approximate $\varphi$ by a map to $\mathbf{R}^n$, to average this approximation over the group to obtain an equivariant approximation in $\mathbf{R}^n$, and then to project normally back into $N$. For most purposes (e.g., $M$ or $N$ compact) this suffices for a proof, but in general we must overcome some technicalities since $N$ cannot generally be embedded in a representation.

Let $K$ be a compact invariant set in $M$ and let $V \supset \varphi(K)$ be an open invariant set in $N$ with $\bar{V}$ compact. Let $\theta\colon N \to \mathbf{R}^n$ be an equivariant smooth map which is an embedding on a neighborhood of $\bar{V}$. Let $U$ be an open invariant neighborhood of $K$ with $\bar{U}$ compact and $\bar{U} \subset \varphi^{-1}(V)$. Now approximate the map $\theta\varphi\colon M \to \mathbf{R}^n$ by a map $\lambda\colon M \to \mathbf{R}^n$ such that

(1) $\lambda$ is smooth on a neighborhood of $K$,
(2) $\lambda = \theta\varphi$ on $(M - U) \cup A$.

[We recall the construction of such approximations (see Milnor [2]): For $x \in M$ let $V_x$ be a neighborhood of $x$ and let $\lambda_x\colon V_x \to \mathbf{R}^n$ be such that:

(i) If $x \in A \cap U$, then $V_x \subset U$ and $\lambda_x$ is a smooth local extension of $\theta\varphi \mid A \cap V_x$ (which exists, by definition, for $V_x$ sufficiently small).
(ii) If $x \in M - U$, then $V_x = M - K$ and $\lambda_x = \theta\varphi \mid V_x$.
(iii) If $x \in U - A$, then $V_x \subset U - A$ and $\lambda_x(y) = \theta\varphi(x)$ (a constant function on $V_x$).

Let $\{f_\alpha\}$ be a smooth partition of unity subordinate to a locally finite refinement of $\{V_x\}$, with refinement projection $\alpha \mapsto x(\alpha)$, and put

$$\lambda(y) = \sum f_\alpha(y)\lambda_{x(\alpha)}(y).$$

This clearly satisfies our requirements and the degree of approximation of $\lambda$ to $\theta\varphi$ can be made as close as we wish by taking the $V_x$ to be sufficiently small for $x \in U$.]

Now define $\mu$: $M \to \mathbf{R}^n$ by

$$\mu(x) = \int_G g\lambda(g^{-1}x)\, dg.$$

Then $\mu(hx) = \int g\lambda(g^{-1}hx)\, dg = \int (hg)\lambda(g^{-1}x)\, dg = h\mu(x)$, so that $\mu$ is equivariant. Now for $y$ ranging over $G(x)$,

$$\|\mu(x) - \theta\varphi(x)\| = \left\| \int \big(g\lambda(g^{-1}x) - g(\theta\varphi)(g^{-1}x)\big)\, dg \right\| \leq \max\|\lambda(y) - (\theta\varphi)(y)\|$$

so that $\mu$ approximates $\theta\varphi$ as closely as we wish. Moreover, $\mu$ is smooth on some neighborhood of $K$ and equals $\theta\varphi$ on $(M - U) \cup A$.

Now since $\theta$ is an embedding on some neighborhood of $\bar{V}$ it follows (as in the *proof* of 2.2) that there is an invariant tubular neighborhood in $\mathbf{R}^n$ of $\theta(V)$. Also, $\mu$ may be assumed to be so close an approximation to $\theta\varphi$ that each line segment between $\mu(x)$ and $\theta\varphi(x)$ lies in this tubular neighborhood for $x \in \varphi^{-1}(V)$. (Recall that $\mu(x) = \theta\varphi(x)$ for $x \in M - U$.) Let $r$ be the (smooth equivariant) retraction of this tubular neighborhood onto $V$. Then the map $\eta$: $M \to \mathbf{R}^n$ given by

$$\eta(x) = \begin{cases} r\mu(x) & \text{for} \quad x \in \varphi^{-1}(V), \\ \mu(x) = \theta\varphi(x) & \text{for} \quad x \in M - U \end{cases}$$

is an approximation to $\theta\varphi$ and the maps $\eta_t$: $M \to \mathbf{R}^n$ given by

$$\eta_t(x) = \begin{cases} r\big(t\mu(x) + (1 - t)\theta\varphi(x)\big) & \text{for} \quad x \in \varphi^{-1}(V), \\ \mu(x) = \theta\varphi(x) & \text{for} \quad x \in M - U \end{cases}$$

define a homotopy between $\eta = \eta_1$ and $\theta\varphi = \eta_0$.

Since $\eta_t = \theta\varphi$ outside $U$ we can pull the $\eta_t$ back uniquely to

$$\psi_t\colon M \to N$$

such that $\theta\psi_t = \eta_t$ and $\psi_t = \varphi$ outside $U$. With $\psi = \psi_1$, note that

(a) $\psi$ is smooth on a neighborhood of $K$,
(b) $\psi = \varphi$ on $(M - U) \cup A$.

Also, of course, $\psi$ is equivariant and can be made to approximate $\varphi$ as closely as we please. (Similarly, the homotopy $\psi_t$ approximates the constant homotopy.)

To complete the proof let $f\colon M \to \mathbf{R}^+$ be an invariant continuous function which is infinite at the point of infinity [e.g., put a metric on the one-point compactification $(M/G)_+$ and put $f(x) = 1/\mathrm{dist}(x^*, \infty)$]. Put $K_i = f^{-1}([2i, 2i+1])$ and $L_i = f^{-1}([2i-1, 2i])$, which are compact invariant sets filling out $M$. Put $K = \bigcup K_i$ and $L = \bigcup L_i$. Clearly we may simultaneously approximate $\varphi$ on each of the disjoint compact sets $K_i$ and obtain an approximation (and the homotopy) $\psi'$ to $\varphi$ such that $\psi'$ is smooth on a neighborhood of $K$ and $\psi' = \varphi$ on $A$. Now similarly approximate $\psi'$ by $\psi''\colon M \to N$ such that $\psi''$ is smooth on a neighborhood of $L$ and $\psi'' = \psi'$ on the closed invariant set $A \cup K$. Since $\psi'' = \psi'$ on $K$ we see that $\psi''$ is *everywhere* smooth and equals $\varphi$ on $A$. ▮

**4.3. Corollary** *Let $G$, $M$, $N$ be as in* 4.2. *Then any equivariant map $M \to N$ is equivariantly homotopic to a smooth equivariant map. Moreover, if two smooth equivariant maps $M \to N$ are equivariantly homotopic, then they are so by a smooth equivariant homotopy.*

*Proof* The first part is immediate from 4.2. For the second part note that a homotopy can be assumed to be constant near the ends and can be extended to $\mathbf{R} \times M \to N$. Then the second part follows from 4.2 with $A = \{0, 1\} \times M$. ▮

## 5. FUNCTIONAL STRUCTURES ON CERTAIN ORBIT SPACES

In the next section we will prove the smooth analog of the theorem of Chapter V, Section 6 concerning actions with two types of orbits. Our version of this result differs from, and is stronger than that of Jänich [1]. This section contains some calculus lemmas we shall need.

Let us first illustrate the difficulties in proving the smooth analog of V.6.1 by discussing the simple case of the involution $x \mapsto -x$ on $\mathbf{R}$; also see Jänich [1]. Call this action $\Theta$. In this example $\mathbf{R}^+ = [0, \infty)$ is a cross section for the action and can be identified with the orbit space. Jänich takes the differentiable structure on the orbit space to be that given by this identification. Thus the orbit map $\mathbf{R} \to \mathbf{R}^+$ can be regarded as $x \mapsto |x|$. Let $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ be the diffeomorphism $f(x) = x + x^2$ (for example). The classification theorem we seek is a classification of actions *over* $\mathbf{R}^+$; that is, the objects to be classified are actions together with diffeomorphisms of the orbit space with $\mathbf{R}^+$, and they are to be classified up to equivariant diffeomorphism, preserving the maps to $\mathbf{R}^+$. Thus $(\Theta, 1)$ and $(\Theta, f)$ are such objects. We claim that they are not equivalent; that is, that there does not exist an equivariant diffeomorphism $\tilde{f}$: $\mathbf{R} \to \mathbf{R}$ which covers $f$ on the orbit space $\mathbf{R}^+$. In fact it is clear that $\tilde{f}$ would have to be $x \mapsto x(1 + |x|)$ or $x \mapsto -x(1 + |x|)$, neither of which is $C^\infty$. On the other hand, we really do not want to distinguish between $(\Theta, 1)$ and $(\Theta, f)$. To overcome this technicality, Jänich classifies only up to diffeomorphisms of the model ($\mathbf{R}^+$ here) for the orbit space which are strongly isotopic to the identity (isotopic with the boundary remaining fixed). From our viewpoint, however, we regard this example as showing that the orbit space has been given the wrong differentiable structure. Rather than using cross sections to define the structure, we use the naturally defined structure *induced* by the orbit map from the structure on the manifold. (In the above example, we claim that this structure will result by regarding $x \mapsto x^2 \in \mathbf{R}^+$ as the orbit map, and we also claim that this removes the anomaly. This requires justification, which we now turn to in the general case.)

Throughout this section we let $G \subset \mathbf{O}(n)$ be a closed subgroup which is *transitive* on $\mathbf{S}^{n-1}$. Let $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be coordinates in $\mathbf{R}^{n+m}$ with $G$ acting on $x_1, \ldots, x_n$ and acting trivially on the $y_i$ coordinates. Let $\pi$: $\mathbf{R}^{n+m} \to \mathbf{R}^{n+m}/G$ be the orbit map and give $\mathbf{R}^{n+m}/G$ the *induced* functional structure (i.e., $f$ is smooth iff $f \circ \pi$ is smooth). Let

$$\theta\colon \mathbf{R}^{n+m} \to \mathbf{R}^+ \times \mathbf{R}^m$$

be the map $\theta(x_1, \ldots, x_n, y_1, \ldots, y_m) = (r^2, y_1, \ldots, y_m)$, where $r^2 = x_1^2 + \cdots + x_n^2$. Clearly there is a unique map $\varphi$ such that the diagram

$$\begin{array}{ccc} \mathbf{R}^{n+m} & \xrightarrow{\theta} & \mathbf{R}^+ \times \mathbf{R}^m \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & \mathbf{R}^{n+m}/G & \end{array}$$

commutes. Moreover $\varphi$ is a *homeomorphism* and is a morphism of functional structures. We shall prove that $\varphi$ is a *diffeomorphism* (and hence that $\mathbf{R}^{n+m}/G$ is a smooth manifold with boundary); see Corollary 5.4.

Suppose that $f$: $\mathbf{R}^+ \times \mathbf{R}^m \to \mathbf{R}$ is any given function and let $h$: $\mathbf{R}^{m+1} = \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ be defined by

$$h(x, y_1, \ldots, y_m) = f(x^2, y_1, \ldots, y_m).$$

Note that also

$$h(x, y_1, \ldots, y_m) = (f\theta)(x, 0, \ldots, 0, y_1, \ldots, y_m).$$

Thus the following implications are clear

$$f \in C^\infty(\mathbf{R}^+ \times \mathbf{R}^m) \Rightarrow (f\theta) \in C^\infty(\mathbf{R}^{n+m}) \Rightarrow h \in C^\infty(\mathbf{R}^{m+1}).$$

**5.1. Theorem**

$$h \in C^\infty(\mathbf{R}^{m+1}) \Rightarrow f \in C^\infty(\mathbf{R}^+ \times \mathbf{R}^m).$$

*Proof* First we shall show that if $h$ is $C^\infty$, then $f$ is $C^\infty$ in the sense that the partial derivatives with respect to the coordinate $x$ of $\mathbf{R}^+$ at $x = 0$ are the *one-sided* derivatives. We then prove the well-known fact (Lemma 5.2) that this implies that $f$ is $C^\infty$ in the usual sense that $f$ extends locally to a smooth function on $\mathbf{R} \times \mathbf{R}^m$.

It is clear that $f$ is smooth except possibly at points of the boundary $x = 0$. Let us make the inductive assumption that $Df$ exists and is continuous when $D$ is a monomial in $\partial/\partial x$, $\partial/\partial y_1, \ldots, \partial/\partial y_m$ which involves $\partial/\partial x$ at most $k - 1$ times. We shall say that such a function is $C^{k-1,\infty}$. This clearly holds for $f$ when $k = 1$.

Since $h$ is an even function of $x$ we know that its partial derivatives with respect to $x$ of *odd* order vanish at $x = 0$. Thus, with $y = (y_1, \ldots, y_m)$, we have the Taylor's expansion

$$h(x, y) = h(0, y) + \frac{h''(0, y)}{2} x^2 + \cdots + \frac{h^{(2n-2)}(0, y)}{(2n-2)!} x^{2n-2} + r(x, y)x^{2n},$$

where $r(x, y)$ is $C^\infty$ and we take $2n > k$, $n \geq 2$. Since $r$ is even in $x$ we can put

$$r(x, y) = s(x^2, y)$$

and $s$ is $C^{k-1,\infty}$ by the inductive assumption. Then

$$f(x, y) = h(0, y) + \frac{h''(0, y)}{2} x + \cdots + \frac{h^{(2n-2)}(0, y)}{(2n-2)!} x^{n-1} + s(x, y)x^n. \quad (1)$$

Let $D$ be a monomial in $\partial/\partial x$, $\partial/\partial y_1, \ldots, \partial/\partial y_m$ involving $\partial/\partial x$, $k$ times. Put $X = \partial/\partial x$ and write $D = TXU$, where $T$ has no $\partial/\partial x$ terms. Applying the operator $U$ to (1) we see that

$$Uf(x, y) = x^n Us(x, y) + g(x, y),$$

where $g$ is $C^{1,\infty}$. Thus it suffices to show that $TX(x^n Us)$ exists and is continuous (at $x = 0$). Now

$$X(x^n Us)_{x=0} = \lim_{x \to 0} x^n Us/x = 0$$

since $n \geq 2$. Thus also

$$TX(x^n Us)_{x=0} = 0.$$

For $x \neq 0$, $s(x, y)$ is $C^\infty$ so that

$$X(x^n Us) = nx^{n-1}Us + x^n XUs \qquad (\text{for} \quad x \neq 0)$$

and hence

$$TX(x^n Us) = nx^{n-1}TUs + x^n Ds \qquad (\text{for} \quad x \neq 0).$$

It suffices to show that this approaches zero uniformly in a compact neighborhood $N$ of $(0, y_0)$ as $x$ approaches zero. Since $TUs$ is continuous (by the inductive assumption) it suffices to show that

$$x^n Ds \xrightarrow{u} 0 \text{ in } N \text{ as } x \to 0.$$

Now since $r(x, y) = s(x^2, y)$ is $C^\infty$ we see that, for $x \neq 0$,

$$(Dr)(x, y) = (2x)^k (Ds)(x^2, y) + \eta(x, y),$$

where $\eta$ involves only derivatives of $s$ which have lower degree in $X$ than $k$ and hence where $\eta$ is the *restriction to* $x \neq 0$ of a *continuous* function (by the inductive assumption). This implies that

$$(2x)^k (Ds)(x^2, y)$$

is bounded in any compact neighborhood of $(0, y_0)$. Since $2n > k$ this implies that in any compact neighborhood of $(0, y_0)$,

$$x^{2n}(Ds)(x^2, y) \xrightarrow{u} 0 \text{ as } x \to 0,$$

and hence that

$$x^n (Ds)(x, y) \xrightarrow{u} 0 \text{ in } N \text{ as } x \to 0$$

as was to be shown. ∎

As remarked, this proof only shows that $f$ is $C^\infty$ in the obvious sense where the partial derivatives $\partial/\partial x$ are the one-sided partials at $x = 0$. To show that this coincides with the usual meaning we must show that $f$ extends locally to a $C^\infty$ function on $\mathbf{R}^{m+1}$. Suppose that we can find a $C^\infty$ function $\varphi$ on $\mathbf{R}^{n+1}$ (coordinates $x, y_1, \ldots, y_m$) such that

$$\frac{\partial^i \varphi}{\partial x^i}(0, y) = \frac{\partial^i f}{\partial x^i}(0, y)$$

for all $i$ and $y$. Then if we define $f(x, y) = \varphi(x, y)$ for $x < 0$, it follows immediately that $f$ is $C^\infty$ on $\mathbf{R}^{m+1}$. (Interchange of order of differentiation is valid for $f$ and for $\varphi$.) Thus it suffices to prove the following lemma.

**5.2. Lemma** *Suppose we are given a sequence of $C^\infty$ functions $g_0$, $g_1$, $g_2, \ldots$ on $\mathbf{R}^m \to \mathbf{R}$. Then there exists a $C^\infty$ function $\varphi$: $\mathbf{R}^{m+1} \to \mathbf{R}$ (coordinates $x, y_1, \ldots, y_m$ on $\mathbf{R}^{m+1}$) such that*

$$\frac{\partial^i \varphi}{\partial x^i}(0, y_1, \ldots, y_m) = g_i(y_1, \ldots, y_m)$$

*for all $i$ and all $y \in \mathbf{R}^m$.*

*Proof* It suffices to find $\varphi$ so that this is true on the cube $|x| < 1$, $|y_1| < 1, \ldots, |y_m| < 1$ since we can patch together using a partition of unity (and we only need the lemma locally in any case). Let $B$: $\mathbf{R} \to \mathbf{R}^+$ be a $C^\infty$ function such that

$$B(x) = \begin{cases} 1 & \text{for } x \text{ in some neighborhood of } 0, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

We fix this "bump function" once and for all.

Let $D$ be a monomial in $\partial/\partial x$, $\partial/\partial y_1, \ldots, \partial/\partial y_m$ of degree $d$ in $\partial/\partial x$ and $b$ over all. Let $n > b$ and $a > 0$ and consider the function

$$D\left(\frac{1}{n!} g_n(y) x^n B(ax)\right).$$

This is a sum of $d + 1$ functions of the form

$$h_{n,j}(y) x^{n-d+j} a^j B^{(j)}(ax) = x^{n-d} h_{n,j}(y)(ax)^j B^{(j)}(ax),$$

where $0 \leq j \leq d$ and the $C^\infty$ functions $h_{n,j}$ depend on $g_n$ and the explicit form of $D$. (There are only finitely many $D$ of degree $b$ and hence only finitely many of the functions $h_{n,j}$ to consider. Their explicit form is unimportant.)

Note that the function $z^j B^{(j)}(z)$ is bounded since it vanishes for $|z| > 1$. It follows that there exists a bound $M_n$ such that

$$|h_{n,j}(y)(ax)^j B^{(j)}(ax)| < \frac{1}{n} M_n$$

for $|y_i| \leq 1$, independently of $a$, $j$ and of $D$ (of degree $b < n$). Since $n - d$

$\geq 1$ it follows, upon multiplying by $x^{n-d}$ and adding the $d+1 \leq n$ functions, that for $|y_i| \leq 1$ and $|x| \leq 1$ we have

$$\left| D\left(\frac{1}{n!} g_n(y) x^n B(ax)\right) \right| < M_n |x|.$$

Putting $a_n = 2^n M_n$ we conclude that, for $|y_i| \leq 1$,

$$\left| D\left(\frac{1}{n!} g_n(y) x^n B(a_n x)\right) \right| \begin{cases} < \dfrac{1}{2^n} & \text{when } |x| \leq \dfrac{1}{2^n M_n} = \dfrac{1}{a_n}, \\ = 0 & \text{when } |x| > \dfrac{1}{a_n}, \end{cases}$$

since $B(a_n x) = 0$ for $a_n |x| > 1$.

Thus for any $D$ as above of degree $b$ and with

$$\varphi_n = \frac{1}{n!} g_n(y) x^n B(a_n x)$$

the series

$$\sum_{n=0}^{\infty} D\varphi_n = \sum_{n=0}^{b} D\varphi_n + \sum_{n=b+1}^{\infty} D\varphi_n$$

has $|D\varphi_n| < 1/2^n$ for $n \geq b+1$ (and $|y_i| \leq 1$). Thus this series converges uniformly (in the cube) and hence converges to $D\varphi$ where

$$\varphi = \sum_{n=0}^{\infty} \varphi_n.$$

Thus $\varphi$ is $C^\infty$.

Putting $D = \partial^d/\partial x^d$ and recalling that $B(a_n x) = 1$ for $x$ sufficiently close to zero, we calculate that $(D\varphi)(0, y) = g_d(y)$. ▮

Recall now the notation introduced preceding Theorem 5.1.

### 5.3. Corollary

$$f \in C^\infty(\mathbf{R}^+ \times \mathbf{R}^m) \quad \text{iff} \quad f\theta \in C^\infty(\mathbf{R}^{n+m}). \quad ▮$$

**5.4. Corollary** *The map $\varphi$: $\mathbf{R}^{n+m}/G \to \mathbf{R}^+ \times \mathbf{R}^m$ is a diffeomorphism and hence $\mathbf{R}^{n+m}/G$ is a smooth manifold with boundary.* ▮

Because of this result we may (for notational convenience) identify $\mathbf{R}^{n+m}/G$ with $\mathbf{R}^+ \times \mathbf{R}^m$ via $\varphi$ and thus the orbit map is identified with $\theta$.

Now suppose that $f\colon \mathbf{R}^+ \times \mathbf{R}^m \to \mathbf{R}^+ \times \mathbf{R}^m$ is a diffeomorphism. (It suffices for this to be defined in some open set rather than globally.) Assume that $f$ fixes the boundary; that is, $f(0, y) = (0, y)$. Put $f(x, y) = \big(f_0(x, y), f_1(x, y)\big) \in \mathbf{R}^+ \times \mathbf{R}^m$.

Define $\psi\colon \mathbf{R}^{n+m} \to \mathbf{R}^{n+m}$ by $\psi(x, y) = \big(\psi_0(x, y),\ \psi_1(x, y)\big) \in \mathbf{R}^n \times \mathbf{R}^m$ where

$$\psi_0(x, y) = \big(f_0(\|x\|^2, y)\big)^{1/2} \cdot \frac{x}{\|x\|},$$
$$\psi_1(x, y) = f_1(\|x\|^2, y).$$

**5.5. Theorem** *The map $\psi$ is an equivariant diffeomorphism covering $f$ (i.e., $\theta\psi = f\theta$).*

***Proof*** Obviously $\psi$ is equivariant and it covers $f$ since

$$\theta\psi(x, y) = (\|\psi_0(x, y)\|^2, \psi_1(x, y)) = \big(f_0(\|x\|^2, y), f_1(\|x\|^2, y)\big) = f\theta(x, y).$$

Since $f$ is a diffeomorphism preserving orbit structure, $\psi$ is clearly one–one and onto. To see that $\psi$ is a diffeomorphism we use Taylor's Theorem to write

$$f_0(x, y) = f_0(0, y) + x\frac{\partial f_0}{\partial x}(0, y) + x^2k(x, y),$$

where $k$ is $C^\infty$. Since $f_0(0, y) = 0$ and $c(y) = (\partial f_0/\partial x)(0, y) > 0$ (because $f$ is a diffeomorphism) we can rewrite this as

$$f_0(x, y) = xc(y)\big(1 + xh(x, y)\big),$$

where $h$ is $C^\infty$. Thus for $x \in \mathbf{R}$ we have

$$\frac{x}{|x|}\,(f_0(x^2, y))^{1/2} = x(c(y))^{1/2}(1 + x^2h(x^2, y))^{1/2}$$

and hence, for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$\psi_0(x, y) = x\,\big(c(y)\big)^{1/2}\big(1 + \|x\|^2h(\|x\|^2, y)\big)^{1/2}$$

which is clearly $C^\infty$. Moreover, it is an easy check, which we leave to the reader, to see that the Jacobian of $\psi$ is nowhere zero. Hence $\psi$ is a diffeomorphism. (Alternatively, one can note that this construction applied to $f^{-1}$ will yield $\psi^{-1}$.) ∎

*Remarks* Theorem 5.1 is due to Whitney [1]. Lemma 5.2 is a very special case of the Whitney Extension Theorem (see Malgrange [1]) but we felt it was desirable to include the relatively simple proof of this special case. There is a general conjecture which includes 5.1 as a very special case and which is formulated as follows. Let the compact Lie group $G$ act on $\mathbf{R}^n$ via some linear representation. From a classical theorem of Hilbert on invariant theory (see Weyl [1]) it is known that there exist a finite number $p_1, \ldots, p_k$ of homogeneous polynomial functions $\mathbf{R}^n \to \mathbf{R}$ which are invariant under $G$ and such that if $p\colon \mathbf{R}^n \to \mathbf{R}$ is any invariant polynomial function, then $p = q(p_1, \ldots, p_k)$ for some polynomial function $q\colon \mathbf{R}^k \to \mathbf{R}$. The conjecture then states that any invariant $C^\infty$ function $g\colon \mathbf{R}^n \to \mathbf{R}$ has the form $g = f(p_1, \ldots, p_k)$ for some $C^\infty$ function $f\colon \mathbf{R}^k \to \mathbf{R}$. This would clearly give one a very firm grasp on the functional structure on the orbit space of a smooth action.

## 6. SPECIAL *G*-MANIFOLDS

Let $G$ be a compact Lie group acting smoothly on a manifold $M$. Let $G/H$ be the principal orbit type and suppose that $P \approx G/K$ is a nonprincipal orbit such that there are exactly two orbit types in a neighborhood of $P$. Then $P$ has a smooth tubular neighborhood which is equivariantly diffeomorphic to $G \times_K \mathbf{R}^k$, where $K$ acts orthogonally on $\mathbf{R}^k$. We may assume (after conjugation) that $H$ is the principal isotropy group of $K$ on $\mathbf{R}^k$. According to IV.6.3 there are exactly two possibilities and we shall assume here that the first of these holds. That is, we assume that $k = n + m$ and that $K$ acts on $\mathbf{R}^k$ via a representation into $\mathbf{O}(n) \subset \mathbf{O}(k)$ and is *transitive* on the unit sphere $S^{n-1}$ in the orthogonal complement $\mathbf{R}^n \times \{0\}$ to the fixed point set $(\mathbf{R}^k)^K = \{0\} \times \mathbf{R}^m$. Then $H$ can be taken to be the inverse image of $\mathbf{O}(n-1)$ in $K$. Now $N(H) \cap N(K)$ acts on the right of $G \times_K \mathbf{R}^k$ (see Exercise 15) and we shall *assume* that this action is smooth.

We shall say that $M$ is a (smooth) **special *G*-manifold** (see Jänich [1]) if there are at most two orbit types in the vicinity of each orbit and if the conditions above hold about each nonprincipal orbit. In this case, $M/G$ is a topological $(m + 1)$-manifold with boundary (corresponding to the nonprincipal orbits).

We give $M/G$ the functional structure induced from that of $M$. Now if $f$ is a real-valued function on $G \times_K \mathbf{R}^k$ which is invariant under the action of $G$, then its pull-back to $G \times \mathbf{R}^k$ is independent of the $G$-coordinate, and it follows that $f$ is smooth iff its restriction to the fiber $\mathbf{R}^k$ is smooth. Thus, near an orbit of type $G/K$ the functional structure of $M/G$ is that induced from $\mathbf{R}^k$ via the orbit map $\mathbf{R}^k \to \mathbf{R}^k/K \approx (G \times_K \mathbf{R}^k)/G$.

From 5.4 we conclude that $M/G$ is a *smooth* $(m + 1)$-manifold with boundary.

Now let $X$ be a given smooth $(m + 1)$-manifold with boundary $B$. We wish to classify smooth special $G$-manifolds over $X$. That is, we consider pairs $(M, \varphi)$, where $M$ is a smooth special $G$-manifold and $\varphi$: $M \to X$ is a smooth map which factors through $M/G$ such that the induced map

$$\varphi^*\colon\ M/G \to X$$

is a *diffeomorphism*. Two such pairs $(M, \varphi)$ and $(N, \psi)$ are said to be equivalent if there exists an equivariant diffeomorphism $\mu$: $M \to N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu} & N \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & X & \end{array}$$

commutes. As in Chapter V, Section 6, we simply use the terminology "$M$ and $N$ are equivalent over $X$" for this relationship, with $\varphi$ and $\psi$ being understood.

We now consider the proof of the Classification Theorem V.6.1. Recall that to state and prove that theorem one first selects a principal isotropy group $H$ and then selects an isotropy group $K_\alpha$ corresponding to each boundary component $B_\alpha$ of $B$. Then one restricts one's attention to *proper* special $G$-manifolds $M$ over $X$ (i.e., near a point of $B_\alpha$, $M$ has the form $G \times_{K_\alpha} \mathbf{R}^{k_\alpha}$ as above with $H$ being an isotropy group of $K_\alpha$ on the slice $\mathbf{R}^{k_\alpha}$). (This selection of the $K_\alpha$ and $H$ is temporary and will not affect the statement of our main theorem 6.2.) In V.6.1 it was assumed, for convenience, that the $K_\alpha$ were constant $= K$, but the generalization is obvious (as was stated there).

A few moments reflection (using, for example, the Smooth Approximation Theorem 4.2 and its Corollary 4.3) should convince the reader that the only difficulty in carrying out the proof of V.6.1 in the *smooth* case is the establishment of the smooth analog of V.4.2 (the "Tube Theorem"). [The reader may, in fact, feel that this also follows trivially from the Invariant Smooth Tubular Neighborhood Theorem 2.2, but this is not so. In fact, it would be false if one uses Jänich's differentiable structure on $M/G$ as would be our main theorem 6.2.]

Let $(M, \varphi)$ be a smooth special $G$-manifold over $X$ and let $A = \varphi^{-1}(B)$, which is the union of the nonprincipal orbits. Then $A$ is a closed submanifold of $M$. Let $\xi$ be a euclidean $G$-bundle over $A$ and $\tau\colon E(\xi) \to M$ an invariant tubular neighborhood of $A$. The canonical map $E(\xi) \to \mathbf{R}^+ \times B$ taking a vector $v$ in the fiber over $a \in A$ to $(\| v \|^2, \varphi(a))$ induces a *diffeomorphism* $E(\xi)/G \xrightarrow{\approx} \mathbf{R}^+ \times B$ by the previous discussion and 5.4. We shall use this to *identify* $E(\xi)/G$ with $\mathbf{R}^+ \times B$. Thus $\tau$ induces

$$\tau^*\colon \mathbf{R}^+ \times B \to X$$

(via $\varphi^*\colon M/G \xrightarrow{\approx} X$ which we also regard, at present, as an identification). Note that $\tau^*(0, b) = b$ and that $\tau^*$ is a diffeomorphism onto its image (an open set). That is, $\tau^*$ is a smooth collar of $B$ in $X$. Now the smooth version of the "Tube Theorem" V.4.2 would say, not only that such a tubular neighborhood $\tau$ exists, but that it exists such that $\tau^*$ equals any *preassigned* smooth collar

$$\lambda\colon \mathbf{R}^+ \times B \to X.$$

For simplicity of notation we shall regard $E(\xi)$ to be a subspace $W$ of $M$ (via the embedding $\tau$) so that $\tau$ will be regarded as the inclusion, and similarly $\tau^*\colon \mathbf{R}^+ \times B \to X$ will be regarded as an inclusion. (Since $\lambda$ is a given preassigned collar it would be more logical to regard $\lambda$ as an inclusion, but this would complicate the notation below.)

Now we know (essentially from 2.6) that there exists a smooth isotopy

$$F\colon (\mathbf{R}^+ \times B) \times \mathbf{I} \to X$$

with

$$F((s, b), t) = \begin{cases} (s, b) & \text{for } \ t = 0, \\ \lambda(s, b) & \text{for } \ t = 1, \\ (0, b) & \text{for } \ s = 0 \ \text{ and } \ \text{all } t \end{cases}$$

(and, of course, $F(\cdot, t)$ is a diffeomorphism onto its image for each $t$; i.e., it is a collar). If we can cover this isotopy by an equivariant isotopy on $M$, then for $t = 1$ we will have our desired invariant tubular neighborhood of $A$ over the given collar $\lambda$ of $B$ in $X$. Thus it will suffice to prove the following result.

**6.1. Theorem** *In the above situation there exists a (smooth) equivariant isotopy*

$$\tilde{F}\colon W \times \mathbf{I} \to M,$$

*of invariant tubular neighborhoods of $A = \varphi^{-1}(B)$, which covers $F$.*

*Proof* Let $b \in B$ and let $\mathbf{R}^m \approx U' \subset B$ be a small coordinate neighborhood of $b$ in $B$. Now the part of $W \subset M$ over $\mathbf{R}^+ \times U' \subset \mathbf{R}^+ \times B \subset X$ has the form

$$G \times_K (\mathbf{R}^n \times U'),$$

where $K \subset \mathbf{O}(n)$ acts on $\mathbf{R}^n$ orthogonally and is transitive on $\mathbf{S}^{n-1}$. Thus the part of $W \times \mathbf{I}$ over $\mathbf{R}^+ \times U' \times \mathbf{I}$ has the form

$$G \times_K (\mathbf{R}^n \times U' \times \mathbf{I})$$

(and one can enlarge $\mathbf{I}$ to $\mathbf{R}$ if so desired to avoid boundary points).

Let $U \subset U'$ be a coordinate neighborhood of $b$ and let $\varepsilon > 0$ be so small that

$$F([0, 2\varepsilon) \times U \times \mathbf{I}) \subset \mathbf{R}^+ \times U'.$$

The part of $M \times \mathbf{I}$ over $[0, 2\varepsilon) \times U \times \mathbf{I}$ is $G \times_K (D \times U \times \mathbf{I})$, where $D \subset \mathbf{R}^n$ is the open ball of radius $(2\varepsilon)^{1/2}$. Let

$$E\colon \mathbf{R}^+ \times B \times \mathbf{I} \to X \times \mathbf{I}$$

be the embedding defined by $E(s, b, t) = \big(F(s, b, t), t\big)$. Then it follows from 5.5 that on $[0, 2\varepsilon) \times U \times \mathbf{I}$, $E$ is covered by a $K$-equivariant embedding $D \times U \times \mathbf{I} \to \mathbf{R}^n \times U' \times \mathbf{I}$ which is the identity on $\{0\} \times U \times \mathbf{I}$ and on $D \times U \times \{0\}$ and which commutes with the projections to $\mathbf{I}$ (since $E$ does). Since $G \times_K (\cdot)$ is functorial, this induces a $G$-equivariant embedding

$$\varphi^{-1}\big([0,2\varepsilon) \times U\big) \times \mathbf{I} \approx G \times_K (D \times U \times \mathbf{I}) \to G \times_K (\mathbf{R}^n \times U' \times \mathbf{I})$$
$$\approx \varphi^{-1}(\mathbf{R}^+ \times U') \times \mathbf{I}$$

covering $E$.

Now $\varphi^{-1}\big((0, \infty) \times U\big) \approx G/H \times (0, \infty) \times U$ and $M - A$ is a smooth $G/H$-bundle over $X - B$ (with structure group $N(H)/H$). A smooth equivariant map $\varphi^{-1}\big((0, \infty) \times U\big) \times \mathbf{I} \to (M - A) \times \mathbf{I}$ covering $E$ would be determined by its restriction to $\{eH\} \times (0, \infty) \times U \times \mathbf{I}$ and this would be a smooth lifting of $E \,|\, (0, \infty) \times U \times \mathbf{I}$ to the principal $N(H)/H$-bundle $(M - A)^H \times \mathbf{I}$ over $(X - B) \times \mathbf{I}$; compare II.5.13. Thus it is clear that the restriction of the above embedding to

$$\varphi^{-1}([0, \varepsilon] \times U) \times \mathbf{I} \to M \times \mathbf{I},$$

together with the identity on $\varphi^{-1}([0, \infty) \times U) \times \{0\}$, can be extended to a smooth equivariant map

$$\tilde{E}_U\colon \varphi^{-1}(\mathbf{R}^+ \times U) \times \mathbf{I} \to M \times \mathbf{I}$$

which covers $E$ on $\mathbf{R}^+ \times U \times \mathbf{I}$. Now $\tilde{E}_U$ is an embedding since it is smooth, it covers an embedding, and it preserves orbit types. Note that $\tilde{E}_U(x, 0) = (x, 0)$ and that $\tilde{E}_U(a, t) = (a, t)$ for $a \in A$.

Let $U$ and $V$ be open sets in $B$ for which we have such embeddings $\tilde{E}_U$ and $\tilde{E}_V$ as above. Then we shall show how to construct such an embedding $\tilde{E}_{U\cup V}$. Let

$$\theta = \tilde{E}_V^{-1}\tilde{E}_U\colon \varphi^{-1}\big(\mathbf{R}^+ \times (U \cap V)\big) \times \mathbf{I} \to \varphi^{-1}\big(\mathbf{R}^+ \times (U \cap V)\big) \times \mathbf{I}$$

which is a self-equivalence covering the identity. This clearly has the form

$$\theta(x, t) = \big(\theta_1(x, t), t\big).$$

Let $f\colon U \cap V \to \mathbf{I}$ be smooth and such that $f = 1$ on a neighborhood of $U - V$ and $f = 0$ on a neighborhood of $V - U$. Let $f'\colon \varphi^{-1}\big(\mathbf{R}^+ \times (U \cap V)\big) \to \mathbf{I}$ be the composition $f' = f \circ \text{projection} \circ \varphi$ and define

$$\theta'(x, t) = \big(\theta_1(x, f'(x)t), t\big)$$

which is another smooth self-equivalence of $\varphi^{-1}\big(\mathbf{R}^+ \times (U \cap V)\big) \times \mathbf{I}$ covering the identity on $\mathbf{R}^+ \times (U \cap V) \times \mathbf{I}$. Then put

$$\tilde{E}_{U\cup V}(x,t) = \begin{cases} \tilde{E}_U(x, t) & \text{if } \varphi(x) \in \mathbf{R}^+ \times (U - V), \\ \tilde{E}_V \circ \theta'(x, t) & \text{if } \varphi(x) \in \mathbf{R}^+ \times (U \cap V), \\ \tilde{E}_V(x, t) & \text{if } \varphi(x) \in \mathbf{R}^+ \times (V - U), \end{cases}$$

which clearly has the desired properties.

Now using a locally finite covering of $B$ and the usual inductive procedure, we can construct an equivariant embedding

$$\tilde{E}\colon W \times \mathbf{I} = \varphi^{-1}(\mathbf{R}^+ \times B) \times \mathbf{I} \to M \times \mathbf{I}$$

covering $E$ and with $\tilde{E}(x, 0) = (x, 0)$ and $\tilde{E}(a, t) = (a, t)$ for $a \in A$. (The inductive argument is the same as that given in the proof of II.7.1 and will not be repeated.) Since $\tilde{E}$ covers $E$ it has the form

$$\tilde{E}(x, t) = \big(\tilde{F}(x, t), t\big)$$

and $\tilde{F}$ is then our desired isotopy covering $F$. ∎

As remarked, this result allows us to repeat the proof of the Classification Theorem V.6.1 in the smooth case. Note that the smooth classification is *exactly* the same as the topological one since, for instance, the set of smooth homotopy classes of smooth cross sections equals the set of homotopy classes of continuous sections (of a smooth bundle). Rather than restating that result, we shall sum up our present knowledge in the following theorem. First we introduce some notation.

Let $\mathscr{M}^\infty(G, X)$ denote the set of (smooth) equivalence classes *over* the smooth manifold $X$ with boundary $B$ of special smooth $G$-manifolds over $X$.

For a *topological* manifold $X$ with boundary $B$ we call a $G$-space $W$ over $X$ a **special topological $G$-manifold** over $X$ if the orbit type over $X - B$ is constant and that over each component of $B$ is constant, and such that each nonprincipal orbit has a neighborhood which can be given the structure of a *smooth* special $G$-manifold. Then it is clear that $W$ is a topological manifold and that $G$ acts locally smoothly on it. Moreover, the group $S(\pi) = [N(H) \cap N(K)]/H$ of Chapter V, Section 4 acts by orthogonal $G$-bundle equivalences on the disk bundle $M_\pi$ over $G/K$. We let $\mathscr{M}^0(G, X)$ denote the set of topological equivalence classes over $X$ of special topological $G$-manifolds over $X$.

If $X$ is smooth, then $X^0$ will denote its underlying topological manifold. The following theorem is now a direct consequence of the Classification Theorem V.6.1 and its smooth analog.

**6.2. Theorem.** *The forgetful map*

$$\mathscr{M}^\infty(G, X) \to \mathscr{M}^0(G, X^0)$$

*is a one–one correspondence.* ▮

For clarity, we restate this result as follows.

**6.3. Theorem** *Let $X$ be a topological manifold with boundary and let $M$ be a special topological $G$-manifold over $X$. For any $C^\infty$ structure on $X$ (if one exists) there exists a $C^\infty$ structure on $M$ on which $G$ acts smoothly and which induces the given structure on $X$. Moreover, given the structure on $X$, this structure on $M$ is unique up to equivariant diffeomorphism over $X$.* ▮

For example, the (special) $\mathbf{O}(n)$-manifold $\Sigma_k^{2n-1}$ over $\mathbf{D}^2$ of Chapter I, Section 7 has, by its construction, a differentiable structure in which $\mathbf{O}(n)$

acts smoothly. Similarly, the Brieskorn manifold $W_k^{2n-1}$ (Chapter V, Section 9) comes with a given smooth structure, and the plumbing construction (Chapter V, Section 8) gives, upon straightening the angle, a smooth structure on $\partial P^{2n}(\mathbf{A}_{k-1})$. By V.8.1 and V.9.2, $\Sigma_k^{2n-1}$, $W_k^{2n-1}$, and $\partial P^{2n}(\mathbf{A}_{k-1})$ are topologically equivalent over $\mathbf{D}^2$ (as $\mathbf{O}(n)$-manifolds). Hence, by 6.3, they are also *smoothly* equivalent over $\mathbf{D}^2$.

Recall from Chapter I, Section 7 that $\Sigma_k^{4m-3}$ is a topological sphere for $k$ odd. Its differentiable structure can be determined from any of the above three descriptions of it (see Hsiang and Hsiang [3], Brieskorn [1], Hirzebruch and Mayer [1], and Bredon [19]). The easiest method is from the description as $\partial P^{4m-2}(\mathbf{A}_{k-1})$ since the Arf invariant is easily computed for this (see the last two references). The result is that

$$\Sigma_k^{4m-3} \approx \begin{cases} \text{the standard sphere for} & k \equiv \pm 1 \pmod 8, \\ \text{the Kervaire sphere for} & k \equiv \pm 3 \pmod 8). \end{cases}$$

The Kervaire sphere is, by definition, $\partial P^{4m-2}(\mathbf{A}_2)$. It is sometimes standard (e.g., in dimensions $4m - 3 = 1, 5, 13, 29, 61$), but is known to be exotic when $m$ is not a power of 2; see Browder [1].

Let us briefly consider the classification of smooth self-equivalences of a special $G$-manifold $M$ over $X$. The topological case was dealt with in V.7.1, and there is little difficulty in proving the smooth analog of that result. Part of the argument would be replaced by the following remarks: Suppose that, for euclidean $G$-bundles $\xi, \eta$ on $A$, we are given two invariant tubular neighborhoods $\sigma: E(\xi) \to M$ and $\tau: E(\eta) \to M$ of $A$ which induce the *same* collar of $B$ in $X$. Then since the induced collar determines lengths of vectors in $\xi, \eta$ we see that $\sigma^{-1}\tau$ preserves norm. The proof of the first part of the Uniqueness Theorem 2.6 of invariant tubular neighborhoods provides an isotopy between $\sigma$ and $\tau\theta$ which induces the *identity* on $X$, where $\theta$ is a vector bundle isomorphism. Since $\theta$ must preserve norms, it is an orthogonal bundle isomorphism. This replaces the argument concerning the homotopy to an $S$-equivalence in the proof of V.7.1. The remainder of the proof is immediate and will be omitted. Again the classification in the smooth case is identical to that in the topological case. Letting $\pi_0\mathbf{Diffeo}_X{}^G(M)$ be the set of smooth isotopy classes over $X$ of self-equivalences of $M$ over $X$, we obtain the following theorem.

**6.4. Theorem** *If $M$ is a smooth special $G$-manifold over $X$, then the forgetful map $\pi_0\mathbf{Diffeo}_X{}^G(M) \to \pi_0\mathbf{Homeo}_X{}^G(M)$ is a one–one correspondence.* ▮

*Remark* There is no difficulty in extending the results of this section to the case of special $G$-manifolds $M$ *with boundary*. In this case $X$ would have a "corner" on its boundary which separates $\partial X$ into two parts, one part $B$ corresponding to the nonprincipal orbits and the other part corresponding to $\partial M$. For example, $X$ could be a half disk $\{x \in \mathbf{D}^{m+1} \mid x_1 \geq 0\}$, with $B = X \cap \mathbf{S}^m$ and with $\varphi(\partial M) = \{x \in \mathbf{D}^{m+1} \mid x_1 = 0\}$.

## 7. SMOOTH KNOT MANIFOLDS

In this section we shall study smooth actions of $G = \mathbf{O}(n)$, $n \geq 2$, on compact manifolds $M$ having three orbit types: fixed points, spheres $\mathbf{O}(n)/\mathbf{O}(n-1)$, and principal orbits $\mathbf{O}(n)/\mathbf{O}(n-2)$. We assume, moreover, that the orbit space $M/G$ is *topologically* a contractible manifold whose boundary corresponds to the singular orbits and with the fixed point set $M^G \subset M/G$ corresponding to a connected, orientable, codimension 2 submanifold of the boundary; see Chapter V, Sections 10 and 11. We shall call such an $\mathbf{O}(n)$-manifold an **$\mathbf{O}(n)$-knot manifold.**

From the discussion in Chapter V, Section 10 we see that about a fixed point in $M$, $\mathbf{O}(n)$ acts by twice the standard representation, which can be regarded as the canonical action of $\mathbf{O}(n)$ on $\mathbf{C}^n = \mathbf{R}^n \times \mathbf{R}^n$. Select an invariant tubular neighborhood $E(\xi) \to M$ of $M^G$, where $\xi$ is a euclidean $G$-bundle on $M^G$. The structure group of $\xi$ is clearly the centralizer of this representation $\mathbf{O}(n) \subset \mathbf{U}(n) \subset \mathbf{O}(2n)$ in $\mathbf{O}(2n)$, and it is easy to see that this centralizer is $\mathbf{O}(2)$ acting by scalar multiplication and by conjugation on $\mathbf{C}^n$. (If we think of the $\mathbf{O}(n)$ representation as matrix multiplication on the space of $n \times 2$ matrices, then the structure group $\mathbf{O}(2)$ acts by right multiplication on this.)

**7.1. Lemma** *$\xi$ is a trivial $G$-bundle over $M^G$.*

*Proof* Consider the set $A$ of vectors $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, with $z = x + iy$, such that $\| x \| = 1 = \| y \|$ and $\langle x, y \rangle = 0$. Then the structure group $\mathbf{O}(2)$ of $\xi$ preserves $A$. Moreover, $A$ is a *principal orbit* of $G$ and this $\mathbf{O}(2)$-action coincides with the right translation of $\mathbf{O}(2) \approx N(H)/H$ on $G/H$, where $H = \mathbf{O}(n-2)$. Regarding $E(\xi) \subset M$ and hence $E(\xi)/G \subset M/G$, the union of these sets $A$ defines a bundle $Q$ over a copy of $M^G$ in $\text{int}(M/G)$, and is just a restriction of the bundle of principal orbits. Since $M/G$ is contractible, this bundle $Q$ is trivial, and so is the associated principal $\mathbf{O}(2)$-

bundle $Q^H$ over this copy of $M^G$. However, by the above remarks, $\xi$ is associated with this principal bundle. ▮

Thus we have a tubular neighborhood of $M^G$ in $M$ of the form $\mathbf{C}^n \times M^G$ with $G = \mathbf{O}(n)$ acting on $\mathbf{C}^n$ as above. Now by V.9.1 the orbit map $\mathbf{C}^n \to \mathbf{C}^n/G$ can be identified with

$$\theta\colon \mathbf{C}^n \to \mathbf{R}^+ \times \mathbf{C} = \mathbf{R}^+ \times \mathbf{R}^2$$

taking $z = (z_1, \ldots, z_n) \mapsto (\| z \|^4 - |\sum z_i^2|^2, z_1^2 + \cdots + z_n^2)$. The reader may make the straightforward check, using Section 5, that $\theta$ gives the induced functional structure on $\mathbf{C}^n/G$ *on the complement of* $0 \in \mathbf{C}^n$. (This is undoubtedly false at the origin, where there should probably be a corner or a cusp. However, we know practically nothing about the induced structure of $\mathbf{C}^n/G$ at the origin and this accounts for the fact that we shall *not* attempt to classify such $\mathbf{O}(n)$-manifolds $M$ *over* $X \approx M/G$, as we did in the topological case in Chapter V, Section 10.)

Now the homeomorphism

$$\theta^* \times 1\colon (\mathbf{C}^n/G) \times M^G \to (\mathbf{R}^+ \times \mathbf{C}) \times M^G$$

defines a differentiable structure on a neighborhood of $M^G$ in $M/G$ (given the chosen tubular neighborhood of $M^G$ in $M$). Since this structure coincides with the induced structure on $M/G - M^G$ we may amalgamate them and obtain a differentiable structure on all of $M/G$ which makes it into a smooth manifold with boundary and with $M^G$ a codimension 2 smooth submanifold of the boundary. Note that $\theta\colon \mathbf{C}^n \to \mathbf{R}^+ \times \mathbf{C}$ is equivariant with respect to the action of the structure group $\mathbf{O}(2)$ of $\xi$ on $\mathbf{C}^n$ and an obvious smooth $\mathbf{O}(2)$-action on $\mathbf{R}^+ \times \mathbf{C}$. Thus the definition of this differentiable structure on $M/G$ does not depend on the fact that $\xi$ is trivial or on the particular (smooth) trivialization used.

Although this structure on $M/G$ is not natural it is clearly well defined up to diffeomorphism (preserving the submanifold $M^G$) by the Uniqueness Theorem for Invariant Tubular Neighborhoods; see the remark at the end of Section 2. (The situation is the same as that involved in "straightening the corner" of a product of two manifolds with boundary.)

**7.2. Theorem** *The assignment to $M$ of the pair $(M/G, M^G)$, with the above differentiable structure, defines a one–one correspondence between the set* *of equivariant diffeomorphism classes of* $\mathbf{O}(n)$*-knot manifolds* $M$ *and the set of diffeomorphism classes of pairs* $(X, \Sigma)$, *where* $X$ *is a compact, contractible, smooth manifold with boundary and* $\Sigma$ *is a connected, orientable, codimension* 2 *submanifold of* $\partial X$.

*Proof* Let us first show that any such pair $(X, \Sigma)$ is realizable. Let $\mathbf{R}^2 \times \Sigma \subset \partial X$ be a tubular neighborhood of $\Sigma$ in $\partial X$. For $p \in \Sigma$ the restriction map $H^1(\partial X - \Sigma) \to H^1(\mathbf{S}^1 \times \{p\})$ is an isomorphism (see Chapter V, Section 10). Let $\alpha \in H^1(\partial X - \Sigma)$ be a generator corresponding to a generator $\iota \in H^1(\mathbf{S}^1)$. The map

$$H^1(\partial X - \Sigma) \to H^1(\mathbf{S}^1 \times \Sigma) \approx H^1(\mathbf{S}^1) \oplus H^1(\Sigma)$$

takes $\alpha$ to $(\iota, \beta)$, for some $\beta$. Now $\beta \in H^1(\Sigma) \approx [\Sigma, \mathbf{S}^1]$ is induced by some smooth map $f\colon \Sigma \to \mathbf{S}^1$. Regarding $\mathbf{R}^2$ as $\mathbf{C}$, it is easy to see that the map $\mathbf{R}^2 \times \Sigma \to \mathbf{R}^2 \times \Sigma$, taking $(z, q) \mapsto (zf(q)^{-1}, q)$, induces $(\iota, \beta) \mapsto (\iota, 0)$ on $H^1(\mathbf{S}^1 \times \Sigma) = H^1(\mathbf{S}^1) \oplus H^1(\Sigma)$. Thus it follows that we can take the tubular neighborhood $\mathbf{R}^2 \times \Sigma \subset \partial X$ in such a way that $\beta = 0$. Also we may assume that this extends to a tubular neighborhood $\mathbf{R}^+ \times \mathbf{R}^2 \times \Sigma$ of $\Sigma$ in $X$. By the discussion in Chapter V, Section 10 and by 6.3 there is a unique $\mathbf{O}(n)$-manifold $M_1$ over $X - \Sigma$ corresponding to the invariant $\pm\alpha \in H^1(\partial X - \Sigma)$. The part of $M_1$ over $(\mathbf{R}^+ \times \mathbf{R}^2 - \{0\}) \times \Sigma$ corresponds to the invariant $(\iota, 0) \in H^1(\mathbf{S}^1) \oplus H^1(\Sigma) = H^1(\mathbf{S}^1 \times \Sigma) \approx H^1(\partial(\mathbf{R}^+ \times \mathbf{R}^2 - \{0\}) \times \Sigma)$. However, the $\mathbf{O}(n)$-manifold $\mathbf{C}^n \times \Sigma$ restricted to $(\mathbf{C}^n - \{0\}) \times \Sigma$ also has this invariant. Thus $M_1$ can be pasted to $\mathbf{C}^n \times \Sigma$ via some equivariant diffeomorphism over $(\mathbf{R}^+ \times \mathbf{R}^2 - \{0\}) \times \Sigma$ and this clearly gives the desired $\mathbf{O}(n)$-manifold corresponding to $(X, \Sigma)$.

Now let $M$ and $N$ be two $\mathbf{O}(n)$-knot manifolds. Select invariant tubular neighborhoods of the fixed point sets and hence the associated differentiable structures on the orbit spaces $M^*$ and $N^*$. Suppose that $h\colon M^* \to N^*$ is a diffeomorphism carrying the submanifold $M^G$ to $N^G$. Now the given tubular neighborhoods in $M$ and $N$ induce tubular neighborhoods of $M^G$ in $M^*$ and of $N^G$ in $N^*$. By an isotopy of $h$ we may suppose that $h$ takes this tubular neighborhood in $M^*$ to that in $N^*$ orthogonally. (Do this for a closed tubular neighborhood and then pass to the interior.) Over these tubular neighborhoods $h$ can clearly be covered by an equivariant diffeomorphism $\tilde{h}_1$ since both of these tubular neighborhoods are trivial: $\mathbf{C}^n \times \Sigma$ where $\Sigma \approx M^G \approx N^G$. Over the complements $M^* - M^G$ and $N^* - N^G$, $h$ can be covered by an equivariant diffeomorphism $\tilde{h}_2\colon M - M^G \to N - N^G$ by the Classification Theorem 6.3 of special $G$-manifolds; see Chapter V, Section 10. Thus

$$\psi = \tilde{h}_2^{-1}\tilde{h}_1$$

is an orbit preserving self-equivalence of the part of $M$ over the complement of $M^G$ in its tubular neighborhood [which is diffeomorphic to $(\mathbf{R}^+ \times \mathbf{R}^2 - \{0\}) \times M^G$]. By 6.4, V.7.1, V.7.2, the discussion in V.7.5, and the fact that

$$[\mathbf{D}^2 \times M^G, \mathbf{S}^1 \times M^G; \mathbf{S}^1, \star] \approx H^1((\mathbf{D}^2, \mathbf{S}^1) \times M^G; \mathbf{Z}) \approx H^{-1}(M^G) = 0$$

it follows that $\psi$ is isotopic through orbit preserving self-equivalences to the identity or to the antipodal map in the fibers of $\mathbf{C}^n \times M^G$. Using such an isotopy we can clearly find an orbit preserving self-equivalence $\varphi$ of the part of $M$ over the tubular neighborhood $\mathbf{R}^+ \times \mathbf{R}^2 \times M^G$ (*including* the 0-section $M^G$) which coincides with $\psi$ outside some smaller closed neighborhood $A$ of $M^G$. Then the map $\tilde{h}$: $M \to N$, defined to be $\tilde{h}_1\varphi^{-1}$ above $\mathbf{R}^+ \times \mathbf{R}^2 \times M^G$ and to be $\tilde{h}_2$ above $M/G - A$, is an equivariant diffeomorphism. ▮

Now we shall restrict our discussion to the case in which the orbit space is a disk. Thus let $\Sigma^k \subset \mathbf{S}^{k+2}$ be a smooth, connected, orientable submanifold. By 7.2 there is an $\mathbf{O}(n)$-knot manifold $M^{2n+k}(\Sigma^k)$ corresponding to $(\mathbf{D}^{k+3}, \Sigma^k)$ and this is unique up to equivariant diffeomorphism.

In the next section we will wish to consider the case of $\mathbf{O}(n)$-knot manifolds $M$ with boundary, where $M/G$ will now be diffeomorphic (with the structure as defined above) to

$$\mathbf{D}_+^{k+4} = \{(x_1, \ldots, x_{k+4}) \in \mathbf{D}^{k+4} \mid x_{k+4} \geq 0\},$$

$(\partial M)/G$ will correspond to $\mathbf{D}^{k+3}$ (i.e., $x_{k+4} = 0$), the set of singular orbits will correspond to $B = \mathbf{S}^{k+3} \cap \mathbf{D}_+^{k+4}$, and the fixed set $M^G$ will correspond to a submanifold $W^{k+1} \subset B$ which is connected, orientable, and transverse to $\partial B = \mathbf{S}^{k+2}$. There is no difficulty in checking that the argument goes through in this case to show that there is an $\mathbf{O}(n)$-manifold $M^{2n+k+1}(W^{k+1})$ with boundary, corresponding to $(\mathbf{D}_+^{k+4}, W^{k+1})$ and which is unique up to equivariant diffeomorphism. Clearly

$$\partial M^{2n+k+1}(W^{k+1}) \approx M^{2n+k}(\partial W^{k+1}).$$

Recall from V.11.2 that if $\Sigma^k$ is a mod 2 homology sphere, then so is $M^{2n+k}(\Sigma^k)$. Similarly it is easily seen that if $W^{k+1}$ is mod 2 acyclic, then so is $M^{2n+k+1}(W^{k+1})$. Similarly, this holds over the integers when $n$ is even.

Let us discuss briefly the question of orientation. Choose an orientation for $\mathbf{S}^{k+2}$ (and hence for $\mathbf{D}^{k+3}$) and choose one for $\Sigma^k \subset \mathbf{S}^{k+2}$. If $n$ is even, then the structure group $\mathbf{O}(2)$ preserves the orientation of $\mathbf{C}^n$ so that the orientation on $\Sigma^k$ induces a canonical orientation on $\mathbf{C}^n \times \Sigma^k$ and hence also on $M^{2n+k}(\Sigma^k)$. If $n$ is odd, then $\mathbf{O}(n)/\mathbf{O}(n-2)$ admits no orientation reversing self-equivalences (by explicit check of the action of $\mathbf{O}(2) \approx N(\mathbf{O}(n-2))/\mathbf{O}(n-2)$ on this), so that we may give it a canonical orientation once and for all. Thus when $n$ is odd the orientation of $\mathbf{D}^{k+3}$ induces an orientation on $M^{2n+k}(\Sigma^k)$. Thus oriented $\Sigma^k$ in oriented $\mathbf{S}^{k+2}$ corresponds to oriented $M^{2n+k}(\Sigma^k)$ with oriented fixed point set $\Sigma^k$ *and* oriented orbit space; but where the orientation of $M^{2n+k}(\Sigma^k)$ and that of $\Sigma^k$ are canonically related, as above, when $n$ is even, and the orientation of $M^{2n+k}(\Sigma^k)$ and of its orbit space are canonically related when $n$ is odd. Thus there are four possible "orientations" on an $\mathbf{O}(n)$-knot manifold, corresponding to the four orientations of the pair $(\mathbf{S}^{k+2}, \Sigma^k)$.

*Remark* The results in this section are due to Jänich [1] and to Hsiang and Hsiang [4]. Analogous results in the case of $\mathbf{U}(n)$ or $\mathbf{Sp}(n)$ actions are false in the smooth case even when $\Sigma$ (codimension 3 or 5, respectively, in the sphere) is a sphere. This contrasts with the topological case; see Exercise 6 of Chapter V.

## 8. GROUPS OF INVOLUTIONS

Let $0 < k < n$ be fixed and consider the set of all smooth involutions $T$, $(T^2 = 1)$ on $\mathbf{S}^n$ (oriented) with *oriented* fixed point set $\Sigma^k$ of dimension $k$. Recall that $\Sigma^k$ is a mod 2 homology $k$-sphere. Up to orientation preserving (for both $\mathbf{S}^n$ and $\Sigma^k$) equivalence, these form an abelian semigroup under the operation of connected sum, which is defined as follows: Suppose we are given two involutions $T_1$ and $T_2$ on $\mathbf{S}^n$ with fixed point sets $\Sigma_1^k$ and $\Sigma_2^k$. Let $\mathbf{R}^n$ have the involution

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, -x_{k+1}, \ldots, -x_n)$$

and pick canonical orientations on $\mathbf{R}^n$ and on the fixed set $\mathbf{R}^k$. Let $\varphi_i$: $\mathbf{R}^n \to \mathbf{S}^n$ be invariant tubular neighborhoods of points $p_i \in \Sigma_i^k$ which preserve both orientations (i.e., of $\mathbf{R}^n \to \mathbf{S}^n$ and of $\mathbf{R}^k \to \Sigma_i^k$). Let $\tau$: $\mathbf{R}^n - \{0\} \to \mathbf{R}^n - \{0\}$ be given by

$$\tau(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)/\sum x_i^2.$$

Then the connected sum $(\mathbf{S}^n, T_1, \Sigma_1^k) \# (\mathbf{S}^n, T_2, \Sigma_2^k)$ (or simply $T_1 \# T_2$) is defined to be

$$(\mathbf{S}^n - \{p_1\}) \cup_\psi (\mathbf{S}^n - \{p_2\})$$

with the involution $T_1 \cup T_2$, where

$$\psi = \varphi_2 \tau \varphi_1^{-1}\colon \varphi_1(\mathbf{R}^n - \{0\}) \to \varphi_2(\mathbf{R}^n - \{0\}).$$

(Slightly less precisely, with $D_i = \varphi_i(\mathbf{D}^n)$, this is $(\mathbf{S}^n - \operatorname{int} D_1) \cup (\mathbf{S}^n - \operatorname{int} D_2)$ via an equivariant, orientation-reversing linear map $\partial D_1 \xrightarrow{\approx} \partial D_2$.) It follows from 3.2, and the remark following it, that connected sum is well defined up to orientation-preserving equivariant diffeomorphism. Of course, $(\mathbf{S}^n, T_1, \Sigma_1^k) \# (\mathbf{S}^n, T_2, \Sigma_2^k)$ is equivalent to an involution on $\mathbf{S}^n$ with fixed set diffeomorphic to $\Sigma_1^k \# \Sigma_2^k$.

**8.1. Lemma** *Let $T$ be a smooth involution on $\mathbf{S}^n \times \mathbf{I}$ preserving both ends. Let $F$ be the fixed set and let $F_i = F \cap (\mathbf{S}^n \times \{i\})$, for $i = 0, 1$. Then $(F, F_0, F_1)$ has the* mod 2 *cohomology of $(\mathbf{S}^k \times \mathbf{I}, \mathbf{S}^k \times \{0\}, \mathbf{S}^k \times \{1\})$ for some $-1 \le k \le n$, and $F$ is orientable. [We mean that this holds for all three absolute cohomology groups and also for the relative groups of $(F, F_0)$, $(F, F_1)$, and $(F, F_0 \cup F_1)$.]*

***Proof*** By Smith theory, $F_0$ has the mod 2 cohomology of $\mathbf{S}^k$ for some $k$, which we now fix. Since $H^*(\mathbf{S}^n \times \mathbf{I}, \mathbf{S}^n \times \{0\}; \mathbf{Z}_2) = 0$ it follows from III. 7.9 that $H^*(F, F_0; \mathbf{Z}_2) = 0$. Note that it follows from this that $F$ is connected for $k > 0$ and must intersect $\mathbf{S}^n \times \{1\}$. (A similar remark holds for $k = 0$.) Similarly $H^*(F, F_1; \mathbf{Z}_2) = 0$. Thus $F$, $F_0$, and $F_1$ all have the mod 2 cohomology of $\mathbf{S}^k$ and the inclusions $F_0 \subset F$ and $F_1 \subset F$ induce mod 2 cohomology isomorphisms. Thus it is clear that $H^i(F, F_0 \cup F_1; \mathbf{Z}_2)$ is zero for $i \neq 1$, $k + 1$ and is $\mathbf{Z}_2$ for $i = 1$, $k + 1$. An easy universal coefficient argument shows that $F$ must be orientable when $k \neq 1$. In the case $k = 1$, $F$ is a 2-manifold with $F_0 \approx \mathbf{S}^1 \approx F_1$, and the cohomology shows that $F \approx \mathbf{S}^1 \times \mathbf{I}$. ▌

We shall let $n \geq 5$ and $0 < k < n$ for the remainder of this section. Let $T$ be a smooth involution on $\mathbf{S}^n \times \mathbf{I}$ (oriented) with oriented fixed point set $F$ of dimension $k+1$ and let $F_0$ and $F_1$ be as in 8.1. We give the pairs $(\mathbf{S}^n \times \{0\}, F_0)$ and $(\mathbf{S}^n \times \{1\}, F_1)$ the *induced* orientations. (Thus, in particular, the canonical identification $\mathbf{S}^n \to \mathbf{S}^n \times \{i\}$ preserves orientation for $i = 0$ and reverses it for $i = 1$.)

Suppose that $(\mathbf{S}^n, T_0, \Sigma_0{}^k)$ and $(\mathbf{S}^n, T_1, \Sigma_1{}^k)$ are involutions with oriented fixed sets $\Sigma_i{}^k$. Then we shall say that they are $L$-**equivalent** if there exists an involution $T$ on $\mathbf{S}^n \times \mathbf{I}$ with oriented fixed set $F$, as above, with $(\mathbf{S}^n, T_0, \Sigma_0{}^k) \approx (\mathbf{S}^n \times \{0\}, T, F_0)$ and $(\mathbf{S}^n, T_1, \Sigma_1{}^k) \approx -(\mathbf{S}^n \times \{1\}, T, F_1)$; where $\approx$ denotes orientation preserving smooth equivalence of involutions (with oriented total space *and* oriented fixed set) and the minus sign indicates reversal of both orientations. We shall let $\Phi_k{}^n$ denote the set of $L$-equivalence classes of all involutions on the oriented $n$-sphere with an oriented $k$-dimensional fixed point set.

**8.2. Theorem** *The set $\Phi_k{}^n$ is an abelian group under connected sum for $0 < k < n$; $n \geq 5$. An involution $(\mathbf{S}^n, T, \Sigma^k)$ represents the zero element of this group iff $T$ extends as a smooth involution on $\mathbf{D}^{n+1} \supset \mathbf{S}^n$. The inverse is given by reversal of orientation.*

*Proof* The main point is to show that the connected sum is a well-defined operation on $\Phi_k{}^n$. Thus suppose that $(\mathbf{S}^n \times \mathbf{I}, T, F)$ and $(\mathbf{S}^n \times \mathbf{I}, T', F')$ realize $L$-equivalences between involutions $T_0$ and $T_1$ and between $T_0'$ and $T_1'$, respectively. Since $k > 0$, $F$ and $F'$ are connected and touch both ends by 8.1. Thus there is a smooth arc $A$ in $F$ from a point of $\mathbf{S}^n \times \{0\}$ to a point of $\mathbf{S}^n \times \{1\}$ transverse to the boundary; and there is a similar arc $A'$ in $F'$. We can then take an invariant tubular neighborhood $N$ of $A$ (necessarily of the form $A \times \mathbf{R}^n$ with the canonical involution on $\mathbf{R}^n$ fixing $\mathbf{R}^k$) and similarly $N'$ of $A'$. In the obvious way we can use $N$ and $N'$ to patch $\mathbf{S}^n \times \mathbf{I} - A$ together with $\mathbf{S}^n \times \mathbf{I} - A'$ equivariantly with respect to $T$ on the first and $T'$ on the second. This yields an involution $J$ on a manifold $W^{n+1}$ (oriented with oriented fixed set) such that $(\partial W, J) \approx T_0 \# T_0' - T_1 \# T_1'$. By the $h$-cobordism theorem (see Milnor [5]) it is easily seen that $W \approx \mathbf{S}^n \times \mathbf{I}$, whence $T_0 \# T_0'$ is $L$-equivalent to $T_1 \# T_1'$.

Clearly $(\mathbf{S}^n, T, \Sigma^k)$ is $L$-equivalent to the *standard linear involution* with fixed set $\mathbf{S}^k \subset \mathbf{S}^n$ iff $T$ extends to an involution on $\mathbf{D}^{n+1}$. (In one direction, cap off $\mathbf{S}^n$ on the standard involution end of an $L$-equivalence; in the other direction, remove an open disk from int $\mathbf{D}^{n+1}$.) If $(\mathbf{S}^n, T, \Sigma^k)$ is any involution, let $\mathbf{D}^n$ be an invariant disk about a point of $\Sigma^k$ and consider the involution $T \times 1$ on $K = (\mathbf{S}^n - \text{int}\, \mathbf{D}^n) \times \mathbf{I}$ with the corner straightened equivariantly. Then $K \approx \mathbf{D}^{n+1}$, by the $h$-cobordism theorem, and on the boundary the involution is just $(\mathbf{S}^n, T, \Sigma^k) \# -(\mathbf{S}^n, T, \Sigma^k)$. Since the linear action obviously acts as an identity under connected sum, the result clearly follows from these remarks. ▌

These groups $\Phi_k{}^n$ are clearly of fundamental interest in the study of smooth involutions of spheres. For $n > 2k$ a good deal of information about them has recently been obtained by L. Jones [2]. We shall prove three general results about these groups, two of which show that many of these groups are infinite.

It is clear that if we consider, more generally, involutions on smooth oriented homotopy spheres we obtain analogous groups $\Theta_k{}^n$. Clearly there is an exact sequence

$$0 \to \Phi_k{}^n \to \Theta_k{}^n \to \Theta^n,$$

where $\Theta^n$ is the group of oriented homotopy spheres under connected sum. Since $\Theta^n$ is finite, some multiple of any element of $\Theta_k{}^n$ is in $\Phi_k{}^n$.

**8.3. Theorem** *For $1 \leq k \leq n$, the involution $T$: $(z_0, \ldots, z_{2n+1}) \mapsto (z_0, \ldots, z_{2k}, -z_{2k+1}, \ldots, -z_{2n+1})$ on the Brieskorn manifold $W_3^{4n+1}$ (see Chapter V, Section 9) represents an element in $\Theta_{4k-1}^{4n+1}$ whose order is either infinite or is divisible by $2^{2k}(2^{2k-1} - 1)a_k$, where $a_k = 1$ or $a_k = 2$ according as k is even or odd. (Note that since $W_3^{4n+1}$ is the Kervaire $(4n + 1)$-sphere, twice this element is always in the subgroup $\Phi_{4k-1}^{4n+1}$.)*

***Proof*** The fixed set of this involution is $W_3^{4k-1}$. (Note that this has 3-torsion in its homology by Chapter I, Section 7.) For the proof we shall use the $\mu$-invariant of Eells and Kuiper [1] (also see Montgomery and Yang [7]). We need only the following properties of $\mu$. It is defined (*in particular*) for smooth oriented mod 2 homology $(4k - 1)$-spheres $\Sigma$ which bound spin manifolds. It takes values in $\mathbf{Q}/\mathbf{Z}$ and is a homomorphism (under connected sum). There is a formula for $\mu$ in terms of Pontriagin classes and the index of any cobounding spin manifold $W$. We only need this in case $W$ is either parallelizable *or* $W$ is a mod 2 (hence rational) homology disk, and in these cases

$$\mu(\Sigma) = \frac{\tau(W)}{b_k} \in \mathbf{Q}/\mathbf{Z},$$

where $\tau(W)$ is the index of $W$ [i.e., the index of the intersection form on

$H_{2k}(W; \mathbf{Q})]$ and $b_k = -2^{2k+1}(2^{2k-1} - 1)a_k$. (Of course $\tau(W) = 0$, and hence $\mu(\Sigma) = 0$, when $W$ is a mod 2 homology disk.)

For the given involution on $W_3^{4n+1}$ we use the fact that the fixed set $W_3^{4k-1} \approx \partial P^{4k}(\mathbf{A}_2)$ and that the intersection matrix of $P^{4k}(\mathbf{A}_2)$ is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(See Chapter V, Section 8 and V.9.2.) Thus $\tau(P^{4k}(\mathbf{A}_2)) = 2$.

Suppose that $m[W_3^{4n+1}; T] = 0$ in $\Theta_{4k-1}^{4n+1}$. Then the involution on the $m$-fold connected sum of $W_3^{4n+1}$ bounds an involution on $\mathbf{D}^{4n+2}$, the fixed point set of which is necessarily a mod 2 homology $4k$-disk. Thus $mW_3^{4k-1} = W_3^{4k-1} \# \cdots \# W_3^{4k-1}$ ($m$ times) bounds a mod 2 homology disk, so that

$$\mu(mW_3^{4k-1}) = 0.$$

However, this is

$$0 = m\mu(W_3^{4k-1}) = \frac{m\tau(P^{4k}(\mathbf{A}_2))}{b_k} = \frac{2m}{b_k}$$

in $\mathbf{Q}/\mathbf{Z}$, which means that $b_k/2$ divides $m$. ▮

*Remark* This is not the best that one can do. For example, one can also use Milnor's $\lambda$-invariant, which is defined for all mod 2 homology $(4k - 1)$-spheres (in particular) and gives a homomorphism

$$\lambda\colon \Theta_{4k-1}^{m} \to \mathbf{Q}/\mathbf{Z}$$

upon application to the fixed set. We refer the reader to Eells and Kuiper [1] for the definition of this and the comparison with $\mu$. In low dimensions $\mu$ gives better information than $\lambda$, but generally both together give better information than either one alone. For example, in case $k = 6$, the $\lambda$ invariant shows that the divisibility in 8.3 can be improved by a factor of 691.

It might be conjectured, in fact, that the elements considered in 8.3 have *infinite* order. When $n = k$ this can be shown by the method we use to prove the next result.

**8.4. Theorem** *For $k \geq 2$, the involution $T$: $(u, v, z_1, \ldots, z_k) \mapsto (u, v, z_1, \ldots, z_{k-1}, -z_k)$ on the Brieskorn sphere $W_{3,5}^{2k+1}$ (see Chapter* V, *Section* 10) *represents an element of infinite order in $\Theta_{2k-1}^{2k+1}$.*

***Proof*** Let $2n \geq k+2$ and let $G = \mathbf{O}(2n-k)$. Consider the $G$-manifold $W_{3,5}^{4n-1}$ with $G$ acting on the last $2n-k$ coordinates. Note that the isotropy types are $G$, $K = \mathbf{O}(2n-k-1)$ and $H = \mathbf{O}(2n-k-2)$. By inspection, it is easily seen that the representation of $G$ about a fixed point is twice the standard representation plus a trivial $(2k-1)$-dimensional representation, and the slice representation of $K$ is a standard representation plus a trivial $(2k+1)$-dimensional representation.

Thus the orbit space $X = W_{3,5}^{4n-1}/G$ is topologically a $(2k+2)$-manifold with boundary $B$. Note that increasing $n$ (and enlarging $G$ correspondingly) does not change the orbits and hence $X$ is independent of $n$. Also note that $F(K, W_{3,5}^{4n-1}) = W_{3,5}^{2k+1}$ with the induced action of $\mathbf{Z}_2 \approx N(K)/K$ corresponding to the given involution $T$ on $W_{3,5}^{2k+1}$. Thus there is a canonical identification of the orbit space of $T$ with the space $B$ of singular orbits:

$$W_{3,5}^{2k+1}/T \approx B = \partial X.$$

Since $k \geq 2$ we know that $W_{3,5}^{2k+1}$ and $W_{3,5}^{4n+1}$ are simply connected by V.11.1. By the existence of fixed points it follows from II.6.3 that $X$ and $B$ are simply connected. Next we claim that $X$ is acyclic and hence is a disk. To see this, note that the orbits of $G$ are all $(2n-k-3)$-connected so that

$$H^i(X; \mathbf{Z}) \approx H^i(W_{3,5}^{4n-1}; \mathbf{Z})$$

for $i \leq 2n-k-3$, by the Vietoris–Begle Mapping Theorem; see Spanier [1, p. 344]. Now $W_{3,5}^{4n-1}$ is a homotopy sphere by V.9.5. Thus, by taking $2n-k-3 \geq \dim X = 2k+2$ (i.e., $2n \geq 3k+5$) we conclude that $X$ is contractible. Since $B$ is simply connected we have

$$X \approx \mathbf{D}^{2k+2}.$$

Thus this situation fits the context of Section 7. Let $\Sigma^{2k-1} \subset \mathbf{S}^{2k+1}$ denote the submanifold corresponding to $F(T, W_{3,5}^{2k+1}) = W_{3,5}^{2k-1}$ in $W_{3,5}^{2k+1}/T = B \approx \mathbf{S}^{2k+1}$. In the notaton of Section 7 we have

$$W_{3,5}^{4n-1} \approx M^{4n-1}(\Sigma^{2k-1}).$$

By taking the $m$-fold equivariant connected sum about fixed points of this it is clear that

$$mW_{3,5}^{4n-1} \approx M^{4n-1}(m\Sigma^{2k-1})$$

for any integer $m > 0$.

Now suppose that $m[W_{3,5}^{2k+1}, T] = 0$ in $\Theta_{2k-1}^{2k+1}$. This means that there is an involution $T'$ on $\mathbf{D}^{2k+2}$ which extends that on $mW_{3,5}^{2k+1} \approx \mathbf{S}^{2k+1}$. Then $F^{2k} = F(T', \mathbf{D}^{2k+2})$ is a mod 2 homology disk. Now $\mathbf{D}^{2k+2}/T'$ is simply connected and its integral homology can be shown to be trivial (see Chapter III, Exercise 3). Thus $\mathbf{D}^{2k+2}/T'$ is a $(2k+2)$-disk and $F^{2k} \subset \mathbf{D}^{2k+2}/T'$ clearly has boundary

$$m\Sigma^{2k+1} \subset \mathbf{S}^{2k+1}/T' = m(W_{3,5}^{2k+1}/T) = mB = \mathbf{S}^{2k+1}.$$

That is, $m\Sigma^{2k-1} \subset \mathbf{S}^{2k+1}$ is the boundary of a mod 2 homology disk $F^{2k} \subset \mathbf{D}^{2k+2}$. By the discussion in Section 7 this implies that

$$mW_{3,5}^{4n-1} \approx M^{4n-1}(m\Sigma^{2k-1}) \approx M^{4n-1}(\partial F^{2k}) \approx \partial M^{4n}(F^{2k}).$$

Moreover, $M^{4n}(F^{2k})$ is mod 2 acyclic as noted in Section 7. In particular, the index of $M^{4n}(F^{2k})$ is 0 so that the Eells–Kuiper $\mu$-invariant

$$m\mu(W_{3,5}^{4n-1}) = \mu(mW_{3,5}^{4n-1}) = 0 \qquad \text{in} \quad \mathbf{Q}/\mathbf{Z}.$$

On the other hand, it can be seen that $W_{3,5}^{4n-1}$ bounds a parallelizable manifold of index $\pm 8$; see Brieskorn [1] and Milnor [7]. (This can be seen by showing that the $\mathbf{O}(2n-1)$-manifold $\partial P^{4n}(\mathbf{E}_8)$ corresponds to the torus knot (3,5) and thus is equivalent to $W_{3,5}^{4n-1}$ (see Chapter V, Sections 10 and 11), and by the easy and well-known computation of the index of $P^{4n}(\mathbf{E}_8)$; see Hirzebruch [2].) Thus

$$\mu(W_{3,5}^{4n-1}) = \pm 8/b_n,$$

so that $b_n$ divides $8m$. However, $b_n$ increases strictly with $n$ and $n$ is arbitrary, independent of $m$. This contradiction shows that *no* multiple of the given involution on $W_{3,5}^{2k+1}$ can be extended to $\mathbf{D}^{2k+2}$. ▌

*Remark* There is a close relationship between the above proof and the notion of the signature of a knot; see Erle [1, 2]. The technique in the above proof can be used, in place of the signature of a knot, to show for example that any nontrivial torus knot in $\mathbf{S}^3$ has infinite order in the Fox–Milnor knot cobordism group; see Milnor [6]. There is some hope that the codimension two case $\Phi_{n-2}^{n}$ can be completely calculated; see Levine [2].

Our next result is essentially due to Browder and Petrie [1], who defined an invariant for certain types of involutions. We shall develop this invariant in a modified, and more geometric, manner. Also the domain of definition of our version of the invariant will differ from that of Browder and Petrie. (In their version (a) below is replaced by the requirement that $M^T$ be a rational homology $(2k - 1)$-sphere with $k \neq n$, and (b) is not needed.)

We assume throughout that $T$ is a smooth orientation preserving involution on a closed oriented $(4n - 1)$-manifold $M$ such that:

(a) $M^T$ is a $(2k - 1)$-manifold with $H_{4k-4n}(M^T; \mathbf{Q}) = 0$.

(b) There exists a nonzero normal vector field $\xi$ to $M^T$ in $M$.

(c) There exists an oriented $4n$-manifold $W$ with $\partial W = M$ and such that $T$ extends to a smooth involution on $W$.

We remark that $W^T$ and $M^T$ need not be orientable. Also $M^T$ (and $M$) need not be connected and $k$ in (a) may vary over the components, with the condition (a) applying to each component separately.

Suppose $W$ is as in (c) and $\xi$ is as in (b). Then $\xi$ can be extended to a normal field $\xi_1$ of $W^T$ in $W$, perhaps with singularities. Assuming $\xi_1$ to be small (this will not affect the argument) we can apply the exponential map to $\xi_1$, thereby obtaining a shifted copy $W^T(\xi_1)$ of $W^T$ in $W$. Moreover, by a slight change of $\xi_1$ it may be assumed that $W^T(\xi_1)$ meets $W^T$ transversely. Then $W^T \cap W^T(\xi_1)$ is a *closed* $(4k - 4n)$-manifold whose normal bundle is naturally the restriction of twice the normal bundle of $W^T$ in $W$, whence $W^T \cap W^T(\xi_1)$ has an orientation canonically induced from that of $W$. Any other extension $\xi_2$ of $\xi$, with $W^T(\xi_2)$ transverse to $W^T$, is homotopic to $\xi_1$ and the homotopy can be altered (making it transverse) to provide a cobordism between $W^T \cap W^T(\xi_1)$ and $W^T \cap W^T(\xi_2)$. Since the index is a cobordism invariant, the index of $W^T \cap W^T(\xi_1)$ depends only on $W^T$ and on the (nonzero) homotopy class of the normal field $\xi$ to $M^T$ in $M$. We denote this index by

$$\operatorname{Index}(W^T \cdot W^T, \xi).$$

Now consider the symmetric bilinear form $\langle \alpha, \beta \rangle = \alpha \cdot T\beta$ (intersection) on $H_{2n}(W; \mathbf{Q})$. The signature of this form is denoted by $\operatorname{Sgn}(T, W)$. We put

$$I(M, T, \xi) = \operatorname{Index}(W^T \cdot W^T, \xi) - \operatorname{Sgn}(T, W).$$

We claim that this is independent of the choice of $W$. In fact if $W_1$ is another cobounding manifold as in (c), then $X = W \cup (-W_1)$ is a closed manifold with involution. If $\xi$ is any nonzero normal field to $M^T$ in $M = W \cap W_1 \subset X$, then it is clear that

$$\operatorname{Index}(X^T \cdot X^T) = \operatorname{Index}(W^T \cdot W^T, \xi) - \operatorname{Index}({W_1}^T \cdot {W_1}^T, \xi).$$

Moreover,

$$\mathrm{Sgn}(T, X) = \mathrm{Sgn}(T, W) - \mathrm{Sgn}(T, W_1);$$

see Atiyah and Singer [2, p. 588] and Hirzebruch [3]. (The case of greatest interest to us is that for which $M = \partial W$ is a rational homology sphere, and in this case the latter formula is obvious.) By the $G$-Signature Theorem of Atiyah and Singer [2, p. 583] (also see Hirzebruch [3] and Jänich and Ossa [1]) we have

$$\mathrm{Sgn}(T, X) = \mathrm{Index}(X^T \cdot X^T)$$

and it follows that $I(M, T, \xi)$ is independent of $W$.

Note that we have not assumed that $M$ is connected. The invariant $I(M, T, \xi)$ is clearly *additive* with respect to disjoint union.

**8.5. Lemma** (1) *If* $(M, T)$ *bounds* $(W, T)$ *such that* $H_{4k-4n}(W^T; \mathbf{Q}) = 0$ *where* dim $W^T = 2k$ (i.e., *this holds separately for each component of* $W^T$, *with k varying*), *then* $\mathrm{Index}(W^T \cdot W^T, \xi) = 0$ *for any* $\xi$, *and hence* $I(M, T, \xi) = -\mathrm{Sgn}(T, W)$.

(2) *In general,* $I(M, T, \xi)$ *is independent of* $\xi$.

***Proof*** First let us show that (1) implies (2). Let $\xi$ and $\xi'$ be two nonzero normal fields on $M^T$ in $M$. Consider $W = M \times \mathbf{I}$ with $\xi$ on $M^T \times \{0\}$ and $\xi'$ on $M^T \times \{1\}$ and with the involution $T \times 1$. The fixed set $W^T = M^T \times \mathbf{I}$ has

$$H_{4k-4n}(W^T; \mathbf{Q}) \approx H_{4k-4n}(M^T; \mathbf{Q}) = 0$$

by assumption (a). Since $H_{2n}(W) = H_{2n}(M \times \mathbf{I}) \approx H_{2n}(M)$, it is geometrically obvious that the intersection form on $W$ is zero. Thus by (1) we have

$$\begin{aligned} 0 = -\mathrm{Sgn}(T \times 1, W) &= I(M \times \{0\} \cup -M \times \{1\}, T \cup T, \xi \cup \xi') \\ &= I(M, T, \xi) - I(M, T, \xi'). \end{aligned}$$

Thus it suffices to prove (1).

To prove part (1) we shall assume familiarity with the $\mathscr{L}$-series and the Index Theorem of Hirzebruch; see Atiyah and Singer [2, p. 577]. Let $Y$ denote the self-intersection $W^T \cap W^T(\xi_1)$ for some suitable extension $\xi_1$ of $\xi$, and recall that $Y$ does not meet $M = \partial W$. Also dim $Y = 4k - 4n$. Let $i$: $Y \to W^T$ be the inclusion. Now the normal bundle of $Y$ in $W^T$ is isomorphic to the restriction to $Y$ of the normal bundle $\nu$ of $W^T$ in $W$. Since $\mathscr{L}$ is multiplicative, this implies that

$$\mathscr{L}(Y) = i^*(\tau),$$

where

$$\tau = \mathscr{L}(W^T)\mathscr{L}(\nu)^{-1}.$$

(These are inhomogeneous, rational, absolute cohomology classes.) Thus

$$\begin{aligned}\operatorname{Index}(W^T \cdot W^T, \xi) = \operatorname{Index}(Y) = L(Y) &= 2^{2k-2n}\mathscr{L}(Y)[Y] \\ &= 2^{2k-2n}i^*(\tau)[Y] = 2^{2k-2n}\tau(i_*[Y]).\end{aligned}$$

However, by assumption, $i_*[Y] \in H_{4k-4n}(W^T; \mathbf{Q}) = 0$, and hence Index$(W^T \cdot W^T, \xi) = 0$. ▮

Since $I(M, T, \xi)$ is independent of $\xi$ we shall now denote it by $I(M, T)$. Note that it follows immediately from the definition that it is additive with respect to *connected sum* as well as disjoint union (where the sum is taken at points whose fixed point set components have the same dimension).

Suppose that $(M, T)$ and $(M', T')$ are involutions on homotopy $(4n - 1)$-spheres with $(2k - 1)$-dimensional fixed point sets. Also suppose that $(M, T)$ is $L$-equivalent to $(M', T')$. If $(M, T)$ bounds an involution on an oriented $4n$-manifold, then so does $(M', T')$. Thus, in this case and for $n \neq k$, $I(M, T)$ and $I(M', T')$ are both defined. Moreover, from the definition of $L$-equivalence and from 8.5, it follows immediately that $I(M, T) - I(M', T') = 0$. Thus $I$ defines an additive integral invariant on its subgroup of definition in $\Theta^{4n-1}_{2k-1}$; $k \neq n$.

**8.6. Theorem** *For $k \neq n$, the involution $(u, v, z_1, \ldots, z_{2n-1}) \mapsto (u, v, z_1, \ldots, z_{k-1}, -z_k, \ldots, -z_{2n-1})$ on the Brieskorn sphere $W^{4n-1}_{3,5}$ represents an element of infinite order in $\Theta^{4n-1}_{2k-1}$.*

*Proof* It suffices to show that the invariant $I$ is nonzero for the involution in question. It will be slightly more convenient to treat the corresponding involution $T$ on the plumbed manifold $\partial P^{4n}(\mathbf{E}_8)$; see Chapter V, Section 8. (As remarked before, $\partial P^{4n}(\mathbf{E}_8)$ can be seen to be equivalent to $W^{4n-1}_{3,5}$ as an $\mathbf{O}(2n - 1)$-manifold by the classification of knot manifolds. We shall assume this but note that the calculation of the invariant $I(\partial P^{4n}(\mathbf{E}_8), T)$ will not use this fact.) The involution $T$ extends canonically to $P^{4n}(\mathbf{E}_8)$. Moreover $H_{2n}(P^{4n}(\mathbf{E}_8))$ is free abelian of rank 8 and is generated by the eight $2n$-spheres used in defining the plumbing; see Chapter V, Section 8. From this it is clear that the induced involution $T_*$ on $H_{2n}(P^{4n}(\mathbf{E}_8))$ is $T_*(\alpha) = (-1)^k\alpha$. Thus the matrix of the bilinear form $\langle\alpha, \beta\rangle = \alpha \cdot T_*\beta$ is $(-1)^k$ times the intersection matrix for $P^{4n}(\mathbf{E}_8)$ and this has index $\pm 8$.

Note that the fixed set of $T$ on $P^{4n}(\mathbf{E}_8)$ is $P^{2k}(\mathbf{E}_8)$. Now $P^{4n}(\mathbf{E}_8)$ and $P^{4n-2}(\mathbf{E}_8)$ are parallelizable and thus the normal bundle of $P^{4n-2}(\mathbf{E}_8)$ in $P^{4n}(\mathbf{E}_8)$ is stably trivial. However, a stably trivial 2-plane bundle is trivial, since it is orientable and is classified by its Chern class $c_1$ which is stable. Thus $P^{2k}(\mathbf{E}_8) \subset P^{4n-2}(\mathbf{E}_8) \subset P^{4n}(\mathbf{E}_8)$ has a nonzero normal field (in fact its normal bundle is trivial) and therefore its self-intersection manifold can be taken to be empty. Thus

$$I(\partial P^{4n}(\mathbf{E}_8), T) = -\operatorname{Sgn}(T, P^{4n}(\mathbf{E}_8)) = \pm 8. \quad \blacksquare$$

## 9. SEMIFREE CIRCLE GROUP ACTIONS

In this section we shall consider smooth semifree (i.e., free outside the fixed point set) actions of the circle group $\mathbf{S}^1$ on disks and homotopy spheres. Suppose that $\mathbf{S}^1$ acts semifreely on $\Sigma^n$ with fixed set $\Sigma^k$. If $k = n - 2$, then the situation is covered by 6.3 and V.6.2 (also see V.2.2). Thus we may assume that $k < n - 2$. Now $\Sigma^k$ is an integral homology sphere, but may not be simply connected (examples are given by the suitable $\mathbf{S}^1 = \mathbf{SO}(2)$ action on the Brieskorn sphere $W_{3,5}^{4m-1}$ fixing $W_{3,5}^3$, and connected sums of this with itself). We shall not concern ourselves with this in this section, however, and will assume throughout that $\Sigma^k$ is a homotopy sphere. Similarly, we consider semifree actions on $\mathbf{D}^n$ fixing some disk $\mathbf{D}^k$. All the results of this section have analogs for semifree $\mathbf{S}^3$-actions, but we leave it to the reader to supply the easy details of this.

**9.1. Theorem** *Let $\mathbf{S}^1$ act semifreely and smoothly on $\mathbf{D}^n$ fixing a k-disk $\mathbf{D}^k$ (embedded in any way in $\mathbf{D}^n$). Assume that $n \geq 7$. Then the action is smoothly equivalent to an orthogonal action.*

*Proof* As remarked, we may assume that $k < n - 2$. Let $p$ be a point in the interior of $\mathbf{D}^k$ and let $B^n \subset \operatorname{int} \mathbf{D}^n$ be an invariant closed disk neighborhood of $p$ on which $\mathbf{S}^1$ acts orthogonally (which exists by the Invariant Tubular Neighborhood Theorem 2.2). Let $X = \mathbf{D}^n - \operatorname{int} B^n$ and let $X_0 = \partial B^n$ and $X_1 = \partial \mathbf{D}^n$ be the two boundary components of $X$. Now $B^n \cap \mathbf{D}^k$

is a closed $k$-disk in int $\mathbf{D}^k$ and, since any such disk is ambient isotopic to a standard concentric disk, we know that

$$X \cap \mathbf{D}^k \approx \mathbf{S}^{k-1} \times [0,1].$$

Thus there is a smooth function $\varphi\colon X \cap \mathbf{D}^k \to [0,1]$ with $\varphi = 0$ on $X_0 \cap \mathbf{D}^k$ and $\varphi = 1$ on $X_1 \cap \mathbf{D}^k$ and with *no critical points.* Clearly, by a standard patching argument, we can extend $\varphi$ to a smooth function

$$\varphi\colon X \to [0,1]$$

such that $\varphi = 0$ on $X_0$, $\varphi = 1$ on $X_1$ and $\varphi$ has no critical points on $X_0$, $X_1$ or on $X \cap \mathbf{D}^k$ (and hence on a neighborhood of these sets). By averaging, we obtain a smooth function $\psi\colon X \to [0,1]$ given by

$$\psi(x) = \int \varphi(gx)\, dg$$

which is *invariant,* is 0 on $X_0$ and 1 on $X_1$, and has no critical points on a neighborhood of $(X \cap \mathbf{D}^k) \cup X_0 \cup X_1$.

Put $W = (X - \mathbf{D}^k)/\mathbf{S}^1$ which is an open $(n-1)$-manifold with boundary components $W_0 = (X_0 - \mathbf{D}^k)/\mathbf{S}^1 = (\partial B^n - \mathbf{D}^k)/\mathbf{S}^1$ and $W_1 = (X_1 - \mathbf{D}^k)/\mathbf{S}^1 = (\partial\mathbf{D}^n - \partial\mathbf{D}^k)/\mathbf{S}^1$. Since $\psi$ is invariant it induces a smooth function

$$\psi^*\colon W \to [0,1]$$

which is 0 on $W_0$, 1 on $W_1$, and whose critical points are contained in a *compact* set in int($W$).

Clearly we can alter $\psi^*$, and hence $\psi$, by changing it only in some compact subset of int($W$) so that the new $\psi^*$ will have only nondegenerate critical points; see Milnor [5]. We assume this to be done.

Now using any *invariant* riemannian metric on $\mathbf{D}^n$ consider the gradient vector field grad $\psi$ on $X$. The trajectories of this field in $\mathbf{D}^k \cap X$ run from $\mathbf{D}^k \cap X_0$ to $\mathbf{D}^k \cap X_1$ and hence there is a neighborhood of $\mathbf{D}^k \cap X$ on which the trajectories run from $X_0$ to $X_1$. (If the reader prefers, one could start with $\varphi$ defined from an invariant tubular *neighborhood* of $\mathbf{D}^k \cap X$ in $X$, and then this will be obvious.) Now grad $\psi$ is invariant under the $\mathbf{S}^1$-action and hence induces a vector field $\xi$ on $W$ which is clearly *gradient-like* for $\psi^*$; see Milnor [5, p. 20]. (In fact, $\xi$ is grad $\psi^*$ with respect to an obvious induced riemannian metric on $W$.) Also, outside some compact set, the trajectories of $\xi$ run from $W_0$ to $W_1$. Thus, even though $W$ is noncompact, the trajectories of $\xi$, or of any vector field which coincides with $\xi$

$\xi$ outside some compact set in int($W$), never run off $W$; that is, they run from $W_0$ or to $W_1$ (at finite time) or from or to critical points (as time goes to $\pm\infty$).

Since $k < n - 2$, $X - \mathbf{D}^k \simeq \mathbf{D}^n - \mathbf{D}^k$ is simply connected and the inclusion $\mathbf{S}^{n-k-1} \to X - \mathbf{D}^k$ of the sphere in a normal plane to $\mathbf{D}^k$ gives a homology isomorphism, and hence is a homotopy equivalence. Similar remarks apply to $X_0 - \mathbf{D}^k$ and $X_1 - \mathbf{D}^k$. Since these are principal $\mathbf{S}^1$-bundles it follows immediately that the inclusions $W_0 \to W$ and $W_1 \to W$ are homotopy equivalences, and they have the homotopy type of $\mathbf{S}^{n-k-1}/\mathbf{S}^1 \approx \mathbf{CP}^{r-1}$, where $r = (n - k)/2$. Thus $(W, W_0, W_1)$ is a simply connected $h$-cobordism.

From these facts it is clear that the proof of the $h$-Cobordism Theorem given in Milnor [5] will go through in the present case to provide a modification $\theta^*\colon W \to [0,1]$ of $\psi^*$ (and a modification of $\xi$ which we now discard), with the modifications taking place only in some compact subset of int($W$), such that the new function $\theta^*$ has no critical points and takes $W_0$ to 0 and $W_1$ to 1. By composing $\theta^*$ with the orbit map and using the fact that $\theta^* = \psi^*$ outside a compact set, we obtain an *invariant* smooth function

$$\theta\colon X \to [0,1]$$

with *no critical points* and taking $X_0$ to 0 and $X_1$ to 1. Then grad $\theta$ is an *invariant* vector field with all trajectories running from $X_0$ to $X_1$. It follows immediately that there is an equivariant diffeomorphism

$$X \approx X_0 \times [0,1] = \partial B^n \times [0,1]$$

extending the identity on $X_0 = \partial B^n$. Since $\mathbf{S}^1$ acts orthogonally on $B^n$ this implies that the given action on $\mathbf{D}^n = B^n \cup X$ is smoothly equivalent to this orthogonal action. ∎

*Remark* It is clear that the method of proof of 9.1 can be used inductively on orbit type to prove a general equivariant $h$-cobordism theorem. Since the necessary hypotheses are so strong, however, it does not seem worth while to give a general statement or proof of such a theorem.

Let $\mathbf{S}^1 = \mathbf{SO}(2)$ act on $\mathbf{R}^n$ by $r$ times the standard representation plus a trivial $k$-dimensional representation, where $n = 2r + k$. It is clear that, up to orthogonal equivalence, this is the unique orthogonal semifree $\mathbf{S}^1$-action on $\mathbf{R}^n$ fixing $\mathbf{R}^k$. Let us denote the restriction of this action to the unit disk $\mathbf{D}^n$ by $\mathbf{D}_k{}^n$.

By removing a disk about a fixed point in an action on a homotopy sphere we see that 9.1 implies the following theorem. By a twisted $k$-sphere we mean a homotopy $k$-sphere which is the union of two $k$-disks via a diffeomorphism on their boundaries. Recall that for $k \neq 3,4$ every homotopy $k$-sphere is a twisted $k$-sphere.

**9.2. Corollary** *Let $\mathbf{S}^1$ act smoothly and semifreely on a homotopy n-sphere $\Sigma^n$ with fixed set a twisted k-sphere and with $n \geq 7$. Then this action is smoothly equivalent to*

$$\mathbf{D}_k^n \cup_\varphi \mathbf{D}_k^n,$$

*where $\varphi$: $\partial\mathbf{D}_k^n \to \partial\mathbf{D}_k^n$ is some equivariant diffeomorphism.* ∎

Since any such $\varphi$ extends to an equivariant *homeomorphism* of $\mathbf{D}_k^n$, by coning, we have the following result.

**9.3. Corollary** *Let $\mathbf{S}^1$ act on $\Sigma^n$ as in 9.2. Then the action is topologically equivalent to an orthogonal action on $\mathbf{S}^n$* (i.e., *the action $\partial\mathbf{D}_{k+1}^{n+1}$*), *and the equivalence can be taken to be a diffeomorphism outside a point.* ∎

*Remarks* The case $k = 0$ of 9.1 and 9.2 is due to Stewart [3]. Connell, Montgomery, and Yang [1] have proved an analogous theorem to 9.1 in the case of actions on $\mathbf{R}^n$ by using the technique of engulfing. Their result implies a stronger version of 9.3. Another way to decompose a semifree action on $\Sigma^n$ would be to remove a tubular neighborhood of the fixed set and to use the $h$-Cobordism Theorem to identify the complement. This method is explored at some length by Browder [2], who uses it to prove the existence of infinitely many exotic examples for $n = 4m + 1$, in particular. Other exotic examples of semifree $\mathbf{S}^1$-actions on homotopy spheres were constructed in Bredon [18, 19] and in Montgomery and Yang [5].

By restricting an $\mathbf{S}^1$-action to the contained $\mathbf{Z}_2$-action, Theorem 8.6 can be used to prove the existence of infinitely many exotic examples of semifree $\mathbf{S}^1$-actions on $\mathbf{S}^{4m-1}$. However, the following theorem gives somewhat better results. (Note that we do not assume semifreeness for this result.)

**9.4. Theorem** *Let $\mathbf{S}^1$ act smoothly and nontrivially on an oriented homotopy sphere $\Sigma^{4n-1}$ and assume that the action extends to a smooth action on some oriented, parallelizable manifold $W^{4n}$ with $\partial W = \Sigma$. Then the index $\tau(W)$ of $W^{4n}$ is an invariant of the action on $\Sigma^{4n-1}$.*

*Proof* Suppose that $\Sigma = \partial W'$ with $W'$ parallelizable and with the $S^1$-action extending to $W'$. We must show that $\tau(W) = \tau(W')$. Put $Y = W \cup -W'$. Then $Y$ is an oriented almost parallelizable manifold on which $S^1$ acts, and we must show that $\tau(Y) = 0$. We shall use the remarkable theorem of Atiyah and Hirzebruch [2] which states that if $Y$ is a closed orientable manifold with $w_2(Y) = 0$ (i.e., a spin-manifold) and which admits a nontrivial smooth $S^1$-action, then the $\hat{A}$-genus $\hat{A}(Y) = 0$. Now an almost parallelizable $4n$-manifold $Y$ has $w_2(Y) = 0$ and the rational Pontriagin classes $p_i(Y) = 0$ for $0 < i < n$. But $\tau(Y) = L(Y)$ and $\hat{A}(Y)$ are then non-zero rational multiples of $p_n(Y)[Y]$; see Hirzebruch [1]. Thus $\tau(Y)$ is a rational multiple of $\hat{A}(Y)$, which is 0 when $S^1$ acts smoothly and nontrivially on $Y$. (In fact

$$\tau(Y) = -2^{2n+1}(2^{2n-1} - 1)\hat{A}(Y)$$

when $Y$ is an almost parallelizable closed $4n$-manifold.) ▮

*Remark* This invariant can be generalized by using the *rational* Eells–Kuiper $\mu$-invariant. This is defined in the same way as in Eells and Kuiper [1] (also see Montgomery and Yang [7]) but one takes it as a rational number without passing to $\mathbf{Q}/\mathbf{Z}$. It is defined for the same class of $(4n - 1)$-manifolds as in the usual definition of $\mu$, but with $S^1$ acting smoothly on the manifolds involved. It is calculated from a certain type of cobounding spin $4n$-manifold $W$ (with $S^1$-action). The difference of these numbers calculated from $W$ and $W'$ with $\partial W = \partial W'$ is just a multiple of the $\hat{A}$-genus of the closed spin manifold $W \cup -W'$ and this is 0 since $S^1$ acts nontrivially on this manifold. Thus this rational number $\mu(\partial W)$ calculated from $W$ is an invariant of the $S^1$-action on $\partial W$. If $\partial W$ is a homotopy sphere, then *any* cobounding spin-manifold with $S^1$-action can be used for the calculation. If $W$ is parallelizable, then, for given $n$, $\mu(\partial W)$ is just a *constant* multiple of $\tau(W)$, so that this invariant does generalize 9.4.

**9.5. Corollary** *There are an infinite number of distinct semifree $S^1$-actions on $S^{4n-1}$ with fixed set diffeomorphic to $S^{4k-1}$ for any $1 < k < n$.*

*Proof* Consider the $(2n + 1)$-dimensional representation of $S^1 = \mathbf{SO}(2)$ given by $n - k$ times the standard representation plus a trivial $(2k + 1)$-dimensional representation. This induces a semifree $S^1$-action on the plumbed manifold $P^{4n}(\mathbf{E}_8)$ fixing $P^{4k}(\mathbf{E}_8)$. (One also has this for $k = 1$, but $\partial P^4(\mathbf{E}_8)$ is not simply connected.) Since $\partial P^{4n}(\mathbf{E}_8)$ and $\partial P^{4k}(\mathbf{E}_8)$ are homotopy spheres, some multiple (by connected sum), say $r$, of them give the standard spheres $S^{4n-1}$ and $S^{4k-1}$ (e.g., if $n = 3$ and $k = 2$, then $r = 6944$ works). Thus the $r$-fold equivariant connected sum of this action on $\partial P^{4n}(\mathbf{E}_8)$ is a semifree action on $S^{4n-1}$ fixing $S^{4k-1}$ and bounding an $S^1$-action on the parallelizable $4n$-manifold $W = rP^{4n}(\mathbf{E}_8)$ which has index $\tau(W) = \pm 8r$. The connected sums of $j$ copies of this action, for $j = 1, 2, 3, \ldots$ are then distinguished by the invariants $\tau(jW) = \pm 8rj$. ∎

*Remark* It follows from 9.1 that for $n \geq 6$, the oriented equivariant diffeomorphism classes of semifree $S^1$-actions on oriented homotopy $n$-spheres fixing a twisted $k$-sphere, form an abelian group $\Theta_k{}^n(S^1)$ under connected sum. The proof of 9.5 shows that the given action on $\partial P^{4n}(\mathbf{E}_8)$ has infinite order in $\Theta_{4k-1}^{4n-1}(S^1)$ for $1 < k < n$.

**9.6. Corollary** *There exists a smoothable manifold $M^{12}$ which admits an effective, locally smooth, semifree $S^1$-action, but which admits no nontrivial smooth $S^1$-action in any differentiable structure.*

*Proof* As in the proof of 9.5 take a connected sum of $r$ copies of the canonical semifree $S^1$-action on $P^{12}(\mathbf{E}_8)$ fixing $P^8(\mathbf{E}_8)$, such that $r\partial P^{12}(\mathbf{E}_8) \approx S^{11}$ ($r = 992$ works). Let $M^{12} = rP^{12}(\mathbf{E}_8) \cup \mathbf{D}^{12}$ which is a smooth almost parallelizable manifold of index $\pm 8r \neq 0$. On $\mathbf{D}^{12}$ we take the cone over the given action on $r\partial P^{12}(\mathbf{E}_8) \approx S^{11}$. By 9.3 the action on $M^{12}$ is then locally smooth.

Since $H^4(M^{12}; \mathbf{Q}) = 0 = H^8(M^{12}; \mathbf{Q})$, it follows that, for *any* differentiable structure on $M^{12}$, $\hat{A}(M^{12})$ is a nonzero multiple of $\tau(M^{12}) \neq 0$, and hence $\hat{A}(M^{12}) \neq 0$. (Alternatively, one can use the fact that $\hat{A}(M)$ is a *topological* invariant of $M$, since the rational Pontriagin classes are topological invariants.) By the theorem of Atiyah and Hirzebruch [2] quoted in the proof of 9.4, it follows that $M^{12}$ carries no smooth $S^1$-action in *any* differentiable structure. (An alternative proof can be based on the fact that $M^{12}$ is almost parallelizable in any differentiable structure, since it is 5-connected and $\pi_5(\mathbf{SO}(12)) = 0$. Thus 9.6 really is a corollary of 9.4.) ∎

## 10. REPRESENTATIONS AT FIXED POINTS

Let $G$ be a compact Lie group and let $\Theta\colon G \times M \to M$ be a smooth action of $G$ on a manifold $M$. If $x \in M$ is a stationary point, then the induced action of $G$ on the tangent space $T_x(M)$ is a representation $\Theta_x$ of $G$.

(Note that if $M$ is given an invariant riemannian metric, then $\Theta_x$ is an orthogonal representation.) If $x$ and $y$ are stationary points, we wish to compare $\Theta_x$ and $\Theta_y$. If $x$ and $y$ are in the same component of $M^G$, then it is clear that $\Theta_x$ and $\Theta_y$ are equivalent, so that the case in which $M^G$ is connected will not concern us. Eventually we shall restrict our attention to the case in which $G = \mathbf{Z}_p$ or $G = \mathbf{S}^1$.

The appropriate tool for this study is equivariant $K$-theory. We shall only need some elementary properties of $K$-theory and, for the most part, the book of Atiyah [2] will suffice. We will make use of both real $K$-theory, denoted by $KO$, and complex $K$-theory, denoted by $KU$. We simply use $K$ for statements applying equally to both cases.

Let us briefly recall some facts we shall use. If $X$ is a compact $G$-space, then $K_G(X)$ is the quotient of the free abelian group on the equivalence classes $[\xi]$ of $G$-vector bundles $\xi$ [real for $KO_G(X)$ and complex for $KU_G(X)$] over $X$ modulo the subgroup generated by the $[\xi \oplus \eta] - [\xi] - [\eta]$. It has a ring structure induced by the tensor product of $G$-vector bundles. For a point $\star$, $K_G(\star)$ is just the representation ring $R(G)$ [$= RO(G)$ or $RU(G)$] which can be regarded additively as the free abelian group on the irreducible representations of $G$. Also, $K_G$ is a contravariant functor. Via the projection $\varepsilon\colon X \to \star$, any representation $\varrho$ of $G$ induces a $G$-vector bundle $\varepsilon^*(\varrho)$ on $X$. This induces a homomorphism

$$\varepsilon^*\colon R(G) = K_G(\star) \to K_G(X).$$

Note that $\varepsilon^*$ is a monomorphism to a direct summand *when* a stationary point exists in $X$, but this is not generally true if $X^G$ is empty. We put

$$\tilde{K}_G(X) = \text{coker } \varepsilon^*.$$

Now for any $G$-vector bundle $\xi$ over a compact $G$-space $X$ there exists a $G$-vector bundle $\eta$ with $\xi \oplus \eta$ equivalent to the product of $X$ with a representation $\varrho$ and hence $[\xi \oplus \eta] = \varepsilon^*[\varrho]$ in $K_G(X)$; see Atiyah [2, pp. 27, 40]. It follows that any element of $K_G(X)$ has the form

$$[\xi] - \varepsilon^*[\varrho]$$

for some $G$-vector bundle $\xi$ and some representation $\varrho$. It also follows that for $G$-vector bundles $\xi$ and $\eta$ over $X$, $[\xi] = [\eta]$ in $K_G(X)$ iff $\xi$ and $\eta$ are stably equivalent; that is, there exists a representation $\varrho$ such that $\xi \oplus \varepsilon^*(\varrho)$ is equivalent to $\eta \oplus \varepsilon^*(\varrho)$.

Let $SX$ be the (unreduced) suspension of $X$ with vertices $w_0$ and $w_1$. The inclusion $b_i$: $\{w_i\} \to SX$ induces a homomorphism

$$b_i^*: K_G(SX) \to K_G(w_i) \approx R(G)$$

and we define

$$\beta: K_G(SX) \to R(G)$$

to be $b_1^* - b_0^*$. Since $b_i^* \varepsilon^* = 1$ we have $\beta\varepsilon^* = 0$ and hence $\beta$ factors through $\tilde{K}_G(SX)$.

**10.1. Proposition** *The sequence*

$$K_G(SX) \xrightarrow{\beta} R(G) \xrightarrow{\varepsilon^*} K_G(X)$$

*is exact.*

***Proof*** We remark that this comes from a Mayer–Vietoris sequence with $R(G)$ replaced by $R(G) \oplus R(G)$, but a direct proof seems desirable. Now $\varepsilon^* b_i^*$ is induced by the constant map $X \to w_i \in SX$ and, since these are equivariantly homotopic to one another, we have $\varepsilon^* b_0^* = \varepsilon^* b_1^*$ and hence $\varepsilon^* \beta = 0$. Suppose that $\varepsilon^*[\varrho_0] = \varepsilon^*[\varrho_1]$. Then $\varepsilon^* \varrho_0$ is stably equivalent to $\varepsilon^* \varrho_1$; that is, $\varepsilon^*(\varrho_0 \oplus \varrho) \approx \varepsilon^*(\varrho_1 \oplus \varrho)$ for some representation $\varrho$. Now $\varrho_i \oplus \varrho$ induces a $G$-vector bundle $\xi_i$ on the cone $CX$ via the projection of $CX$ to the vertex, and $\xi_0$ is equivalent to $\xi_1$ on the boundary $X$. Thus $\xi_0$ and $\xi_1$ can be pasted together, forming a $G$-vector bundle $\xi$ on $SX$. Clearly

$$\beta(\xi) = b_1^*(\xi) - b_0^*(\xi) = [\varrho_1 \oplus \varrho] - [\varrho_0 \oplus \varrho] = [\varrho_1] - [\varrho_0]. \quad \blacksquare$$

If $\xi$ is a vector bundle over $X/G$, then the pull-back of $\xi$ via the orbit map $X \to X/G$ is a $G$-vector bundle over $X$. This induces a homomorphism

$$\gamma: K(X/G) \to K_G(X)$$

which is an isomorphism when $G$ acts *freely* on $X$, the inverse being given by passage to the orbit space of $G$ on the total space of a $G$-vector bundle over $X$.

Now let $E_G \to B_G$ be a universal principal $G$-bundle whose classifying space $B_G$ is a CW-complex with finite skeletons. Let $B_G^{(k)}$ be the $k$-skeleton of $B_G$ and let $E_G^{(k)}$ be its inverse image. In the main cases $G = \mathbf{Z}_p$ or $G = \mathbf{S}^1$ of interest to us, $E_G$ can be taken to be the infinite sphere $\mathbf{S}^\infty = \bigcup \mathbf{S}^{2n+1}$.

In the special case for which $X = E_G^{(k)}$, $\varepsilon^*$ will be denoted by $\alpha_k$ and $\beta$ will be denoted by $\beta_k$, so that the sequence of 10.1 has the form

$$K_G(SE_G^{(k)}) \xrightarrow{\beta_k} R(G) \xrightarrow{\alpha_k} K_G(E_G^{(k)}) = K(B_G^{(k)}).$$

We shall use a superscript $c$ on $\alpha_k$ and $\beta_k$ in the case of complex $K$-theory and a superscript $r$ for the real case.

**10.2. Theorem** *Let $\Theta$: $G \times M \to M$ be a smooth action of the compact Lie group $G$ on the simply connected manifold $M$ and let $x$ and $y$ be stationary points. Assume that*

$$H^i(B_G;\ \pi_i(M)) = 0 \qquad \text{for} \quad 1 \leq i \leq m,$$

*where $\pi_i(M)$ is the system of local coefficients defined by the action of $\pi_1(B_G) \approx G/G_0$ on $\pi_i(M)$ via $\Theta$. Then*

$$\alpha_m(\Theta_x - \Theta_y) = 0.$$

*Proof* Note that for $G$ connected, the coefficients $\pi_i(M)$ are constant. If $G$ is finite, then these cohomology groups are the same as the algebraic cohomology groups $H^i(G, \pi_i(M))$ of $G$ with coefficients in the $G$-module $\pi_i(M)$.

Consider the constant maps $f_0$: $E_G \to \{x\} \subset M$ and $f_1$: $E_G \to \{y\} \subset M$. We try to construct an equivariant homotopy through the $m$-skeleton

$$F:\ E_G^{(m)} \times \mathbf{I} \to M$$

between $f_0$ and $f_1$. By II.2.6 this problem is the same as that of constructing a cross section over $B_G^{(m)} \times \mathbf{I}$ of the bundle over $B_G \times \mathbf{I}$ with fiber $M$ associated with the principal $G$-bundle $E_G \times \mathbf{I} \to B_G \times \mathbf{I}$, and extending a given cross section on $B_G \times \{0,1\}$. The obstructions to doing this are in

$$H^{i+1}(B_G \times (\mathbf{I}, \partial\mathbf{I}); \pi_i(M)) = H^i(B_G; \pi_i(M))$$

for $1 \leq i \leq m$. By assumption, these obstruction groups are zero, so that $F$ exists.

Then $F$ induces an equivariant map $\psi$: $SE_G^{(m)} \to M$ taking the vertex $w_0$ to $x$ and $w_1$ to $y$. Pulling back the tangent bundle of $M$ via $\psi$ gives a $G$-vector bundle $\tau$ on $SE_G^{(m)}$. The representation in the fiber of $\tau$ at $w_0$ is $\Theta_x$ and at $w_1$ it is $\Theta_y$. Thus $b_0^*[\tau] = \Theta_x$ and $b_1^*[\tau] = \Theta_y$ whence $\beta_m[\tau] = \Theta_y - \Theta_x$ and $\alpha_m$ must kill this. ∎

We shall now discuss the application of this theorem to the specific cases $G = \mathbf{Z}_p$ and $G = \mathbf{S}^1$ ($p$ need not be prime for the moment). We regard $\mathbf{Z}_p$ as a subgroup of the group $\mathbf{S}^1$ of complex numbers of absolute value 1. Let $t^n$ denote the 1-dimensional complex representation given by $t^n(z) \cdot (w) = z^n w$, where $z \in \mathbf{Z}_p$ (or $\mathbf{S}^1$) and $w \in \mathbf{C}$. For $G = \mathbf{Z}_p$ we have $t^n = t^{n+p}$ and $1, t, \ldots, t^{p-1}$ are a complete set of irreducible complex representations of $\mathbf{Z}_p$. Moreover $t^n$ is the tensor product of $n$ copies of $t$. Thus the representation ring of $\mathbf{Z}_p$ is

$$RU(\mathbf{Z}_p) = \mathbf{Z}[t]/(1 - t^p).$$

Similarly

$$RU(\mathbf{S}^1) = \mathbf{Z}[t, t^{-1}],$$

the ring of *finite* Laurent series in $t$. The augmentation

$$RU(G) \to \mathbf{Z}$$

is $RU(G) = KU_G(\star) \to KU(\star) \approx \mathbf{Z}$, which assigns to a representation its dimension. The kernel of this is called the "augmentation ideal" and is denoted by $I(G)$. Clearly

$$I(G) = (1 - t)RU(G)$$

for $G = \mathbf{Z}_p$ or $G = \mathbf{S}^1$.

For later reference let us discuss briefly the real case. For $G = \mathbf{Z}_p$ or $\mathbf{S}^1$ the representation $t^n + t^{-n}$ is the complexification of the real 2-dimensional representation taking $z = e^{i\theta}$ to

$$\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

in $\mathbf{O}(2)$. For $p$ odd or for $G = \mathbf{S}^1$ these are all the irreducible real representations. For $p$ even, $t^{p/2}$ is the complexification of the real representation $z = e^{2\pi i/p} \mapsto (-1) \in \mathbf{O}(1)$. Thus in all cases $G = \mathbf{Z}_p$ or $G = \mathbf{S}^1$, the complexification

$$RO(G) \to RU(G)$$

is an injection (regarded as *inclusion*) whose image consists of those elements $\varrho \in RU(G)$ such that $\bar{\varrho} = \varrho$. Here $\bar{\varrho}$ is the complex conjugate of $\varrho$ and is given by $t \mapsto t^{-1}$.

Now for $G = \mathbf{Z}_p$, $E_G^{(2n-1)}$ can be taken to be $\mathbf{S}^{2n-1}$ (and for $G = \mathbf{S}^1$, $E_G^{(2n-1)} = E_G^{(2n-2)} = \mathbf{S}^{2n-1}$) with the action given by the complex $n$-dimensional representation $nt$. In Atiyah [2, p. 105] the following exact sequence is derived from the exact sequence of the pair $(\mathbf{D}^{2n}, \mathbf{S}^{2n-1})$:

$$0 \to KU_G(SE_G^{(2n-1)}) \to RU(G) \xrightarrow{\varphi} RU(G) \xrightarrow{\alpha_{2n-1}^c} KU_G(E_G^{(2n-1)}) \to 0.$$

The map $\varphi$ is multiplication by $\lambda_{-1}(nt) = \Sigma(-1)^i\lambda^i(nt) = (1-t)^n$. Thus for $G = \mathbf{S}^1$

$$\ker \alpha_{2n-1}^c = \ker \alpha_{2n-2}^c = I(\mathbf{S}^1)^n = (1-t)^n RU(\mathbf{S}^1)$$

since $B_{\mathbf{S}^1}^{(2n-1)} = B_{\mathbf{S}^1}^{(2n-2)}$. In Atiyah [2, p. 107] it is shown, for the case $G = \mathbf{Z}_2$, that the restriction $KU_G(E_G^{(2n-1)}) \to KU_G(E_G^{(2n-2)})$ is an isomorphism. The same proof also gives this result for $G = \mathbf{Z}_p$ in general. Thus in the case $G = \mathbf{Z}_p$, we also have

$$\ker \alpha_{2n-1}^c = \ker \alpha_{2n-2}^c = I(\mathbf{Z}_p)^n = (1-t)^n RU(\mathbf{Z}_p).$$

From 10.2 we derive the following consequence of these facts.

**10.3. Theorem** *Let $\Theta$ be a smooth action of $G$ on the simply connected manifold $M$ with stationary points $x$ and $y$ (at least). If $G = \mathbf{Z}_p$ or if $G = \mathbf{S}^1$ and if*

$$H^i(B_G; \pi_i(M)) = 0 \qquad \text{for} \quad 1 \le i \le 2n-2,$$

*then $\Theta_x - \Theta_y \in RU(G)$ is divisible by $(1-t)^n$.* ∎

We remark that for $G = \mathbf{Z}_p$ then

$$H^i(B_G; \pi_i(M)) = H^i(G; \pi_i(M)) = \begin{cases} (\ker T)/(\operatorname{Im} N) & \text{for} \quad i \text{ even}, \\ (\ker N)/(\operatorname{Im} T) & \text{for} \quad i \text{ odd}, \end{cases}$$

where $T = 1 - g$ and $N = 1 + g + \cdots + g^{p-1}$ operating on $\pi_i(M)$, with $g$ a generator of $G = \mathbf{Z}_p$. For $G = \mathbf{S}^1$, $B_G = \mathbf{CP}^\infty$ and $H^i(B_G; \pi_i(M)) = 0$ for $i$ odd and is $\pi_i(M)$ for $i$ even. Actually, for $G = \mathbf{S}^1$ it suffices in 10.3 for $\pi_i(M)$ to be *finite* for $i$ even and $i \le 2n-2$. This, and other improvements, will be discussed in Section 11.

In any specific case it is always possible to check for divisibility by $(1-t)^n$, but for $G = \mathbf{Z}_p$ this is usually quite tedious since $RU(\mathbf{Z}_p)$ is *not* a unique factorization domain. For most cases the following corollary suffices and it is easily applicable. (A more general result for prime power orders can be found in Bredon [22].)

**10.4. Corollary** *Let $G = \mathbf{Z}_p$ for $p$ prime in the situation of* 10.3. *Then $\Theta_x - \Theta_y$ is divisible by $p^h$, where $h = [(n-1)/(p-1)]$ is the greatest integer less than or equal to $(n-1)/(p-1)$.*

*Proof* Since $t^p = 1$, $(1-t)^p$ is divisible by $p$. Thus $I(\mathbf{Z}_p)^p \subset pI(\mathbf{Z}_p)$. (Actually these can be shown to be equal; see Bredon [22].) Also

$$I(\mathbf{Z}_p)^{p+(p-1)} \subset pI(\mathbf{Z}_p)I(\mathbf{Z}_p)^{p-1} = pI(\mathbf{Z}_p)^p \subset p^2 I(\mathbf{Z}_p)$$

and an induction shows that, in general,

$$I(\mathbf{Z}_p)^{hp-h+1} = I(\mathbf{Z}_p)^{p+(h-1)(p-1)} \subset p^h I(\mathbf{Z}_p).$$

Thus for $n-1 \geq h(p-1)$ we have that $(1-t)^n$ is divisible by $(1-t)^{hp-h+1}$ which is divisible by $p^h$. ∎

In the next section we shall obtain several improvements of 10.3 and 10.4, by using *KO*-theory. In the case $p = 2$ these improvements are quite substantial.

*Remarks* In the situation of 10.4 note that, in particular, the difference of the dimensions of the fixed point set at $x$ and at $y$ is divisible by $p^h$, and by $2p^h$ when $p$ is odd. One should also notice that the proof of 10.4 provides an easily applied criterion for an element $\varphi(t) \in I(\mathbf{Z}_p)$, such as $\Theta_x - \Theta_y$, to be divisible by $(1-t)^n$ in $RU(\mathbf{Z}_p) = \mathbf{Z}[t]/(1-t^p)$ for a prime $p$. Namely, if we know abstractly that $\varphi(t)$ must be divisible by $(1-t)^n$ (e.g., from application of 10.3 in a particular situation), then, if we multiply $\varphi(t)$ by $(1-t)^i$ for some $i$ such that $n+i$ has the form $h(p-1)+1$ for some integer $h$, the result must be divisible by $p^h$. The converse is also true as the reader may verify by an easy argument using the fact that $I(\mathbf{Z}_p)^p = pI(\mathbf{Z}_p)$; that is, if $p^h$ divides $(1-t)^i\varphi(t)$ and if $\varphi(1) = 0$, then $(1-t)^n$ divides $\varphi(t)$ where $n = h(p-1) - i + 1$. It is desirable to illustrate this by an example. Suppose that $\mathbf{Z}_5$ acts on a 10-manifold $M$ and we wish to check whether it is possible to have the representations $\Theta_x = 8 + t^2 + t^3$ and $\Theta_y = 6 + 2t^2 + 2t^3$ at two fixed points. Then put $\varphi(t) = \Theta_x - \Theta_y = 2 - t^2 - t^3 = (1-t)(2+2t+t^2)$. If $M$ is 2-connected then $\varphi(t)$ must be divisible by $(1-t)^2$. The fact that $1-t$ does not divide $2+2t+t^2$ does not preclude this. In fact, applying the above criterion, we compute that

$$(1-t)^3\varphi(t) = (1-3t+3t^2-t^3)(2-t^2-t^3) = -5t+5t^2$$

which is divisible by 5. This is consistent with (and actually implies) divisibility of $\varphi(t)$ by $(1-t)^2$. In fact one may check that

$$\varphi(t) = (2 - t^2 - t^3) = (1-t)^2(1 + 2t + 2t^2 + t^3).$$

On the other hand, if $M$ is 4-connected, then $\varphi(t)$ must be divisible by $(1-t)^3$. We compute that

$$(1-t)^2\varphi(t) = (1 - 2t + t^2)(2 - t^2 - t^3) = 1 - 4t + t^2 + t^3 + t^4,$$

which is not divisible by 5 and hence precludes divisibility of $\varphi(t)$ by $(1-t)^3$.

## 11. REFINEMENTS USING REAL $K$-THEORY

In this section we shall give several refinements of the results of Section 10 by making use of $KO$-theory. We also shall obtain some improvements under the assumption that the tangent bundle of $M$ is stably divisible by $p$ over the $k$-skeleton; by which we mean that the restriction of $T(M)$ to the $k$-skeleton of $M$ plus some trivial real vector bundle is equivalent to the direct sum of $p$ copies of some real vector bundle.

For future reference, we recall that $\widetilde{KO}(\mathbf{S}^n) \approx \pi_{n-1}(\mathbf{O}(\infty))$ is given by the following table; see Husemoller [1, p. 222] for example.

| $n$ (mod 8): | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\widetilde{KO}(\mathbf{S}^n) \approx \pi_{n-1}(\mathbf{O}(\infty))$: | $\mathbf{Z}$ | $\mathbf{Z}_2$ | $\mathbf{Z}_2$ | 0 | $\mathbf{Z}$ | 0 | 0 | 0 | $\mathbf{Z}$ |

We shall be concerned with $G$-vector bundles over a $G$-space $X$ whose underlying vector bundles are trivial. To study this, consider the space $\mathbf{Map}(G, e; \mathbf{O}(N), I)$ of maps $\theta\colon G \to \mathbf{O}(N)$ with $\theta(e) = I$, with the compact-open topology. Let $G$ act on this by

$$(g\theta)(h) = \theta(hg)\theta(g)^{-1}.$$

The reader may make the straightforward check that this does define an action. The following result is also an easy computation which we leave to the reader.

**11.1. Proposition** *If $X$ is a $G$-space, then there is a natural one–one correspondence between orthogonal $G$-vector bundle structures on $X \times \mathbf{R}^N$ over $X$ and equivariant maps $\theta\colon X \to \mathbf{Map}(G, e; \mathbf{O}(N), I)$. If $\theta\colon x \mapsto \theta_x$, then the* *corresponding action of $G$ on $X \times \mathbf{R}^N$ is given by*

$$g(x, v) = \big(gx, \theta_x(g) \cdot v\big). \quad \blacksquare$$

Clearly a homotopy of such maps gives a $G$-bundle structure on $\mathbf{I} \times X \times \mathbf{R}^N$, so that homotopic maps give equivalent $G$-vector bundles.

Consider the forgetful map

$$\psi^{(m)}\colon \widetilde{KO}_G(SE_G^{(m)}) \to \widetilde{KO}(SE_G^{(m)}).$$

**11.2. Lemma** *If $G = \mathbf{Z}_p$ for $p$ odd, then $\psi^{(2n)}$ is trivial.*

*Proof* For the $G$-action on $\widetilde{KO}(SE_G^{(m)})$ it is clear that the image of $\psi^{(m)}$ consists of elements *invariant* under the action. Thus it suffices to show that $\widetilde{KO}(SE_G^{(2n)})^G$ is trivial for $G = \mathbf{Z}_p$, $p$ odd. Now we can take $E_G^{(2n)}$ to be the join $\mathbf{Z}_p \star \mathbf{S}^{2n-1}$ which has the homotopy type of the one-point union of $p - 1$ copies of $\mathbf{S}^{2n}$. Moreover (using the pointed category),

$$\widetilde{KO}(SE_G^{(2n)}) \approx [E_G^{(2n)}; \mathbf{O}(\infty)] \approx H^{2n}\big(E_G^{(2n)}; \pi_{2n}(\mathbf{O}(\infty))\big).$$

The only nontrivial case is that for which $n \equiv 0 \pmod 4$, in which case

$$\widetilde{KO}(SE_G^{(2n)}) \approx H^{2n}(E_G^{(2n)}; \mathbf{Z}_2) \approx (\mathbf{Z}_2)^{p-1}.$$

This can be thought of as being generated by the dual cohomology classes $\gamma_i$ to the $2n$-spheres

$$(g^i \star \mathbf{S}^{2n-1}) \cup (g^{i+1} \star \mathbf{S}^{2n-1}) \subset \mathbf{Z}_p \star \mathbf{S}^{2n-1} = E_G^{(2n)},$$

where $g$ is a generator of $\mathbf{Z}_p$. Now $\gamma_1, \ldots, \gamma_{p-1}$ form a basis and $\gamma_p = -(\gamma_1 + \cdots + \gamma_{p-1})$. Also $g$ acts on this by taking $\gamma_i$ to $\gamma_{i+1}$ (indexes modulo $p$). Since $p$ is odd it is an easy check that there are no invariants. $\blacksquare$

For $G = \mathbf{Z}_p$ we have $E_G^{(2n-1)} = \mathbf{S}^{2n-1}$ and $\widetilde{KO}(SE_G^{(2n-1)}) = \widetilde{KO}(\mathbf{S}^{2n})$, which is $\mathbf{Z}_2$ for $n \equiv 1 \pmod 4$, 0 for $n \equiv 3 \pmod 4$, and is $\mathbf{Z}$ for $n$ even.

Let $\psi_p^{(m)}$ denote the induced map ($G = \mathbf{Z}_p$)

$$\psi_p^{(m)}\colon \widetilde{KO}_G(SE_G^{(m)}) \to \frac{\widetilde{KO}(SE_G^{(m)})}{p\widetilde{KO}(SE_G^{(m)})}$$

and note that, for $p$ odd, this is the trivial map unless $m \equiv 3 \pmod 4$.

**11.3. Theorem** *If $p$ is odd, then*

$$\beta^r_{k-1}(\ker \psi_p^{(k-1)}) \subset I(\mathbf{Z}_p)^{\mu(k)},$$

*where* $\mu(k) = 2[k/4] + 2$.

*Proof* Let us first show that $\beta^r_{k-1}(\ker \psi^{(k-1)}) \subset I(\mathbf{Z}_p)^{\mu(k)}$. Let $\xi$ be a $G$-vector bundle over $SE_G^{(k-1)}$ with $\psi[\xi] = 0$. Then the underlying vector bundle of $\xi$ is stably trivial. Adding a trivial $G$-bundle to $\xi$ does not affect its image under $\beta^r_{k-1}$ so that we may as well assume that the underlying vector bundle of $\xi$ is $SE_G^{(k-1)} \times \mathbf{R}^N$, and $N$ can be taken as large as we please. Thus $\xi$ corresponds, by 11.1, to an equivariant map

$$\theta\colon SE_G^{(k-1)} \to Y = \mathbf{Map}(G, e; \mathbf{O}(N), I).$$

Since $G = \mathbf{Z}_p$ we have $Y \approx \mathbf{O}(N) \times \cdots \times \mathbf{O}(N)$, $p-1$ times. We can regard $\theta$ as an equivariant homotopy

$$E_G^{(k-1)} \times \mathbf{I} \to Y$$

between the constant maps to $\theta(w_0)$ and $\theta(w_1)$ on the ends. By II.2.6, this in turn can be regarded as a cross section over $(E_G \times \partial\mathbf{I}) \cup (E_G^{(k-1)} \times \mathbf{I})$ of the associated $Y$-bundle over $B_G \times \mathbf{I}$. Thus the obstructions to extending the restriction $\theta \mid SE_G^{(k-2)}$ to $\theta'\colon SE_G^{(n)} \to Y$ for some $n \geq k$ lie in

$$H^i(\mathbf{Z}_p; \pi_i(Y)) \qquad \text{for} \quad k \leq i \leq n.$$

However, $\pi_i(Y) = \pi_i(\mathbf{O}(N)) \oplus \cdots \oplus \pi_i(\mathbf{O}(N))$ and we may take $N$ large enough so that these are all stable groups. Since $p$ is *odd*, these obstruction groups are nonzero only for $i \equiv 3 \pmod 4$. Thus the extension exists for $n$ the smallest integer with $n \geq k-1$ and $n \equiv 2 \pmod 4$. The extension $\theta'$ gives rise to a $G$-vector bundle structure $\eta$ on $SE_G^{(n)} \times \mathbf{R}^N$ with the *same* representations at the vertices $w_0$ and $w_1$ as $\xi$. Thus

$$\beta^r_{k-1}[\xi] = \beta_n{}^r[\eta] \in \ker \alpha_n{}^r \subset \ker \alpha_n{}^c = I(\mathbf{Z}_p)^{(n+2)/2}.$$

It is easy to check that $(n+2)/2 = \mu(k)$.

Since $\psi^{(k-1)} = 0$ for $k$ odd by 11.2 it now suffices to show that $\beta^r_{k-1}$ (ker $\psi_p^{(k-1)}) = \beta^r_{k-1}$ (ker $\psi^{(k-1)}$) for $k$ *even*. Here $E_G^{(k-1)} = \mathbf{S}^{k-1}$ and $B_G^{(k-1)} = \mathbf{S}^{k-1}/\mathbf{Z}_p$ is a lens space. Thus there is a map $B_G^{(k-1)} \to E_G^{(k-1)}$ of degree 1, and the composition

$$\mathbf{S}^k = SE_G^{(k-1)} \xrightarrow{\pi} SB_G^{(k-1)} \longrightarrow SE_G^{(k-1)} = \mathbf{S}^k$$

then has degree $p$. This implies that the image of the map $\pi^*\colon \widetilde{KO}(SB_G^{(k-1)}) \to \widetilde{KO}(SE_G^{(k-1)})$ contains $p\widetilde{KO}(SE_G^{(k-1)})$. The commutative diagram

$$\begin{array}{ccccc}
KO(SB_G^{(k-1)}) & & \xrightarrow{\quad 0 \quad} & & KO(\star) \\
\Big\downarrow {\scriptstyle \pi^*} & \searrow^{\gamma} & & & \Big\downarrow {\scriptstyle \gamma} \\
 & & KO_G(SE_G^{(k-1)}) & \xrightarrow{\beta r} & KO_G(\star) = RO(G) \\
 & \swarrow^{\psi} & & & \\
\widetilde{KO}(SE_G^{(k-1)}) & & & &
\end{array}$$

shows that $\beta(\ker \psi_p) \subset \beta(\ker \psi + \operatorname{Im} \gamma) = \beta(\ker \psi)$ which is the desired result. ∎

We remark that the latter part of this proof also applies to the case $p = 2$ to show that $\beta(\ker \psi_2) = \beta(\ker \psi)$ since $\psi_2^{(k-1)} = \psi^{(k-1)}$ for $k$ odd.

**11.4. Corollary** *If $p$ is odd and $\Theta$ is a smooth action of $\mathbf{Z}_p$ on the simply connected manifold $M$ with*

$$H^i(\mathbf{Z}_p;\, \pi_i(M)) = 0 \qquad \textit{for} \quad 1 \le i \le k-1,$$

*then, for any two stationary points $x$ and $y$, $\Theta_x - \Theta_y$ is divisible by $(1-t)^{\mu(k-1)}$ in $RU(\mathbf{Z}_p)$. If, moreover, the tangent bundle of $M$ is stably divisible by $p$ over the $k$-skeleton, then $\Theta_x - \Theta_y$ is divisible by $(1-t)^{\mu(k)}$. Here $\mu(k) = 2[k/4] + 2$.*

*Proof* As before, there is an equivariant map $SE_G^{(k-1)} \to M$ and $T(M)$ induces a $G$-vector bundle $\tau$ on $SE_G^{(k-1)}$ with $\beta_{k-1}^r[\tau] = \Theta_x - \Theta_y$. For $k \not\equiv 0 \pmod 4$, $\psi_p^{(k-1)} = 0$ so that 11.3 applies. For $k \equiv 0 \pmod 4$, $\mu(k-1) = k/2$ and 10.3 already gives the result. In the last part, $\psi_p[\tau] = 0$ by assumption, so that 11.3 applies. ∎

By the argument in 10.4 and a straightforward calculation, we obtain the following improvement of 10.4.

**11.5. Corollary** *With the assumptions of* 11.4, *and with $p$ an odd prime, $\Theta_x - \Theta_y$ is divisible by $p^{\lambda(k-1)}$ in general and by $p^{\lambda(k)}$ when the tangent bundle of $M$ is stably divisible by $p$ over the $k$-skeleton. Here $\lambda(k) = [k/(2p-2)]$.* ∎

Note that the remarks following 10.4 provide, more generally, an easily applied method of utilizing the much more precise information contained in 11.4.

We now take up the case $p = 2$. In this case $E_G^{(n)} = \mathbf{S}^n$ and $B_G^{(n)} = \mathbf{RP}^n$. Let $\Phi(n)$ denote the number of integers $i \equiv 0, 1, 2, 4 \pmod 8$ such that $1 \leq i \leq n$. This is given by the following table.

| $n$: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $k+8$ |
|---|---|---|---|---|---|---|---|---|---|
| $\Phi(n)$: | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | $\Phi(k)+4$ |

Now $\widetilde{KO}(\mathbf{RP}^n)$ is known to be cyclic of order $2^{\Phi(n)}$ and is generated by $\alpha_n{}^r(1 - t)$; see Husemoller [1, p. 223] for example. Thus $\alpha_n{}^r\colon R(\mathbf{Z}_2) \to KO_{\mathbf{Z}_2}(\mathbf{S}^n)$ has

$$\ker \alpha_n{}^r = 2^{\Phi(n)} I(\mathbf{Z}_2) \subset RO(\mathbf{Z}_2) = RU(\mathbf{Z}_2).$$

**11.6. Theorem** *Let $\Theta$ be a smooth action of $\mathbf{Z}_2$ on the simply connected manifold $M$ with $H^i(\mathbf{Z}_2; \pi_i(M)) = 0$ for $1 \leq i \leq k - 1$. Then for any stationary points $x$ and $y$, $\Theta_x - \Theta_y$ is divisible by $2^{\Phi(k-1)}$. If the tangent bundle of $M$ is stably divisible by 2 over the $k$-skeleton, then $\Theta_x - \Theta_y$ is divisible by $2^{\Phi(k)}$.*

*Proof* The first part follows from the above remarks and 10.2. Note that $I(\mathbf{Z}_2) \approx \mathbf{Z}$ *additively*. Also a representation of $\mathbf{Z}_2$ is completely determined stably by the multiplicity of the eigenvalue $-1$, and $\Theta_x - \Theta_y$ can be regarded as the difference of these multiplicities at $x$ and $y$. Moreover this is just the difference of the dimensions of the components of $M^{\mathbf{Z}_2}$ containing $y$ and $x$.

To prove the second part of the theorem we recall from the proof of 11.3 and of 11.4 that the hypothesis implies that there is a $\mathbf{Z}_2$-vector bundle $\xi$ on $\mathbf{S}^k = SE_{\mathbf{Z}_2}^{(k-1)}$ (with fixed points $w_0$ and $w_1$) with *trivial* underlying vector bundle and such that $\beta_{k-1}^r[\xi] = \Theta_x - \Theta_y$. This gives rise to an equivariant map

$$\theta\colon \mathbf{S}^k \to \mathbf{Map}(\mathbf{Z}_2, e;\ \mathbf{O}(N), I) \approx \mathbf{O}(N)$$

[with involution $A \mapsto A^{-1}$ on $\mathbf{O}(N)$] such that the difference of the multiplicities of the eigenvalue $-1$ of $\theta(w_0)$ and $\theta(w_1)$ is $\Theta_x - \Theta_y$.

Thus it suffices to show that for any such equivariant map $\theta\colon \mathbf{S}^k \to \mathbf{O}(N)$, the difference of the multiplicities of the eigenvalue $-1$ of $\theta(w_0)$ and $\theta(w_1)$ is divisible by $2^{\Phi(k)}$. To prove this we can take $N$ as large as we please. As before, we attempt to extend $\theta$ to $\theta'\colon \mathbf{S}^{k+1} = SE_{\mathbf{Z}_2}^{(k)} \to \mathbf{O}(N)$. (In the present case it is simpler to think of the obstruction to the extension as the class $[\theta] \in \pi_k(\mathbf{O}(N))$ since if this is 0, then $\theta$ can be extended to one hemisphere as a map and then to the other hemisphere by equivariance.) If $k \equiv 2, 4, 5, 6 \pmod 8$, then $\pi_k(\mathbf{O}(N)) = 0$ and the extension exists. Otherwise the map $\theta_1$: $\mathbf{S}^k \to \mathbf{O}(2N)$ taking $q$ to

$$\begin{bmatrix} \theta(q) & 0 \\ 0 & \theta(q)^{-1} \end{bmatrix}$$

is equivariant and has $0 = [\theta_1] \in \pi_k(\mathbf{O}(2N))$. Thus an extension $\theta_1'$: $\mathbf{S}^{k+1} \to \mathbf{O}(2N)$ exists. The difference of the multiplicities of the eigenvalue $-1$ at $w_0$ and $w_1$ for $\theta_1'$ is twice that for $\theta$; but $2^{\Phi(k+1)} = 2 \cdot 2^{\Phi(k)}$ in this case also. By successive application of this argument, we can reduce the question eventually to the case in which $\Phi(k) = \Phi(k-1)$; i.e., $k \equiv 3, 5, 6, 7 \pmod 8$. In this case the (new) map $\theta$ corresponds to a $\mathbf{Z}_2$-vector bundle $\eta$ on $\mathbf{S}^k = SE_{\mathbf{Z}_2}^{(k-1)}$ and

$$\beta_{k-1}^r[\eta] \in \ker \alpha_{k-1}^r = 2^{\Phi(k-1)} I(\mathbf{Z}_2) = 2^{\Phi(k)} I(\mathbf{Z}_2).$$

However, this is just the difference of the multiplicities of the eigenvalue $-1$ of $\theta(w_0)$ and $\theta(w_1)$, which gives the desired result. ▌

We now turn to the case of $\mathbf{S}^1$-actions. Recall that for $G = \mathbf{S}^1$, $E_G^{(2n-1)} = E_G^{(2n-2)} = \mathbf{S}^{2n-1}$ which can be regarded as the join $\mathbf{S}^1 * \cdots * \mathbf{S}^1$ of $n$ copies of $\mathbf{S}^1$ with the diagonal action. Then $\mathbf{S}^{2n}$ will always have the $\mathbf{S}^1$-action as the suspension $\mathbf{S}^{2n} = S\mathbf{S}^{2n-1} = SE_G^{(2n-2)}$ which has the two fixed points $w_0$ and $w_1$. As with $\mathbf{Z}_2$, it will be more convenient to give the needed obstruction arguments directly. Thus if $X$ is an $\mathbf{S}^1$-space and $\varphi$: $\mathbf{S}^{2n} \to X$ is equivariant, then $\varphi$ extends equivariantly to $\mathbf{S}^{2n+2} = \mathbf{S}^{2n} * \mathbf{S}^1 \to X$ only if $0 = [\varphi] \in \pi_{2n}(X)$. Conversely, if $[\varphi] = 0$, then $\varphi$ extends as a map to the cone $\mathbf{S}^{2n} * \{e\} \subset \mathbf{S}^{2n+2}$ and extends uniquely from this to an equivariant map $\mathbf{S}^{2n+2} \to X$ by I.3.3. Thus $[\varphi] \in \pi_{2n}(X)$ is the only obstruction to extending $\varphi$ to $\mathbf{S}^{2n+2} \to X$.

Now we shall give an argument which allows us to kill the obstruction $[\varphi]$ when it has *finite order*. Let $\varrho_q$: $\mathbf{S}^{2n} \to \mathbf{S}^{2n}$ be the map which is the suspension of the join of $n$ copies of the map $z \mapsto z^q$ of $\mathbf{S}^1 \to \mathbf{S}^1$. For an $\mathbf{S}^1$-space $M$ let $M_{(q)}$ denote $M$ with the $\mathbf{S}^1$-action induced from the old action via the homomorphism $z \mapsto z^q$ of $\mathbf{S}^1 \to \mathbf{S}^1$. Clearly if $\varphi$: $\mathbf{S}^{2n} \to M$ is equivariant, then so is $\varphi \circ \varrho_q$: $\mathbf{S}^{2n} \to M_{(q)}$. Since $\varrho_q$ has degree $q^n$ we have that $[\varphi \circ \varrho_q] = 0$ when $[\varphi]$ has order $q$ in $\pi_{2n}(M)$.

Moreover, if $\omega(t) \in \mathbf{Z}[t, t^{-1}] = RU(\mathbf{S}^1)$ is the representation at a fixed point $x$ in $M$, then the representation at $x$ in $M_{(q)}$ is just $\omega(t^q)$. The following lemma shows that this change will not affect divisibility by $(1-t)^k$.

**11.7. Lemma** *A polynomial $f(t)$ is divisible by $(1-t)^k$ iff $f(t^q)$ is divisible by $(1-t)^k$.*

*Proof* Put $s = 1 - t$ and $g(s) = f(1-s) = f(t)$. Put $t^q = (1-s)^q = 1 - h(s)$, where $h$ has a simple zero at $s = 0$. Then the order of the zero at $s = 0$ of $f(t^q) = f((1-s)^q) = f(1-h(s)) = g(h(s))$ clearly equals that of $f(t) = g(s)$. ∎

If $\mathbf{S}^1$ acts smoothly on $M$ with stationary points $x$ and $y$ and if $\pi_i(M)$ is finite for $i$ *even* and $i \le 2n-2$, then the above arguments allow us to construct an equivariant map $\mathbf{S}^{2n} \to M_{(k)}$ taking $w_0$ to $x$ and $w_1$ to $y$, for *some* $k$. Also, the change of action does not change the order of divisibility of $\Theta_x - \Theta_y$ by $(1-t)$, and hence $\Theta_x - \Theta_y$ is divisible by $(1-t)^n$ by the discussion in Section 10.

Let $f(t)$ be the *polynomial* of smallest degree obtained from $\Theta_x - \Theta_y \in \mathbf{Z}[t, t^{-1}]$ by multiplying by a power of $t$. Since $\Theta_x - \Theta_y$ is *real*, it is invariant under $t \mapsto t^{-1}$ and this implies that $f(t)$ has *even* degree and that its coefficients enjoy a formal Poincaré duality. The quotient $g(t) = f(t)(1-t)^{-n}$ clearly also satisfies duality. If $n$ is *odd*, then $g(t)$ has odd degree and thus $g(1) = 0$ by duality. Thus $1 - t$ divides $g(t)$. This shows that $\Theta_x - \Theta_y$ is divisible by $(1-t)^{n+1}$, under the above assumptions, when $n$ is *odd*.

Assume now that $n$ is *even* and, in addition to the above conditions, that the tangent bundle of $M$ is stably trivial over the $2n$-skeleton. A map $\mathbf{S}^{2n} \to M_{(k)}$ then induces an $\mathbf{S}^1$-vector bundle on $\mathbf{S}^{2n}$ whose underlying vector bundle is stably trivial. By 11.1 this gives rise to an equivariant map

$$\theta\colon \mathbf{S}^{2n} \to \mathbf{Map}(\mathbf{S}^1, e;\ \mathbf{O}(N), I) = \Omega\mathbf{O}(N)$$

for any large $N$.

Then $[\theta] \in \pi_{2n}(\Omega\mathbf{O}(N)) = \pi_{2n+1}(\mathbf{O}(N))$ is the only obstruction to extending $\theta$ to an equivariant map $\mathbf{S}^{2n+2} \to \Omega\mathbf{O}(N)$. Since $n$ is even, this group, for $N$ large, is either 0 or $\mathbf{Z}_2$. We shall now show how to remove this obstruction. Let

$$\lambda\colon \Omega\mathbf{O}(N) \to \Omega\mathbf{O}(N)$$

be given by $(\lambda(\omega))(g) = \omega(g^2)$. Define $\psi\colon \mathbf{S}^{2n} \to \Omega\mathbf{O}(N)$ to be the composition $\psi = \lambda \circ \theta \circ \varrho_2$. Thus $\psi_x(g) = \theta_{\varrho_2(x)}(g^2)$ and it is easy to check that $\psi$ is equivariant. The $\mathbf{S}^1$-vector bundle structure on $\mathbf{S}^{2n} \times \mathbf{R}^N$ induced by $\psi$ is given by

$$g(x, v) = (gx, \theta_{\varrho_2(x)}(g^2) \cdot v)$$

and thus the representations on the fibers at $x = w_0$ and $x = w_1$ are obtained from the old representations by replacing $t$ by $t^2$. Since $\varrho_2$ has degree $2^n$ we have $0 = [\psi] \in \pi_{2n}(\Omega\mathbf{O}(N))$ and thus $\psi$ can be extended to $\mathbf{S}^{2n+2}$. This shows that $(\Theta_x - \Theta_y)(t^2)$, and hence $(\Theta_x - \Theta_y)(t)$, is divisible by $(1 - t)^{n+1}$. However, then it is also divisible by $(1 - t)^{n+2}$ by the Poincaré duality argument given in the case of odd $n$. Thus we have proved the following theorem.

**11.8. Theorem** *Let $\Theta$ be a smooth action of $\mathbf{S}^1$ on $M$. Assume that $\pi_i(M)$ is finite for all even $i$ with $2 \leq i \leq 2n - 2$. Then $\Theta_x - \Theta_y$ is divisible by $(1 - t)^{\omega(n-1)}$ for any two stationary points $x$ and $y$. If the tangent bundle of $M$ is stably trivial over the $2n$-skeleton of $M$, then $\Theta_x - \Theta_y$ is divisible by $(1 - t)^{\omega(n)}$. Here $\omega(n) = 2[n/2] + 2$.* ▮

*Remarks* Most of the material in Sections 10 and 11 is taken from Bredon [22], but we have incorporated several improvements here. Some further situations are also studied in this reference and several explicit examples are discussed there. We will meet some of these examples and applications in the next chapter.

## EXERCISES FOR CHAPTER VI

**1.** Let $G = \mathbf{Z}_2$ act on $\mathbf{R}$ by $x \mapsto -x$. For a real number $r > 0$ let $f_r$: $\mathbf{R} \to \mathbf{R}^+$ be $f_r(x) = |x|^r$. Then $f_r$ factors $f_r$: $\mathbf{R} \xrightarrow{\pi} \mathbf{R}/G \xrightarrow{\varphi_r} \mathbf{R}^+$, where $\varphi_r$ is a homeomorphism. Put a differentiable structure $\mathscr{F}_r$ on $\mathbf{R}/G$ by demanding that $\varphi_r$ be a diffeomorphism. Show that $\mathscr{F}_r$ is *natural* iff $r$ has the form $r = 2/n$ for some *integer n*. (By "natural" we mean that each equivariant diffeomorphism $\mathbf{R} \to \mathbf{R}$ should induce a *diffeomorphism* on $\mathbf{R}/G$.)

**2.** In Exercise 1 let us say that a differentiable structure on $\mathbf{R}/G$ is *liftable* iff each diffeomorphism $\mathbf{R}/G \to \mathbf{R}/G$ is induced by an equivariant diffeomorphism $\mathbf{R} \to \mathbf{R}$. Show that $\mathscr{F}_r$ is liftable iff $r$ has the form $r = 2m$ for some integer $m$. (Thus the *induced* structure $\mathscr{F}_2$ is the only one of these which is both natural and liftable.)

**3.** Prove the statement, made at the end of Section 7, that the analogs of the results of that section are false for $\mathbf{U}(n)$- or $\mathbf{Sp}(n)$-actions. (*Hint*: Consider, for example, the restriction of the $\mathbf{O}(2n + 1)$-action on $\Sigma_k^{4n+1}$ to $\mathbf{U}(n) \subset \mathbf{O}(2n) \subset \mathbf{O}(2n + 1)$.)

**4.** Show that the group $\Phi_3{}^6$ of Section 8 has an element whose order is either infinite or is divisible by 8. (*Hint*: Extend the involution on $W_3{}^5 \approx \Sigma_3{}^5$ with fixed set $\mathbf{L}(3, 1)$ to an involution on $\mathbf{S}^6$.)

**5.** Show that the group $\Phi^n_{n-2}$ contains a subgroup isomorphic to $\ker\{2: \Theta^n \to \Theta^n\}$. (Thus $\Phi_5{}^7$ contains an element of order 2.)

**6.** Define a homomorphism $\Phi^n_{n-2} \to \Theta^n$ which extends the identity on the subgroup of Exercise 5. Similarly, show that $\Theta^n$ is isomorphic to some direct summand of $\Theta^n_{n-2}$.

**7.** Show that there are an infinite number of distinct smooth semifree $\mathbf{S}^3$-actions on $\mathbf{S}^{4n-1}$ with fixed set a sphere of any codimension divisible by 8.

**8.** Let $p$ be an odd prime and let $\eta$ be the real 2-dimensional representation $\eta = t + t^{-1}$ of $\mathbf{Z}_p$. Let $n$ be a given integer and put $h = [n/(p-1)]$. For sufficiently large $N$ show that there exists a smooth action $\Theta$ of $\mathbf{Z}_p$ on $\mathbf{S}^{2n} \times \mathbf{S}^N$ whose fixed point set is a disjoint union $\mathbf{S}^a + \mathbf{S}^b$ of spheres, and such that if $x \in \mathbf{S}^a$ and $y \in \mathbf{S}^b$, then $\Theta_x - \Theta_y = p^h(2 - \eta)$ in $RO(\mathbf{Z}_p)$. It follows that if $a \geq b$, then $a - b = 2p^h$. Note that this shows that 11.5 is best possible. [*Hint*: Use the fact that $I(\mathbf{Z}_p)^{h(p-1)+1} = p^h I(\mathbf{Z}_p)$; compare the proof of 10.4. Also note that if $\varrho \in RU(\mathbf{Z}_p)$ is in $\operatorname{Im} \beta_m{}^c$, then $\varrho \oplus \bar{\varrho}$ is in $\operatorname{Im} \beta_m{}^r$. First consider the case $n = h(p-1) + 1$.]

**9.** Suppose that $\mathbf{Z}_3$ acts smoothly on $\mathbf{S}^8 \times \mathbf{S}^8$. Show that the representations of $\mathbf{Z}_3$ at all of the fixed points are equivalent to one another.

**10.** Consider the Stiefel manifold $\mathbf{V}_{n,2} = \mathbf{SO}(n)/\mathbf{SO}(n-2)$ for $n \geq 10$. For any smooth involution on this with an isolated fixed point, show that all other fixed points are also isolated.

**11.** If $\mathbf{Z}_3$ acts smoothly on the underlying manifold of $\mathbf{SU}(n)$ for $n \neq 3,5$ show that the representations at fixed points are all equivalent. (*Hint*: Use the fact that, for $i < 2n$, $\pi_i(\mathbf{SU}(n))$ is 0 for $i$ even and is $\mathbf{Z}$ for $i$ odd.)

**12.** Let $M^8$ be the connected sum of several copies of $\mathbf{S}^4 \times \mathbf{S}^4$. Suppose that $\mathbf{Z}_5$ acts smoothly on $M^8$ with isolated fixed points only. Show that the representations at the fixed points are all equivalent to one another.

**13.** Let $M = \mathbf{S}^{n_1} \times \mathbf{S}^{n_2} \times \cdots \times \mathbf{S}^{n_k}$ with $n_i \neq 2$. Suppose that $\mathbf{S}^1$ acts smoothly and semifreely on $M$. Show that the components of the fixed set $F$ have equal dimensions. Construct an example showing that this is false if some $n_i = 2$ (e.g., construct a smooth semifree $\mathbf{S}^1$-action on $\mathbf{S}^2 \times \mathbf{S}^5$ such that $F$ is the disjoint union $\mathbf{S}^1 + \mathbf{S}^5$).

**14.** Let $X$ be a $G$-space and suppose that we are given two orthogonal $G$-bundle structures on $X \times \mathbf{R}^N$ over $X$ corresponding to the equivariant maps $\theta$ and $\theta'$ of $X$ into $\mathbf{Map}(G, e; \mathbf{O}(N), I)$ as in 11.1. Show that these $G$-bundles are orthogonally equivalent over $X$ iff there is a map $\varphi\colon X \to \mathbf{O}(N)$ such that $\theta_x'(g) = \varphi(gx)\theta_x(g)\varphi(x)^{-1}$ for all $x \in X$ and $g \in G$.

**15.** Let $G \supset K$ be compact Lie groups and let $K$ act orthogonally and transitively on $\mathbf{S}^{n-1}$. Let $v_0 = (0, \ldots, 0, 1) \in \mathbf{S}^{n-1}$ and put $H = K_{v_0}$. If $\pi\colon G/H \to G/K$ is the projection, then the mapping cylinder $M_\pi$ is equivalent to $G \times_K \mathbf{D}^n$ as a $G$-space. Every point of $G \times_K \mathbf{D}^n$ has the form $[g, v]$ where $v \in \mathbf{R}v_0$. Show that, in this description, the right action of $S(\pi) = [N(H) \cap N(K)]/H$ is induced by $([g, v], s) \mapsto [gs, v]$ for $s \in N(H) \cap N(K)$ and $v \in \mathbf{R}v_0$. Show that this maps the fiber $\mathbf{D}^n$ over $gK$ to that over $gsK$ by $kv \mapsto s^{-1}ksv$ and hence that this is smooth iff the map $kv_0 \mapsto s^{-1}ksv_0$ of $\mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is orthogonal. [We remark that it seems to be possible to prove that this condition is, in fact, always satisfied. There does not appear to be any elegant or easy proof of this, and it seems to depend on the difficult classification of those homogeneous spaces $K/H$ which are spheres. On the other hand, this condition is easily checked for all specific cases of interest to us in this book.]

# CHAPTER VII

## COHOMOLOGY STRUCTURE OF FIXED POINT SETS

In this chapter we return to the study of the relationships between the cohomological structure of a space and that of the fixed set of a $\mathbf{Z}_p$-action or $\mathbf{S}^1$-action on the space, and shall obtain deeper results than those studied in Chapter III. The method we use is originally due to Borel [4, 5], and the background for this method is explained in Section 1. Some general inequalities on ranks of cohomology groups, due to Heller, Swan, and the author, are proved in Section 2.

In Sections 3 and 5 we investigate actions on projective spaces (real, complex, quaternionic, and Cayley) and examples illustrating the theorems are given in Section 4.

A general theorem concerning actions on Poincaré duality spaces is proved in Section 6. In Section 7 a general theorem about involutions is proved which implies that an involution on a Poincaré duality space of odd euler characteristic has a nonempty, high-dimensional, fixed point set.

Actions on a product of two spheres are studied in Sections 8 and 9, and circle actions on a product of several odd-dimensional spheres are studied in Section 10.

In Section 11 the cohomological method is applied to mapping cones to prove some results about equivariant maps between spheres with linear involutions.

### 1. PRELIMINARIES

For a compact Lie group $G$ let $E_G \to B_G$ be a universal principal $G$-bundle whose classifying space $B_G$ is a CW-complex with finite $N$-skeleton $B_G{}^N$ for all $N$. Let $E_G{}^N$ be the inverse image of $B_G{}^N$ and note that $E_G{}^N$ is compact and $N$-universal. The cases of main interest for us in this chapter are $G = \mathbf{Z}_p$ and $G = \mathbf{S}^1$. For a paracompact $G$-space $X$ let

$$X_G{}^N = X \times_G E_G{}^N$$

which is the associated bundle over $B_G{}^N$ with fiber $X$.

Since $X_G^N$ is paracompact and since Čech and sheaf cohomology theories coincide for paracompact spaces, there is the Leray spectral sequence (with $K$ an arbitrary coefficient group)

$$E_2^{r,q} = \check{H}^r(B_G^N; \check{H}^q(X; K)) \Rightarrow \check{H}^{r+q}(X_G^N; K)$$

in which the coefficients $\check{H}^q(X; K)$ are locally constant, but are twisted via the canonical action of $\pi_1(B_G^N) \approx \pi_0(G)$ on $\check{H}^q(X; K)$. (For background on this, in this generality, see Bredon [13, p. 140–145].)

From the spectral sequences of these fibrations it follows that the restriction $\check{H}^n(X_G^{N+1}; K) \to \check{H}^n(X_G^N; K)$ is an isomorphism for $n < N$. We shall *define*

$$\check{H}^n(X_G; K) = \varprojlim \check{H}^n(X_G^N; K).$$

(It is not clear that this coincides with the usual Čech groups of $X_G$ unless $X \times E_G$ is paracompact, but this is immaterial since the latter will never be used.)

Because of the stability, there is no difficulty in defining the limit spectral sequence

$$E_2^{r,q} = \check{H}^r(B_G; \check{H}^q(X; K)) \Rightarrow \check{H}^{r+q}(X_G; K).$$

This passage to the limit is technical and not really necessary. The reader may simply verify that all arguments are valid with $X_G^N$ replacing $X_G$ for sufficiently large $N$.

Similarly if $A \subset X$ is closed and invariant, then there is a spectral sequence

$$E_2^{r,q} = \check{H}^r(B_G; \check{H}^q(X, A; K)) \Rightarrow \check{H}^{r+q}(X_G, A_G; K).$$

Throughout this chapter we shall denote the fixed point set of $G$ on $X$ by

$$F = X^G,$$

and the orbit space by

$$X^* = X/G.$$

We regard $F$ as embedded in both $X$ and $X^*$. Note that

$$X_G \supset F_G = F \times B_G.$$

Upon passage to orbit spaces the equivariant projection $X \times E_G \to X$ induces the map

$$\varphi\colon X_G \to X^*.$$

**1.1. Proposition** *Let $G = \mathbf{Z}_p$ for $p$ prime. Then*

$$\varphi^*: \check{H}^i(X^*, A^* \cup F) \to \check{H}^i(X_G, A_G \cup F_G)$$

*is an isomorphism for arbitrary coefficients and for all closed invariant $A \subset X$. This also holds for $G = \mathbf{S}^1$ and rational coefficients.*

*Proof* For $G = \mathbf{Z}_p$ this follows from Exercise 8 of Chapter III and a similar proof applies to $G = \mathbf{S}^1$. For a direct proof, note that since

$$(G/H) \times_G E_G \approx (G \times_G E_G)/H \approx E_G/H \approx B_H$$

we see that for $x \in X$, with image $x^* \in X^*$, $\varphi^{-1}(x^*) \approx B_{G_x}$. Thus for $x^* \notin F$, $\varphi^{-1}(x^*)$ is acyclic (over $\mathbf{Q}$ for $G = \mathbf{S}^1$) and the result follows from the Vietoris–Begle Mapping Theorem; see Bredon [13, p. 142]. (This uses the fact that, for example, $\check{H}^i(X^*, F)$ is the cohomology of $X^* - F$ with supports in the family of subsets of $X^* - F$ which are closed in $X^*$.) ∎

Using 1.1 and the exact sequence of the pair $(X_G, F_G)$ we obtain the basic exact sequence

**(1.2)** $$\cdots \to \check{H}^k(X^*, F) \xrightarrow{i^*} \check{H}^k(X_G) \xrightarrow{j^*} \check{H}^k(F_G) \xrightarrow{\delta^*} \check{H}^{k+1}(X^*, F) \to \cdots$$

which is defined for $G = \mathbf{Z}_p$, $p$ prime, with arbitrary coefficients and also for $G = \mathbf{S}^1$ with rational coefficients.

All the terms in the cohomology sequence of $(X_G, F_G)$ are left modules over $\check{H}^*(X_G)$ via the cup product and the homomorphisms are module homomorphisms (up to sign for the connecting homomorphism). The projection $X_G \to B_G$ induces $\check{H}^*(B_G) \to \check{H}^*(X_G)$ and this gives (via 1.1) an $\check{H}^*(B_G)$-module structure to all the terms in 1.2 and $i^*$, $j^*$, and $\delta^*$ are module homomorphisms (only up to sign for $\delta^*$).

More generally, if $A \subset X$ is closed and invariant, then, in the same situation as (1.2), we have the exact sequence

**(1.3)** $$\cdots \to \check{H}^k(X^*, A^* \cup F) \to \check{H}^k(X_G, A_G) \to \check{H}^k(F_G, F_G \cap A_G) \to$$
$$\to \check{H}^{k+1}(X^*, A^* \cup F) \to \cdots$$

of $\check{H}^*(B_G)$-modules. We shall usually take $A = \varnothing$ or take $A$ to be one point in $F$.

Let us now digress to discuss the Leray–Hirsch Theorem. Let $(Y, Y')$ be a fiber bundle pair over a space $B$ with projection $p: Y \to B$ and fiber pair $(X, A)$. Let $\Lambda$ be a principal ideal domain. By a **cohomology extension of the fiber** we mean a $\Lambda$-module homomorphism of degree 0

$$\theta\colon \check{H}^*(X, A; \Lambda) \to \check{H}^*(Y, Y'; \Lambda)$$

such that for any $b \in B$ the composition

$$\check{H}^*(X, A; \Lambda) \xrightarrow{\theta} \check{H}^*(Y, Y'; \Lambda) \to \check{H}^*(X_b, A_b; \Lambda)$$

with the restriction to the fiber $(X_b, A_b)$ over $b$ is an isomorphism; see Spanier [1, p. 256]. It is important to notice that $\theta$ is not required to preserve products.

The following theorem holds in somewhat more generality but there are difficult technicalities in the Čech case unless some strong assumptions are placed on $(X, A)$ or on $B$. When $B$ is a finite CW-complex the proof is easy and this case suffices for our purposes.

**1.4. Theorem** (Leray–Hirsch) *In the above situation assume that $X$ is paracompact and $A$ is closed and that $B$ is a finite* CW-*complex. Let $\theta$ be a cohomology extension of the fiber, with respect to the base ring $\Lambda$, and assume that $\check{H}^*(X, A; \Lambda)$ is a torsion free $\Lambda$-module. Then the map*

$$\check{H}^*(B; \Lambda) \otimes_\Lambda \check{H}^*(X, A; \Lambda) \to \check{H}^*(Y, Y'; \Lambda),$$

*taking $\beta \otimes \alpha \mapsto p^*(\beta) \cup \theta(\alpha)$, is an isomorphism of $\check{H}^*(B; \Lambda)$-modules.*

*Proof* Note that $Y$ is paracompact and that $Y'$ is closed in $Y$. We use induction on the dimension and number of cells in $B$. Thus it suffices to prove the theorem for an adjunction $B = B' \cup_\varphi \mathbf{D}^n$, assuming it for $B'$. Using homotopy invariance we may as well assume that $B = B' \cup \mathbf{D}^n$; $B' \cap \mathbf{D}^n = \mathbf{S}^{n-1}$. Since $\mathbf{S}^{n-1}$ has smaller dimension we can assume the theorem for it. The theorem is true trivially over $\mathbf{D}^n$. There is the exact Mayer–Vietoris sequence

$$\cdots \to \check{H}^i(B) \to \check{H}^i(B') \oplus \check{H}^i(\mathbf{D}^n) \to \check{H}^i(\mathbf{S}^{n-1}) \to \check{H}^{i+1}(B) \to \cdots$$

(coefficients in $\Lambda$) and this remains exact upon tensoring with $\check{H}^*(X, A)$ since this is torsion free. The homomorphism defined in the theorem maps this sequence into the Mayer–Vietoris sequence for the parts of $(Y, Y')$ over $B'$ and $\mathbf{D}^n$ and the result follows upon application of the 5-lemma. ∎

In particular this implies, in the situation that concerns us, that the Künneth formula holds for $F_G^N = F \times B_G^N$. Thus

$$\check{H}^*(F_G, F_G \cap A_G; \Lambda) \approx \check{H}^*(B_G; \Lambda) \otimes_\Lambda \check{H}^*(F, F \cap A; \Lambda),$$

when $\Lambda$ is a field, or generally when $\check{H}^*(F, F \cap A; \Lambda)$ is torsion free.

The main case of interest of 1.4 is that of the fibering $X_G \to B_G$ or $(X_G, A_G) \to B_G$. (More accurately, that of $X_G^N \to B_G^N$ for each $N$.) Also we are mainly concerned with the case of a *field* $\Lambda$ of coefficients. In this case a cohomology extension of the fiber clearly exists iff the restriction to a typical fiber

$$\check{H}^*(X_G, A_G; \Lambda) \to \check{H}^*(X, A; \Lambda)$$

is *surjective*. In this case we say that $(X, A)$ is **totally nonhomologous to zero** in $(X_G, A_G)$.

Recall that for $p$ odd

$$\check{H}^*(B_{\mathbf{Z}_p}; \mathbf{Z}_p) \approx \mathbf{Z}_p[s, t]/(s^2)$$

with $\deg s = 1$, $\deg t = 2$, and $t = \beta(s)$, where $\beta$ is the Bockstein homomorphism associated with the coefficient sequence

$$0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0.$$

For $p = 2$ we have

$$\check{H}^*(B_{\mathbf{Z}_2}; \mathbf{Z}_2) \approx \mathbf{Z}_2[t], \qquad \deg t = 1.$$

Also

$$\check{H}^*(B_{\mathbf{S}^1}; \mathbf{Z}) \approx \mathbf{Z}[t], \qquad \deg t = 2.$$

We shall retain this notation throughout the chapter.

For a general coefficient group $K$ on which $G = \mathbf{Z}_p$ operates we have

$$\check{H}^i(B_G; K) \approx \begin{cases} \ker T & \text{for } i = 0, \\ (\ker T)/(\operatorname{Im} N) & \text{for } i > 0 \text{ even}, \\ (\ker N)/(\operatorname{Im} T) & \text{for } i > 0 \text{ odd}, \end{cases}$$

where $g$ generates $G$, $T = 1 - g$ and $N = 1 + g + g^2 + \cdots + g^{p-1}$. Note, in particular, that if $K$ is a vector space over $\mathbf{Z}_p$, then

$$\operatorname{rk} \check{H}^i(B_G; K) \leq \operatorname{rk} K^G = \operatorname{rk} \check{H}^0(B_G; K)$$

for all $i$.

**1.5. Theorem** *Suppose that $p$ is prime and let $G = \mathbf{Z}_p$ act on the finitistic space $X$. Let $A \subset X$ be closed and invariant and suppose that $\check{H}^i(X, A; \mathbf{Z}_p) = 0$ for $i > n$. Then*

$$j^*: \check{H}^k(X_G, A_G; \mathbf{Z}_p) \to \check{H}^k(F_G, F_G \cap A_G; \mathbf{Z}_p)$$

*is an isomorphism for $k > n$. If $(X, A)$ is totally nonhomologous to zero* (mod $p$) *in $(X_G, A_G)$, then $j^*$ is a monomorphism for all $k$ so that the sequence*

$$0 \to \check{H}^k(X_G, A_G; \mathbf{Z}_p) \xrightarrow{j^*} \check{H}^k(F_G, F_G \cap A_G; \mathbf{Z}_p) \to \check{H}^{k+1}(X^*, A^* \cup F; \mathbf{Z}_p) \to 0$$

*is exact. This also holds for $G = \mathbf{S}^1$ and rational coefficients if $X^*$ is also finitistic and the number of orbit types is finite.*

*Proof* The first part follows from the exact sequence (1.3) and the fact from III.7.9, III.7.6, and III.10.9 (for $G = \mathbf{S}^1$), that $\check{H}^i(X^*, A^* \cup F) = 0$ for $i > n$. (For a proof of the latter fact from the present point of view rather than from Smith theory, see Bredon [21, p. 249]. This also gives the case $G = \mathbf{S}^1$ without the assumption of finiteness of number of orbit types.)

If $(X, A)$ is totally nonhomologous to zero, then $\check{H}^*(X_G, A_G)$ is a *free* $H^*(B_G)$-module by 1.4. If $\alpha \in \check{H}^k(X^*, A^* \cup F)$, then $t^r\alpha = 0$ for sufficiently large $r$ so that $t^r i^*(\alpha) = 0$. Thus $i^*(\alpha) = 0$ for all $\alpha$, which implies that $j^*$ is a monomorphism. ∎

Recall from Chapter III that a space is said to be "finitistic" if each open covering has a finite-dimensional refinement.

**1.6. Theorem** *Let $p$ be a prime and let $G = \mathbf{Z}_p$ act on the finitistic space $X$. Let $A \subset X$ be closed and invariant. Suppose that $\sum \operatorname{rk} \check{H}^i(X, A; \mathbf{Z}_p) < \infty$. Then the following statements are equivalent:*

(a) *$(X, A)$ is totally nonhomologous to zero* (mod $p$) *in $(X_G, A_G)$.*

(b) *$G$ acts trivially on $\check{H}^*(X, A; \mathbf{Z}_p)$ and the spectral sequence $E_2^{r,q} = \check{H}^r(B_G; \check{H}^q(X, A; \mathbf{Z}_p)) \Rightarrow \check{H}^{r+q}(X_G, A_G; \mathbf{Z}_p)$ of $(X_G, A_G) \to B_G$ degenerates.*

(c) $\sum \operatorname{rk} \check{H}^i(X, A; \mathbf{Z}_p) = \sum \operatorname{rk} \check{H}^i(F, F \cap A; \mathbf{Z}_p)$.

*This also holds for $G = \mathbf{S}^1$ and rational coefficients, provided that $X^*$ is also finitistic and there are only finitely many orbit types.*

*Proof* The edge homomorphism

$$\check{H}^q(X_G, A_G) \to E_2^{0,q} \to \check{H}^q(X, A)$$

is just restriction to a fiber. Also

$$E_2^{0,q} = \check{H}^0(B_G; \check{H}^q(X, A)) = \check{H}^q(X, A)^G.$$

Thus (a) holds iff $G$ acts trivially on $\check{H}^q(X, A)$ and $E_2^{0,q}$ consists of permanent cocycles. From the multiplicative structure of the spectral sequence this is equivalent to (b).

By 1.5 we have (over $\mathbf{Z}_p$)

$$\text{rk } \check{H}^k(X_G, A_G) = \text{rk } \check{H}^k(F_G, F_G \cap A_G)$$

for $k > n$. But

$$\begin{aligned}\text{rk } \check{H}^k(X_G, A_G) = \sum \text{rk } E_\infty^{k-i,i} &\leq \sum \text{rk } E_2^{k-i,i} \\ &= \sum \text{rk } \check{H}^{k-i}(B_G; \check{H}^i(X, A)) \\ &\leq \sum \text{rk } \check{H}^i(X, A)^G \leq \sum \text{rk } \check{H}^i(X, A)\end{aligned}$$

with equality iff (b) holds (by the multiplicative properties of the differentials in the spectral sequence). Since

$$\check{H}^k(F_G, F_G \cap A_G) = \bigoplus (\check{H}^{k-i}(B_G) \otimes \check{H}^i(F, F \cap A))$$

has rank equal to $\sum$ rk $\check{H}^i(F, F \cap A)$, it follows that (b) is equivalent to (c). ∎

## 2. SOME INEQUALITIES

In this section we shall prove some generalizations of the inequalities III.7.9 of Floyd.

**2.1. Theorem** *Let $p$ be an odd prime and let $G = \mathbf{Z}_p$ act on the finitistic space $X$ with $A \subset X$ closed and invariant. Assume that $\sum$ rk $\check{H}^i(X, A; \mathbf{Z}_p) < \infty$ and that $(X, A)$ is totally nonhomologous to zero in $(X_G, A_G)$ mod $p$. Then, for each $k$,*

$$\sum_{i \geq 0} \text{rk } \check{H}^{k+2i}(F, F \cap A; \mathbf{Z}_p) \leq \sum_{i \geq 0} \text{rk } \check{H}^{k+2i}(X, A; \mathbf{Z}_p).$$

*In particular, if $\check{H}^*(X, A; \mathbf{Z}_p)$ vanishes in odd degrees, then so does $\check{H}^*(F, F \cap A; \mathbf{Z}_p)$.*

***Proof*** For notational convenience we shall omit $A$ and the coefficients. Recall that $\check{H}^m(X_G)$ is filtered by submodules $F_k\check{H}^m(X_G)$ with

$$E_\infty^{r,q} \approx \frac{F_q\check{H}^{r+q}(X_G)}{F_{q-1}\check{H}^{r+q}(X_G)}.$$

The isomorphism of the Leray–Hirsch Theorem 1.4 clearly preserves the filtration. It is, in fact, an isomorphism of *filtered* groups, since the map on the associated graded groups is just the isomorphism $E_2^{r,q} \approx E_\infty^{r,q}$. (For background on these facts see Bredon [13, Appendix]. Our present $F_k$ is $F_{m-k}$ of that reference.) The filtration of $\check{H}^*(B_G) \otimes \check{H}^*(X)$ is given by

$$F_k = \bigoplus_{q\le k} (\check{H}^*(B_G) \otimes \check{H}^q(X)).$$

The $\check{H}^*(B_G)$-operation preserves filtration, since $\check{H}^*(B_G)$ has filtration zero.

The map $\check{H}^m(X_G) \to \check{H}^m(F_G)$ also preserves (or decreases) filtration. Since this is an isomorphism for $m$ large (but *not* an isomorphism of *filtered* groups) the horizontal maps in the following diagram are surjective for large $m$

$$\begin{array}{ccc} \dfrac{\check{H}^m(X_G)}{F_{k-1}\check{H}^m(X_G)} & \twoheadrightarrow & \dfrac{\check{H}^m(F_G)}{F_{k-1}\check{H}^m(F_G)} \\ \varphi\downarrow & & \downarrow\psi \\ \dfrac{\check{H}^{m+1}(X_G)}{F_{k-1}\check{H}^{m+1}(X_G)} & \twoheadrightarrow & \dfrac{\check{H}^{m+1}(F_G)}{F_{k-1}\check{H}^{m+1}(F_G)} \end{array}$$

The vertical maps $\varphi$ and $\psi$ are multiplication by $s \in \check{H}^1(B_G)$. Thus the image of $\varphi$ maps onto that of $\psi$. However

$$\frac{\check{H}^{m+1}(X_G)}{F_{k-1}\check{H}^{m+1}(X_G)} \approx \bigoplus_{q\ge k} (\check{H}^{m-q+1}(B_G) \otimes \check{H}^q(X))$$

and the image of $\varphi$ consists of those terms with $m - q$ *even.* Taking $m - k$ even, the dimension of the image of $\varphi$ is $\sum \operatorname{rk} \check{H}^{k+2i}(X)$ for $i \ge 0$. Similarly for $\psi$, and the inequality follows. ∎

With some assumptions on integral cohomology one can improve on 2.1 as in the following theorem due to Heller [2] and Swan [2]. Note that this applies to the case $p = 2$ while 2.1 definitely is false for $p = 2$.

**2.2. Theorem** *Suppose that p is prime and that $G = \mathbf{Z}_p$ acts on the finite-dimensional space X with $A \subset X$ closed and invariant. Suppose that G acts* *trivially on* $\check{H}^*(X, A; \mathbf{Z})$. *Then*

$$\sum_{i\geq 0} \operatorname{rk} \check{H}^{k+2i}(F, F\cap A; \mathbf{Z}_p) \leq \sum_{i\geq 0} \operatorname{rk} \check{H}^{k+2i}(X, A; \mathbf{Z}_p).$$

***Proof*** As before we shall omit $A$ from the notation during the proof. Since $X$ is finite-dimensional, we have that $\check{H}^m(X^*, F; \mathbf{Z}) = 0$ for large $m$ and hence

$$j^*\colon \check{H}^m(X_G; \mathbf{Z}) \to \check{H}^m(F_G; \mathbf{Z})$$

is an isomorphism for large $m > \dim X$. Since, for $m$ large, $\check{H}^m(X \times E_G; \mathbf{Z}) = 0$, the composition (multiplication by $p$)

$$\check{H}^m(X_G; \mathbf{Z}) \to \check{H}^m(X \times E_G; \mathbf{Z}) \to \check{H}^m(X_G; \mathbf{Z})$$

with the transfer shows that $p$ kills $\check{H}^m(X_G; \mathbf{Z})$ and hence this is a *vector space over* $\mathbf{Z}_p$.

Since $G$ acts trivially on $\check{H}^*(X; \mathbf{Z})$, we compute

$$E_2^{r,q} = \check{H}^r\big(B_G; \check{H}^q(X; \mathbf{Z})\big) \approx \begin{cases} \check{H}^q(X; \mathbf{Z}) & \text{for } r = 0,\\ \check{H}^q(X; \mathbf{Z}) \otimes \mathbf{Z}_p & \text{for } r > 0 \text{ even},\\ \check{H}^q(X; \mathbf{Z}) * \mathbf{Z}_p & \text{for } r > 0 \text{ odd}.\end{cases}$$

Computing dimensions of $\mathbf{Z}_p$-vector spaces, we obtain (for $m - k$ even and $m$ large)

$$\begin{aligned}\dim \frac{\check{H}^m(X_G; \mathbf{Z})}{F_{k-1}\check{H}^m(X_G; \mathbf{Z})} &= \sum_{i\geq k} \dim E_\infty^{m-i,i}\\ &\leq \sum_{i\geq k} \dim E_2^{m-i,i}\\ &= \sum_{j\geq 0} [\dim\big(\check{H}^{k+2j}(X; \mathbf{Z}) \otimes \mathbf{Z}_p\big)\\ &\qquad + \dim\big(\check{H}^{k+2j+1}(X; \mathbf{Z}) * \mathbf{Z}_p\big)]\\ &= \sum_{j\geq 0} \dim \check{H}^{k+2j}(X; \mathbf{Z}_p).\end{aligned}$$

Similarly

$$\dim \frac{\check{H}^m(F_G; \mathbf{Z})}{F_{k-1}\check{H}^m(F_G; \mathbf{Z})} = \sum_{j\geq 0} \dim \check{H}^{k+2j}(F; \mathbf{Z}_p)$$

and the inequality follows, since $j^*$ maps $\check{H}^m(X_G; \mathbf{Z})/F_{k-1}$ *onto* $\check{H}^m(F_G; \mathbf{Z})/F_{k-1}$. ▮

*Remark* Theorem 2.2 can be proved for $X$ finitistic rather than finite-dimensional, but there are some technical difficulties with the present approach since $\check{H}^*(X^*, F \cup A^*; \mathbf{Z})$ (and $\check{H}^*(X, A; \mathbf{Z})$, etc.) need not vanish in high degrees. The difficulties can be overcome by using the fact that these groups have no $p$-torsion and are $p$-divisible. However, these difficulties are not present in the method of Swan [2], which is similar to, but more algebraic than, the present approach. Hence that method is preferable for this particular result. The present method has some other advantages over Swan's method, however, and it applies to the case $G = \mathbf{S}^1$, unlike the Swan method.

## 3. $\mathbf{Z}_p$-ACTIONS ON PROJECTIVE SPACES

For a prime $p$ we shall write $X \sim_p Y$ if $X$ and $Y$ have isomorphic mod $p$ cohomology rings. Similarly, we use the notation

$$X \sim_p \mathbf{P}^h(n)$$

to mean that the cohomology ring

$$\check{H}^*(X; \mathbf{Z}_p) \approx \mathbf{Z}_p[a]/(a^{h+1}),$$

where $n = \deg a$. Thus for $n = 1$, 2, or 4, $X$ has the mod $p$ cohomology ring of $\mathbf{RP}^h$, $\mathbf{CP}^h$ or $\mathbf{QP}^h$, respectively. For $n = 8$ and $h = 2$, $X$ has the mod $p$ cohomology ring of the Cayley projective plane $\mathbf{Cay\,P}^2$. If $p = 2$, then these are the only possibilities for $n$ if $h \geq 2$. When $p$ is odd then $n$ must be even for $h \geq 2$ and there are examples (see Section 4) of $X \sim_p \mathbf{P}^2(n)$ for all odd $p$ and even $n$. The following theorem from Bredon [10, 21] is the basic result for $\mathbf{Z}_p$-actions on projective spaces.

**3.1. Theorem** *Suppose that $p$ is prime and that $G = \mathbf{Z}_p$ acts on the finitistic space $X \sim_p \mathbf{P}^h(n)$. Then either $F$ is empty or $F$ is the disjoint union of components $F_i \sim_p \mathbf{P}^{h_i}(n_i)$ with $h + 1 = \sum (h_i + 1)$ and $n_i \leq n$. The number of components is at most $p$. For $p$ odd and $h \geq 2$, $n$ and the $n_i$ are all even. Moreover, if $n_i = n$ for some $i$, then the restriction $\check{H}^n(X; \mathbf{Z}_p) \to \check{H}^n(F_i; \mathbf{Z}_p)$ is an isomorphism.*

*Proof* Because of the Smith Theorems we may assume that $h \geq 2$. Let $F_0$ be a component of $F$ and let $x \in F_0$. Consider the spectral sequence of

$(X_G, x_G)$ with $\mathbf{Z}_p$ coefficients. Since $\mathbf{Z}_p$ has no automorphisms of period $p$, it follows that $G$ acts trivially on $\check{H}^*(X, x)$. Let $a \in \check{H}^n(X, x)$ be a generator. Then

$$1 \otimes a \in \check{H}^0(B_G) \otimes \check{H}^n(X, x) = E_2^{0,n}$$

is a permanent cocycle since $E_2^{0,q} = 0$ for $q < n$. However, then each $1 \otimes a^i$ is a permanent cocycle and this implies that the spectral sequence degenerates. Thus $(X, x)$ is totally nonhomologous to zero in $(X_G, x_G)$ by 1.6. There is a unique element $\alpha \in \check{H}^n(X_G, x_G)$ restricting to $a \in \check{H}^n(X, x)$. By 1.4, $\check{H}^*(X_G, x_G)$ is the free $\check{H}^*(B_G)$-module generated by

$$\alpha, \alpha^2, \ldots, \alpha^h.$$

If $p$ is odd, then $\check{H}^*(F, x)$ vanishes in odd degrees by 2.1. Our arguments below will take place in even degrees for $p$ odd, and this explains the absence of $s \in \check{H}^1(B_G)$ from the formulas that we will write down.

Let $j_0$: $B_G \times (F_0, x) = ((F_0)_G, x_G) \to (X_G, x_G)$ be the inclusion and put

$$j_0^*(\alpha) = 1 \otimes b_n + t \otimes b_{n-d} + t^2 \otimes b_{n-2d} + \cdots + t^k \otimes b_{n-kd}, \qquad (1)$$

where

$$b_i \in \check{H}^i(F_0, x), \qquad b_{n-kd} \neq 0,$$

and $d = 1, 2$ according as $p = 2$ or $p$ is odd. Put $b = b_{n-kd}$ and note that $n - kd = \deg b > 0$. We claim that $\check{H}^*(F_0, x)$ is generated by $b$ as an algebra over $\mathbf{Z}_p$.

Now $j_0^*$ is just $j^*$ followed by the projection of $\check{H}^*(F_G, x_G)$ onto its direct summand $\check{H}^*((F_0)_G, x_G)$. Since $j^*$ is onto in high degrees by 1.5, this is also true for $j_0^*$. However, the image of $j_0^*$ is generated over $\check{H}^*(B_G)$ by $j_0^*(\alpha), j_0^*(\alpha^2), \ldots, j_0^*(\alpha^h)$ and it follows that $\check{H}^*(F_0, x)$ is generated by the $b_i$ of (1) as a $\mathbf{Z}_p$-algebra.

Since $j_0^*$ is onto in high degrees we can write

$$t^{r+k} \otimes b = j_0^*(A_1 t^r \alpha + A_2 t^{r-m} \alpha^2 + \cdots + A_h t^{r-hm+m} \alpha^h) \qquad (2)$$

for sufficiently large $r$, where $m = n/d$ and $A_i \in \mathbf{Z}_p$. If we expand the right-hand side of (2) using (1) and compare coefficients of $t^{r+k}$ we see that $A_1 = 1$. Subtracting this term from both sides of (2) we obtain

$$t^{r+k} \otimes b - j_0^*(t^r \alpha) = j_0^*(A_2 t^{r-m} \alpha^2 + \cdots + A_h t^{r-hm+m} \alpha^h).$$

On the other hand, multiplying (1) by $t^r$ and rearranging, leads to

$$j_0^*(t^r\alpha) - t^{r+k} \otimes b = t^r \otimes b_n + t^{r+1} \otimes b_{n-d} + \cdots + t^{r+k-1} \otimes b_{n-(k-1)d}.$$

Thus from the last two equations we have

$$t^r \otimes b_n + \cdots + t^{r+k-1} \otimes b_{n-(k-1)d} = -j_0^*(A_2 t^{r-m}\alpha^2 + \cdots + A_h t^{r-hm+m}\alpha^h).$$

If we think of expanding the right-hand side of this equation using (1) and then comparing coefficients of powers of $t$, we obtain expressions for each of the elements

$$b_n, b_{n-d}, \ldots, b_{n-(k-1)d}$$

as polynomials in the $b_i$ (including $b = b_{n-kd}$) *with no linear terms.* Thus $b_{n-(k-1)d}$ is a polynomial in $b = b_{n-kd}$ and hence is a scalar multiple of a power of $b$. Similarly, $b_{n-(k-2)d}$ is a polynomial in $b$ and $b_{n-(k-1)d}$, hence in $b$, and must then be a scalar multiple of a power of $b$; and so on. This shows that the $b_i$ are all scalar multiples of powers of $b$ and hence that $b$ generates $\check{H}^*(F_0, x)$ as a $\mathbf{Z}_p$-algebra.

It follows that $F_0 \sim_p \mathbf{P}^{h_0}(n_0)$ for some $h_0$ and for $n_0 = \deg b \leq n$. If $p$ is odd, then $n_0 = n - 2k$ is even, since $n$ is, even if $h_0 = 1$.

The equality $h + 1 = \sum(h_i + 1)$ simply expresses 1.6(c) and the last statement of the theorem is an immediate consequence of the proof.

It remains to show that there are at most $p$ components of $F$. For this, consider the projection

$$\pi\colon \check{H}^*(B_G) \otimes \check{H}^*(F, x) \to \check{H}^*(B_G) \otimes \check{H}^0(F, x)$$

which is a ring homomorphism. Put

$$\pi j^*(\alpha) = t^m \otimes c$$

so that

$$\pi j^*(t^r\alpha^i) = t^{im+r} \otimes c^i.$$

Since $\pi$ is onto and $j^*$ is onto in high degrees, we see that $c, c^2, \ldots, c^h$ generate $\check{H}^0(F, x)$ as a $\mathbf{Z}_p$-vector space. Since $c^p = c$, rk $\check{H}^0(F, x) \leq p - 1$ and hence $F$ has at most $p$ components. ∎

Note that if one uses the canonical ring isomorphisms to write

$$c = (c_1, c_2, \ldots) \in \mathbf{Z}_p \oplus \mathbf{Z}_p \oplus \cdots \approx \check{H}^0(F_1) \oplus \check{H}^0(F_2) \oplus \cdots,$$

then it follows from the above remarks that the $c_i \in \mathbf{Z}_p$ are all *distinct* and nonzero. (Also, one can delete $x$ and regard $c_0$ as zero.)

*Remark* In the case of $\mathbf{Z}_2$-actions on $\mathbf{RP}^h$ the above theorem was first proved by Smith [10] using a geometrical argument to reduce it to the Smith Theorem on spheres. That argument was extended by Su [3] to the case of a cohomology $\mathbf{RP}^h$. The general case was proved in Bredon [10, 21] as were the rest of the results in this section.

*Remark* Theorem 3.1 can be regarded as saying that if $\check{H}^*(X; \mathbf{Z}_p)$ is generated as an algebra by one element, then the same holds for each component of $F$ for a $\mathbf{Z}_p$-action on $X$. In Bredon [21] it was conjectured that if $\check{H}^*(X; \mathbf{Z}_p)$ is generated by at most $k$ elements as an algebra, then the same is true of each component of $F$. A counterexample to this is constructed at the end of Section 10. However, this may hold when $X$ is totally nonhomologous to zero in $X_G$ and this remains a nontrivial unsolved problem.

To make a closer study of actions on projective spaces we will apply the Steenrod operations. Let us briefly recall the properties of these that we will need. The standard reference is Steenrod and Epstein [1].

For $p = 2$, there exist natural homomorphisms

$$Sq^i\colon \check{H}^n(X, A; \mathbf{Z}_2) \to \check{H}^{n+i}(X, A; \mathbf{Z}_2)$$

for $i \geq 0$, with the following properties:

(a) $Sq^0 = 1$;

(b) $Sq^i(x) = \begin{cases} x^2 & \text{if } i = \deg x, \\ 0 & \text{if } i > \deg x; \end{cases}$

(c) $Sq^k(xy) = \sum_{i=0}^{k} Sq^i(x)\, Sq^{k-i}(y)$ (the Cartan formula).

It follows that, in particular,

$$\deg x = 1 \Rightarrow Sq^i(x^k) = \binom{k}{i} x^{k+i}.$$

The Adem relations (loc. cit.), which are difficult consequences of the above facts, imply that if $X \sim_2 \mathbf{P}^h(n)$ and $h \geq 2$, then $n$ is a power of 2. (In fact, it is a well-known theorem of Adams [1] that $n = 1, 2, 4,$ or $8$.) Another fact about these operations is that $Sq^1$ is the Bockstein homomorphism of the coefficient sequence

$$0 \to \mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2 \to 0.$$

For an odd prime $p$ there are natural homomorphisms

$$P_p{}^i\colon \check{H}^n(X, A; \mathbf{Z}_p) \to \check{H}^{n+2i(p-1)}(X, A; \mathbf{Z}_p)$$

such that

(a) $P_p{}^0 = 1$;

(b) $P_p{}^i(x) = \begin{cases} x^p & \text{if } 2i = \deg x, \\ 0 & \text{if } 2i > \deg x; \end{cases}$

(c) $P_p{}^k(xy) = \sum_{i=0}^k P_p{}^i(x)\, P_p^{k-i}(y)$ (the Cartan formula).

In particular

$$\deg x = 2 \Rightarrow P_p{}^i(x^k) = \binom{k}{i} x^{k+i(p-1)}.$$

Of the Adem relations (loc. cit.) we shall only have occasion to use the fact that

$$P_p{}^1 P_p{}^1 = 2P_p{}^2.$$

We will need to know the value of $P_p{}^1$ on $\check{H}^4(\mathbf{QP}^h; \mathbf{Z}_p)$. For this let $x \in \check{H}^2(\mathbf{CP}^{2h+1}; \mathbf{Z}_p)$ be a generator and note that $P_p{}^1(x^2) = 2x^{p+1}$ (from the above remarks). Now there is a canonical fibration $f$: $\mathbf{CP}^{2h+1} \to \mathbf{QP}^h$ with fiber $\mathbf{S}^2$ and $f^*$: $\check{H}^4(\mathbf{QP}^h; \mathbf{Z}_p) \to \check{H}^4(\mathbf{CP}^{2h+1}; \mathbf{Z}_p)$ is an isomorphism. We choose a generator $y \in \check{H}^4(\mathbf{QP}^h; \mathbf{Z}_p)$ by demanding that

$$f^*(y) = x^2.$$

Then, by naturality,

$$P_p{}^1(y) = 2y^{(p+1)/2}.$$

Suppose now that we only know that $X \sim_p \mathbf{QP}^h$ and let $z \in \check{H}^4(X; \mathbf{Z}_p)$. Then

$$P_p{}^1 P_p{}^1(z) = 2P_p{}^2(z) = 2z^p$$

by (b). However, $P_p{}^1(z) = Az^{(p+1)/2}$ for some $A \in \mathbf{Z}_p$ since there are no other elements of $\check{H}^{2p+2}(X; \mathbf{Z}_p)$. From the Cartan formula we compute

$$2z^p = P_p{}^1(Az^{(p+1)/2}) = A\,\frac{p+1}{2}\, z^{(p-1)/2}\, P_p{}^1(z) = \frac{p+1}{2}\, A^2 z^p.$$

If $h \geq p$ (so that $z^p \neq 0$), then it follows that $A^2 = 4$ and hence that $A = \pm 2$. Thus $P_p{}^1(z) = \pm 2z^{(p+1)/2}$ provided that $h \geq p$. On the other hand there are examples of spaces $X \sim_3 \mathbf{QP}^2$ (so that $h = 2 < 3 = p$) for which $P_3{}^1(z) = 0$; see Section 4.

**3.2. Theorem** *Suppose that $G = \mathbf{Z}_2$ acts on the finitistic space $X \sim_2 \mathbf{P}^h(n)$, $h \geq 2$. Then one of the following possibilities must hold*:

(1) *$F = \emptyset$ and $h$ is odd.*

(2) *$F$ is connected and $F \sim_2 \mathbf{P}^h(m)$, where $n = m$ or $n = 2m$.*

(3) *$F$ has two components $F_1$ and $F_2$, where $F_i \sim_2 \mathbf{P}^{h_i}(n)$ and $h = h_1 + h_2 + 1$. (Here $h_i$ may be 0 or 1.)*

*Moreover, in case (2) the restriction $\check{H}^n(X; \mathbf{Z}_2) \to \check{H}^n(F; \mathbf{Z}_2)$ is an isomorphism.*

*Proof* By 3.1, $F$ has at most two components and not both can be acyclic. Let $F_0$ be a component of $F$ and assume that $F_0 \sim_2 \mathbf{P}^k(m)$, where $k \geq 1$ and $m < n$. It then suffices to show that case (2) holds with $n = 2m$. Since $h \geq 2$ we know that $n$ is a power of 2, and in fact is one of 1, 2, 4, 8. Rather than give the general proof (which can be found in Bredon [10]), we shall restrict our attention to the case $n = 4$, which is typical.

We retain the notation of the proof of 3.1. Thus $\alpha \in \check{H}^4(X_G, x_G)$ represents the generator $a \in \check{H}^4(X, x)$, where $x \in F_0$. Put

$$j^*(\alpha) = t^4 \otimes c_0 + t^3 \otimes c_1 + t^2 \otimes c_2 + t \otimes c_3 + 1 \otimes c_4,$$

where $c_i \in \check{H}^i(F, x)$ and $c_m \neq 0$. By the proof of 3.1, $c_0$ generates $\check{H}^0(F, x)$ and hence $c_0 = 0$ iff $F$ is connected. Suppose that $c_0 \neq 0$. If $m = 1$ or $m = 3$, put $i = 1$, and if $m = 2$ put $i = 2$. Consider $Sq^i(\alpha)$. Since $i < 4$, $\check{H}^{4+i}(X_G, x_G)$ has only two elements 0 and $t^i\alpha$. However, computation shows that $j^*(Sq^i\alpha) = Sq^i(j^*(\alpha))$ has no term involving $c_0$. Thus $Sq^i(\alpha) = 0$. Now recall that

$$j_0^*(\alpha) = t^{4-m} \otimes b + (\star),$$

where $0 \neq b \in \check{H}^m(F_0, x)$ and $(\star)$ denotes a sum of terms $t^{4-k} \otimes b_k$ whose second degree $k > m$. Then, by the choice of $i$,

$$j_0^*(Sq^i\alpha) = Sq^i(j_0^*(\alpha)) = t^{4-m+i} \otimes b + (\star) \neq 0$$

and hence $Sq^i(\alpha) \neq 0$. This contradiction shows that $F$ is connected.

Since $F$ is connected we have $F \sim_2 \mathbf{P}^h(m)$ and $m$ is a power of 2 since $h \geq 2$. We must show that $m \neq 1$. If $m = 1$, then

$$j^*(\alpha) = t^3 \otimes b + t^2 \otimes b_2 + t \otimes b_3 + 1 \otimes b_4,$$

where $b$ generates $\check{H}^1(F, x; \mathbf{Z}_2)$ and $b_i$ is either 0 or $b^i$. (In fact $b_3$ and $b_4$ will not concern us.) As above, $Sq^1(\alpha)$ is either 0 or $t\alpha$ and hence $Sq^1(j^*(\alpha)) = j^*(Sq^1\alpha)$ is either 0 or $tj^*(\alpha)$. However, we calculate

$$Sq^1 j^*(\alpha) = t^4 \otimes b + t^3 \otimes b^2 + (\star),$$

where $(\star)$ involves only higher powers of $b$. Thus we must have $b_2 = b^2 \neq 0$. Similarly $Sq^2(\alpha)$ is either 0 or $t^2\alpha$ and we calculate

$$j^*(Sq^2\alpha) = Sq^2(j^*(\alpha)) = t^5 \otimes b + (\star)$$

which is neither 0 nor $t^2 j^*(\alpha)$ since it has no term in $b^2$; a contradiction.

For the last statement, it suffices to show that

$$j^*(\alpha) = t^2 \otimes b + 1 \otimes b^2.$$

The only other possibility is $j^*(\alpha) = t^2 \otimes b$, but in this case $Sq^2(j^*(\alpha)) = t^4 \otimes b + t^2 \otimes b^2$ contrary to the fact that $Sq^2(\alpha)$ must be either 0 or $t^2\alpha$.

That $F \neq \varnothing$ when $h$ is even follows from Floyd's result III.7.10 that $\chi(F) \equiv \chi(X) \equiv 1 \pmod 2$ when $h$ is even. A direct proof is also easy to give by showing that the spectral sequence must degenerate. ∎

*Remark* If $X$ is a finite CW-complex whose integral cohomology ring is that of $\mathbf{QP}^h$, $h \geq 2$, then $F$ must be nonempty. This follows from the stronger fact that these spaces have the fixed point property, and the proof of this is an elementary exercise in the use of the operation $P_3{}^1$ and the Lefschetz Fixed Point Theorem.

*Remark* Case (2) of 3.2 with $n = m$ occurs, of course, for the trivial action and this is the only possibility when $X$ is a closed manifold. However, for spaces like $\mathbf{CP}^h \times \mathbf{D}^r$ there can exist nontrivial examples of this situation. See the next section for some strange examples of this case.

**3.3. Theorem** *Suppose that $G = \mathbf{Z}_2$ acts on the finitistic space $X \sim_2 \mathbf{P}^h(n)$ with $h \geq 2$, $n \geq 2$, and $F \neq \varnothing$. Then $\check{H}^n(X; \mathbf{Z}_4) \approx \mathbf{Z}_4$. If $n > 2$, then $G$ acts trivially on $\check{H}^n(X; \mathbf{Z}_4)$. If $n = 2$ (i.e., $X \sim_2 \mathbf{CP}^h$), then $G$ acts nontrivially on $\check{H}^2(X; \mathbf{Z}_4)$ iff $F \sim_2 \mathbf{RP}^h$.*

*Proof* Since $n \geq 2$, $\check{H}^*(X; \mathbf{Z}_2) = 0$ in odd degrees. Thus we have the exact sequence

$$0 \to \check{H}^n(X; \mathbf{Z}_2) \xrightarrow{\varphi} \check{H}^n(X; \mathbf{Z}_4) \xrightarrow{\psi} \check{H}^n(X; \mathbf{Z}_2) \to 0.$$

This, and the fact that $\varphi\psi = 2$ on $\check{H}^n(X; \mathbf{Z}_4)$, shows that $\check{H}^n(X; \mathbf{Z}_4) \approx \mathbf{Z}_4$. Considering the spectral sequence of $(X_G, x_G) \to B_G$ with $\mathbf{Z}_4$ coefficients, we see that

$$\check{H}^n(X_G, x_G; \mathbf{Z}_4) \approx E_2^{0,n} = \check{H}^0(B_G; \check{H}^n(X, x; \mathbf{Z}_4)) \approx \check{H}^n(X, x; \mathbf{Z}_4)^G$$

is $\mathbf{Z}_2$ or $\mathbf{Z}_4$ according as $G$ acts nontrivially or trivially. As before, let $\alpha$ generate $\check{H}^n(X_G, x_G; \mathbf{Z}_2) \approx \mathbf{Z}_2$, Then the sequence

$$0 \to \check{H}^n(X_G, x_G; \mathbf{Z}_2) \to \check{H}^n(X_G, x_G; \mathbf{Z}_4) \to \check{H}^n(X_G, x_G; \mathbf{Z}_2) \xrightarrow{Sq^1} \check{H}^{n+1}(X_G, x_G; \mathbf{Z}_2)$$

shows that $Sq^1(\alpha) = 0$ iff $G$ acts trivially on $\check{H}^n(X, x; \mathbf{Z}_4)$. If $n > 2$, then $\check{H}^*(F, x; \mathbf{Z}_2) = 0$ in odd degrees by 3.2, so that $j^*(Sq^1\alpha) = Sq^1(j^*(\alpha)) = 0$ and hence $Sq^1(\alpha) = 0$ since $j^*$ is a monomorphism.

If $n = 2$ and $\check{H}^1(F, x; \mathbf{Z}_2) = 0$, then $Sq^1(\alpha) = 0$ for similar reasons. If $n = 2$ and $\check{H}^1(F, x; \mathbf{Z}_2) \neq 0$, then, as noted in the proof of 3.2, $j^*(\alpha) = t \otimes b + 1 \otimes b^2$ and thus

$$j^*(Sq^1(\alpha)) = Sq^1(j^*(\alpha)) = t^2 \otimes b + t \otimes b^2 \neq 0. \quad \blacksquare$$

We shall now turn to the case of odd primes. Our first result improves the bound on the number of components of $F$.

**3.4. Theorem** *Let $p$ be an odd prime and let $G = \mathbf{Z}_p$ act on the finitistic space $X \sim_p \mathbf{QP}^h$. Assume either that $h \geq p$ or that $X = \mathbf{QP}^h$ or, more generally, that $P_p^1(a) = \pm 2a^{(p+1)/2}$, where $a$ is some generator of $\check{H}^4(X; \mathbf{Z}_p)$. Then $F$ has at most $(p+1)/2$ components.*

*Proof* The result is trivially true when $(p+1)/2 > h$ and hence we may assume that $a^{(p+1)/2} \neq 0$. With $\alpha \in \check{H}^4(X_G, x_G; \mathbf{Z}_p)$ a generator, let

$$j^*(\alpha) = t^2 \otimes c + (\star),$$

where $c \in \check{H}^0(F, x)$ and $(\star)$ involves terms with positive second degree. By the proof of 3.1, $\check{H}^0(F, x)$ is spanned by $c, c^2, c^3, \ldots$ as a $\mathbf{Z}_p$-vector space.

Since $P_p^1(\alpha)$ restricts to $P_p^1(a) = \pm 2a^{(p+1)/2}$ we have

$$P_p^1(\alpha) = A_0\alpha^{(p+1)/2} + A_2 t^2 \alpha^{(p-1)/2} + \cdots + A_{p-1} t^{p-1} \alpha,$$

where $A_i \in \mathbf{Z}_p$ and $A_0 = \pm 2$. Thus

$$j^*(P_p^1(\alpha)) = t^{p+1} \otimes (A_0 c^{(p+1)/2} + A_2 c^{(p-1)/2} + \cdots + A_{p-1} c) + (\star)$$

while

$$P_p^1(j^*(\alpha)) = P_p^1(t^2 \otimes c + (\star)) = 2t^{p+1} \otimes c + (\star).$$

Thus

$$A_0 c^{(p+1)/2} + A_2 c^{(p-1)/2} + \cdots + A_{p-1} c = 2c$$

and this equation can be solved for $c^{(p+1)/2}$ since $A_0 = \pm 2$. Thus $c, c^2, \ldots, c^{(p-1)/2}$ span $\check{H}^0(F, x; \mathbf{Z}_p)$ and the result follows. ▮

In the following two theorems we restrict our attention to the case $p = 3$. There are analogs for all odd primes $p$, but they seem to require stronger hypotheses; see Bredon [10].

**3.5. Theorem** *Let $G = \mathbf{Z}_3$ act on the finitistic space $X \sim_3 \mathbf{QP}^h$. Then there is at most one component $F_0$ of $F$ with $F_0 \sim_3 \mathbf{QP}^k$, $k \geq 1$; all other components having* **CP**-*type (or acyclic).*

***Proof*** If $h < 3$, then this holds trivially by the equality $\operatorname{rk} \check{H}^*(X; \mathbf{Z}_3) = \operatorname{rk} \check{H}^*(F; \mathbf{Z}_3)$. If $h \geq 3$, then $a^3 \neq 0$ for $a \in \check{H}^4(X; \mathbf{Z}_3)$ a generator, and hence $P_3^1(a) = \pm 2a^2$. By a sign change we may assume that $P_3^1(a) = 2a^2$. Assume that $F_0$ and $F_1$ are two components of $F$, both of **QP**-type. Let $x \in F_0$ and let $\alpha \in \check{H}^4(X_G, x_G; \mathbf{Z}_3)$ represent $a$. Then

$$j^*(\alpha) = t^2 \otimes c + 1 \otimes b_0 + 1 \otimes b_1 + (\star),$$

where $c \in \check{H}^0(F, x)$, $b_0 \in \check{H}^4(F_0, x)$, $b_1 \in \check{H}^4(F_1)$, and $(\star)$ consists of terms not involving these groups. We have $cb_0 = 0$ and $cb_1 \neq 0$ (since $c$ generates $\check{H}^0(F, x)$ multiplicatively). Now

$$P_3^1(\alpha) = 2\alpha^2 + At^2\alpha$$

for some $A \in \mathbf{Z}_3$. Since $P_3^1(t^2) = 2t^4$ we compute

$$P_3^1(j^*(\alpha)) = 2t^4 \otimes c + (\star)$$

while

$$\begin{aligned} j^*(P_3^1(\alpha)) &= 2(j^*(\alpha))^2 + At^2 j^*(\alpha) \\ &= 2(t^4 \otimes c^2 + 2t^2 \otimes cb_1) + A(t^4 \otimes c + t^2 \otimes b_0 + t^2 \otimes b_1) + (\star). \end{aligned}$$

Comparing terms in $b_0$ shows that $A = 0$ and comparing terms in $b_1$ then gives a contradiction. ▮

**3.6. Theorem** *Let* $\mathbf{Z}_3$ *act on the finitistic space* $X \sim_3 \mathbf{QP}^h$ *with* $h \geq 3$ *or with* $h = 2$ *and* $P_3^1(a) = \pm 2a^2$ (e.g., $X = \mathbf{QP}^2$). *Suppose that* $F_0$ *is a component of* $F$ *with* $F_0 \sim_3 \mathbf{CP}^k$, $k \geq 2$. *Then* $\check{H}^4(X; \mathbf{Z}_3) \to \check{H}^4(F_0; \mathbf{Z}_3)$ *is an isomorphism.*

*Proof* Take $x \in F_0$ and let $\alpha$ be as before. As above

$$P_3^1(\alpha) = 2\alpha^2 + At^2\alpha$$

for some $A \in \mathbf{Z}_3$. Also we can write

$$j_0^*(\alpha) = t \otimes b + B \otimes b^2,$$

where $b$ generates $H^2(F_0, x; \mathbf{Z}_3)$ and $B \in \mathbf{Z}_3$. We must show that $B \neq 0$. We compute

$$P_3^1(j_0^*(\alpha)) = t^3 \otimes b + t \otimes b^3 + 2B \otimes b^4$$

which must equal

$$\begin{aligned} j_0^*(P_3^1(\alpha)) &= 2j_0^*(\alpha)^2 + At^2 j_0^*(\alpha) \\ &= At^3 \otimes b + (2 + AB)t^2 \otimes b^2 + 4Bt \otimes b^3 + 2B^2 \otimes b^4. \end{aligned}$$

Thus $A = 1$ and $2 + AB = 0$, whence $B = -2$. ∎

**3.7. Corollary** *Suppose that* $G$ *is a finite group with order a power of* 3, *and that* $G$ *acts on the finitistic space* $X \sim_3 \mathbf{QP}^h$ *with* $h \geq 3$ (*or* $h = 2$ *and* $P_3^1(a) = \pm 2a^2$). *Then there is at most one component of* $F = X^G$ *of* $\mathbf{QP}$*-type. Moreover, if* $F_0$ *is any component of* $F$ *with* $F_0 \sim_3 \mathbf{CP}^k$, $k \geq 2$, *then the restriction* $\check{H}^4(X; \mathbf{Z}_3) \to \check{H}^4(F_0; \mathbf{Z}_3)$ *is an isomorphism.*

*Proof* We can write

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_r = \{e\}$$

with $G_i/G_{i+1} \approx \mathbf{Z}_3$. The first statement of the theorem is an easy induction. For the second statement let $F_i$ denote the component of $F(G_i, X)$ which contains $F_0$, so that

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = X.$$

Then $F_i$ is a component of the action of $\mathbf{Z}_3 \approx G_i/G_{i+1}$ on $F_{i+1}$. Since $\operatorname{rk} \check{H}^*(F_0) \geq 3$, the same follows for each $F_i$ and hence none of these is a cohomology $\mathbf{S}^2$ or $\mathbf{S}^4$. If $F_i$ and $F_{i+1}$ are both of **CP**-type or both of **QP**-type, then

$$\check{H}^4(F_{i+1}; \mathbf{Z}_3) \to \check{H}^4(F_i; \mathbf{Z}_3)$$

is an isomorphism by 3.1. By naturality of $P_3^1$ it also follows inductively that $P_3^1(a_i) = \pm 2a_i^2$ for a generator $a_i \in \check{H}^4(F_i; \mathbf{Z}_3)$ when $F_i$ has **QP**-type. Thus 3.6 is applicable for the passage from $F_i$ of **QP**-type to $F_{i-1} \subset F(\mathbf{Z}_3, F_i)$ of **CP**-type, and the result follows. ▮

*Remark* It is natural to ask whether the analog of 3.7 holds for actions of $p$-groups, $p \neq 2, 3$, on $\mathbf{QP}^h$. The answer is unknown.

## 4. SOME EXAMPLES

In this section we shall give several examples to illustrate the theorems of the last section.

On $\mathbf{RP}^h$, $\mathbf{CP}^h$, and $\mathbf{QP}^h$ we denote the homogeneous coordinates of a point by $(x_0 : x_1 : \cdots : x_h)$. Also, $\mathbf{QP}^h$ is taken to be the *right* projective space, so that

$$(q_0 \colon q_1 \colon \cdots \colon q_h) = (q_0 q \colon q_1 q \colon \cdots \colon q_h q)$$

for $q \neq 0$.

The "linear" actions of the circle group $\{z \in \mathbf{C} \mid |z| = 1\}$, and hence of each $\mathbf{Z}_p$, on $\mathbf{CP}^h$ and $\mathbf{QP}^h$ are given by

$$z(q_0 \colon q_1 \colon \cdots \colon q_h) = (z^{a_0} q_0 \colon z^{a_1} q_1 \colon \cdots \colon z^{a_h} q_h),$$

where $a_0, a_1, \ldots, a_h$ are integers. (This action may have a kernel, which should be factored out.) There are analogous actions of $\mathbf{Z}_2$ on $\mathbf{RP}^h$. In particular $(q_0 \colon q_1 \colon \cdots \colon q_h) \mapsto (iq_0 \colon iq_1 \colon \cdots \colon iq_h)$ defines an involution on $\mathbf{QP}^h$ with $F = \mathbf{CP}^h$. There is also the involution $(z_0 \colon z_1 \colon \cdots \colon z_h) \mapsto (\bar{z}_0 \colon \bar{z}_1 \colon \cdots \colon \bar{z}_h)$ on $\mathbf{CP}^h$ with $F = \mathbf{RP}^h$.

It is a pedestrian exercise to analyze the above actions and to see that, for $X = \mathbf{RP}^h$, $\mathbf{CP}^h$, or $\mathbf{QP}^h$, they give examples of most (all for $p = 2, 3$) of the possibilities for $F$ consistent with 3.1, 3.2, 3.4, and 3.5.

We shall now consider some actions on the Cayley projective plane $\mathbf{Cay\,P}^2$. Let us recall its definition. The Cayley numbers can be defined to be pairs $(a, b)$ of quaternions with coordinatewise addition and where multiplication is defined by

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}).$$

If $x = (a, b)$ is a Cayley number, then we put $\bar{x} = (\bar{a}, -b)$ and $|x| = (|a|^2 + |b|^2)^{1/2}$, so that $x\bar{x} = |x|^2 = \bar{x}x$. Also $|xy| = |x||y|$. The Cayley numbers are not associative but any two elements lie in an associative subalgebra. Moreover, there is the "Moufang identity"

$$(zx)(yz) = z(xy)z.$$

For background on all of this we recommend Curtis [1].

For any map $A \times B \to C$ there is an obvious induced map of the join $A \star B$ to the suspension $SC$ [which factors through $S(A \times B)$]. For group actions on $A$, $B$, and $C$ with $A \times B \to C$ equivariant, the induced map $A \star B \to SC$ is also equivariant. The multiplication of Cayley numbers of norm 1 is a map $\mathbf{S}^7 \times \mathbf{S}^7 \to \mathbf{S}^7$ and hence induces the "Hopf map"

$$\varphi\colon \mathbf{S}^{15} = \mathbf{S}^7 \star \mathbf{S}^7 \to S\mathbf{S}^7 = \mathbf{S}^8.$$

Then $\mathbf{Cay\,P}^2$ is defined to be the mapping cone of $\varphi$. We prefer to use this description to produce examples of actions on $\mathbf{Cay\,P}^2$. An alternative approach would be to exploit the fact that $\mathbf{Cay\,P}^2$ is the homogeneous space $\mathbf{F}_4/\mathbf{Spin}(9)$.

We shall now describe some actions on $\mathbf{Cay\,P}^2$. For a quaternion $\tau$ of norm 1 and a Cayley number $c = (a, b)$ let

$$\tau(c) = (\tau a \tau^{-1}, \tau b \tau^{-1}).$$

It is clear that this is an automorphism of the Cayley algebra and defines an action of $\mathbf{S}^3$ on $\mathbf{S}^7$ [but $-1 \in \mathbf{S}^3$ acts trivially so that, effectively, this is an action of $\mathbf{SO}(3)$]. Multiplication $\mathbf{S}^7 \times \mathbf{S}^7 \to \mathbf{S}^7$ is equivariant for this action and hence there is an induced action on $\mathbf{S}^{15} \to \mathbf{S}^8$ and on its mapping cone $\mathbf{Cay\,P}^2$. If $\tau$ is of order $p$ (or 4 for $p = 2$), then this generates an action of $\mathbf{Z}_p$ on $\mathbf{Cay\,P}^2$ with $F = \mathbf{QP}^2$.

For another example, let $\omega$ be a Cayley number of order $p$ and let $\omega$ act on $\mathbf{S}^7$ by $c \mapsto \omega c \omega$. Also let $\omega$ act on $\mathbf{S}^7 \times \mathbf{S}^7$ by $(c_1, c_2) \mapsto (\omega c_1, c_2\omega)$. Then multiplication $\mathbf{S}^7 \times \mathbf{S}^7 \to \mathbf{S}^7$ is equivariant by the Moufang identity. Thus there is an induced action of $\mathbf{Z}_p$ on $\mathbf{Cay\,P}^2$. To identify $F$ note that if $c \in \mathbf{S}^7$ is perpendicular to 1 and $\omega$, then $\bar{c} = -c$ and

$$2 = |c + \omega|^2 = (\bar{c} + \bar{\omega})(c + \omega) = 2 + \bar{c}\omega + \bar{\omega}c$$

so that $c\omega = \bar{\omega}c$. That is, $\omega c \omega = c$ for $c$ in this 5-sphere. If $c \in \mathbf{S}^7$ is in the plane spanned by 1 and $\omega$, then $\omega c \omega = \omega^2 c$ which equals $c$ iff $p = 2$. Since there are no fixed points on $\mathbf{S}^7 \times \mathbf{S}^7$ we see that $F$ is the vertex of the mapping cone of $\mathbf{S}^7 \star \mathbf{S}^7 \to S\mathbf{S}^7$ together with the suspension of the fixed set of $c \mapsto \omega c \omega$ on $\mathbf{S}^7$. Thus

$$F = \begin{cases} \text{point} + \mathbf{S}^8 & \text{if } p = 2, \\ \text{point} + \mathbf{S}^6 & \text{if } p \text{ is odd.} \end{cases}$$

One can see that this action is smooth in some smooth structure on **Cay** $\mathbf{P}^2$. Note that for $p = 3$ it illustrates VI.10.4 and shows that the condition in the last part of VI.11.5 cannot be dropped. Also note that $\mathbf{Z}_3$ cannot act smoothly on **Cay** $\mathbf{P}^2$ with $F = \text{point} + \mathbf{S}^r$ for any $r \neq 6$, because 3 must divide $r$ by VI.10.4 and $r$ must be even and at most 8 by 3.1. (There is also a cohomological proof of this fact; see Exercise 10.) The case $r = 0$ can be ruled out by applying a theorem of Conner and Floyd [8, p. 94]; compare the proof of 9.3.

We now turn to another class of examples in the case of odd primes $p$. Recall from Chapter I, Section 7 the map

$$\theta\colon \mathbf{S}^{n-1} \to \mathbf{O}(n)$$

taking $x$ to $\theta_x$, the reflection through the line $\mathbf{R}x$. Also recall that the map

$$\psi\colon \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}, \qquad \psi(x, y) = \theta_x(y)$$

has bidegree $(2, -1)$ for $n$ even and bidegree $(0, 1)$ for $n$ odd. Thus the induced map

$$\varphi\colon \mathbf{S}^{2n-1} = \mathbf{S}^{n-1} \star \mathbf{S}^{n-1} \to S\mathbf{S}^{n-1} = \mathbf{S}^n$$

has Hopf invariant $-2$ or 0 according as $n$ is even or odd. Let $X_n$ denote the mapping cone of $\varphi$. Then for *p odd* we have

$$X_n \sim_p \begin{cases} \mathbf{P}^2(n) & \text{for } n \text{ even}, \\ \mathbf{S}^n \vee \mathbf{S}^{2n} & \text{for } n \text{ odd}. \end{cases}$$

The space $X_0$ consists of three points. Now $\psi$ is equivariant with respect to the usual $\mathbf{O}(n)$-action on $\mathbf{S}^{n-1}$ and the diagonal action on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. Thus $X_n$ inherits an $\mathbf{O}(n)$-action. If $G \subset \mathbf{O}(n)$ and $F(G, \mathbf{R}^n) = \mathbf{R}^k$, then it is clear that $F(G, X_n) = X_k$. Thus for $k, n$ even, $0 \leq k \leq n$, and $p$ odd, there is a $\mathbf{Z}_p$-action on $X_n \sim_p \mathbf{P}^2(n)$ with $F = X_k \sim_p \mathbf{P}^2(k)$. These examples give limitations on what one can expect to prove by purely cohomological methods. For example $\mathbf{Z}_3$ acts on $X_4 \sim_3 \mathbf{QP}^2$ with $F = X_0$ consisting of three points. This shows that the condition on $P_p{}^1$ is needed for 3.4 and it also shows that $P_3{}^1 = 0$ on $X_4$. Similarly, the $\mathbf{Z}_3$-action on $X_4$ with $F = X_2$

shows that the $P_3{}^1$ condition in 3.6 is necessary, since $H^4(X_4; \mathbf{Z}_3) \to H^4(X_2; \mathbf{Z}_3)$ is trivial because it factors through $H^4(X_3; \mathbf{Z}_3) = 0$. Also see Exercise 2.

We shall conclude this section by showing that there can be some rather interesting actions even in the case $X \sim_p F$; i.e., when the restriction $\check{H}^*(X; \mathbf{Z}_p) \to \check{H}^*(F; \mathbf{Z}_p)$ is an isomorphism.

For the first such example consider the action of $\mathbf{S}^1$ on $\mathbf{C}^2 = \mathbf{R}^4$ by

$$z(z_1, z_2) = (z^2 z_1, z^3 z_2).$$

By adding the point at infinity, this gives a linear action on $\mathbf{S}^4$. Let $A \subset \mathbf{S}^4$ be the orbit of the point (1, 1), so that

$$A = \{(z^2, z^3) \mid |z| = 1\},$$

and note that this touches neither of the 2-spheres

$$F_2 = \{(z_1, 0)\}, \qquad F_3 = \{(0, z_2)\}$$

which are the fixed point sets of the subgroups $\mathbf{Z}_2$ and $\mathbf{Z}_3$, respectively. Now $A$ has an invariant tubular neighborhood $U \approx A \times \mathbf{R}^3 \approx \mathbf{S}^1 \times \mathbf{R}^3$ in $\mathbf{S}^4$ and

$$\mathbf{S}^4 - U \approx \mathbf{S}^2 \times \mathbf{D}^2.$$

Both of the 2-spheres $F_2$ and $F_3$ are contained in this $\mathbf{S}^2 \times \mathbf{D}^2$. We wish to find the *degree* (up to sign) of the inclusion $\mathbf{S}^2 \approx F_2 \subset \mathbf{S}^2 \times \mathbf{D}^2$. However, this is clearly just the linking number of $F_2 \approx \mathbf{S}^2$ and $A \approx \mathbf{S}^1$ in $\mathbf{S}^4$. Now $F_2$ bounds the 3-disk

$$\mathbf{D}^3 = \{(z_1, z_2) \mid z_2 \geq 0 \text{ (real)}\}.$$

This intersects $A$ in the three points $\{(\omega^2, 1) \mid \omega^3 = 1\}$ and the intersections all have the same sign. Thus the linking number is 3, which means that the inclusion $F_2 \subset \mathbf{S}^2 \times \mathbf{D}^2$ has degree 3. Similarly $F_3 \subset \mathbf{S}^2 \times \mathbf{D}^2$ has degree 2. Thus the subgroup $\mathbf{Z}_2$ (respectively, $\mathbf{Z}_3$) acts on $\mathbf{S}^2 \times \mathbf{D}^2$ with $F \approx \mathbf{S}^2$ contained in $\mathbf{S}^2 \times \mathbf{D}^2$ with degree 3 (respectively, 2). Similar constructions yield actions of $\mathbf{Z}_p$ on $\mathbf{S}^2 \times \mathbf{D}^2$ with $F \approx \mathbf{S}^2$ lying in $\mathbf{S}^2 \times \mathbf{D}^2$ with degree $q$, for all relatively prime $p$ and $q$.

These are, of course, nontrivial examples of the Smith Theorems and also of 3.1 and 3.2(2) with $h = 1$. We shall now construct generalizations of this for $h > 1$. Consider the Brieskorn manifold $W_{p,q}^{4n+1}$ defined by the equations

$$u^p + v^q + z_1{}^2 + \cdots + z_{2n}^2 = 0,$$

$$|u|^2 + |v|^2 + |z_1|^2 + \cdots + |z_{2n}|^2 = 1$$

in $\mathbf{C}^{2n+2}$, where $p$ and $q$ are distinct primes. Recall that this is always a homotopy sphere. Let $\mathbf{S}^1 \subset \mathbf{SO}(2) \times \cdots \times \mathbf{SO}(2) \subset \mathbf{SO}(2n)$ be the diagonal circle subgroup and note that this acts semi-freely on $W_{p,q}^{4n+1}$ with fixed set $W_{p,q}^1 \approx \mathbf{S}^1$ (given by $z_i = 0$). Multiplication of $u$ by $p$th roots of unity defines a $\mathbf{Z}_p$-action on this, commuting with the $\mathbf{S}^1$-action, and its fixed point set is $W_q^{4n-1}$; the set where $u = 0$. Note that

$$W_q^{4n-1} \cap W_{p,q}^1 = \varnothing.$$

An invariant tubular neighborhood $U$ of the circle $W_{p,q}^1$ has the form $\mathbf{S}^1 \times \mathbf{R}^{4n}$ and its complement $W_{p,q}^{4n+1} - U \approx \mathbf{S}^{4n-1} \times \mathbf{D}^2$ contains the fixed set $W_q^{4n-1}$ of $\mathbf{Z}_p$. We shall now divide by the action of $\mathbf{S}^1$. There is an equivariant inclusion $\mathbf{S}^{4n-1} \to \mathbf{S}^{4n-1} \times \mathbf{D}^2$ which is a homotopy equivalence. It follows that

$$X = (\mathbf{S}^{4n-1} \times \mathbf{D}^2)/\mathbf{S}^1$$

has the homotopy type of $\mathbf{CP}^{2n-1}$. (In fact, it can be shown that $X \approx \mathbf{CP}^{2n-1} \times \mathbf{D}^2$ by using the $s$-Cobordism Theorem in an appropriate way; at least for $n \neq 2$.) There is an induced $\mathbf{Z}_p$-action on $X$ with

$$F = F(\mathbf{Z}_p, X) = W_q^{4n-1}/\mathbf{S}^1.$$

Recall that $H^{2n}(W_q^{4n-1}; \mathbf{Z}) \approx \mathbf{Z}_q$ and that otherwise $W_q^{4n-1}$ resembles a $(4n-1)$-sphere. The spectral sequence (or the Gysin sequence) of the $\mathbf{S}^1$-bundle $W_q^{4n-1} \to F$ shows easily that

$$H^*(F; \mathbf{Z}) \approx \mathbf{Z}[x, y]/(x^n - qy, y^2)$$

where $\deg x = 2$ and $\deg y = 2n$. This $\mathbf{S}^1$-bundle is contained in that of $\mathbf{S}^{4n-1} \times \mathbf{D}^2 \to X$ and comparison of the spectral sequences shows that the inclusion $F \subset X$ has degree $\pm q$; that is,

$$\mathbf{Z} \approx H^{4n-2}(X; \mathbf{Z}) \to H^{4n-2}(F; \mathbf{Z}) \approx \mathbf{Z}$$

is multiplication by $\pm q$. (This also implies that the inclusion $W_q^{4n-1} \subset \mathbf{S}^{4n-1} \times \mathbf{D}^2$ has degree $\pm q$.) Since $F \sim_p \mathbf{CP}^{2n-1} \sim_p X$ this illustrates 3.1 and 3.2(2) in the "apparently trivial" case. We hope that these examples will prevent the reader from reading more into the theorems of Section 3 than is there.

## 5. CIRCLE ACTIONS ON PROJECTIVE SPACES

For $S^1$-actions on *rational* cohomology projective spaces it is clear that one can prove the obvious analog of 3.1. In this section we shall study $S^1$-actions on *integral* cohomology projective spaces. The symbol $X \sim_{\mathbf{Z}} Y$ means that $X$ and $Y$ have isomorphic graded integral cohomology rings and $X \sim_{\mathbf{Z}} \mathbf{P}^h(n)$ means that $\check{H}^*(X; \mathbf{Z}) \approx \mathbf{Z}[a]/(a^{h+1})$ where $\deg a = n$. Recall that for $h \geq 2$, $n$ must be 2, 4, or 8, and for $h \geq 3$, $n \neq 8$; see Adams [1] and Steenrod and Epstein [1].

**5.1. Theorem** *Let $X \sim_{\mathbf{Z}} \mathbf{P}^h(n)$, $n$ even, and assume that $\mathbf{S}^1$ acts on $X$ with finitely many orbit types and with $X$ and $X/\mathbf{S}^1$ finitistic. Then $F = X^{\mathbf{S}^1}$ is nonempty and consists of components $F_i \sim_{\mathbf{Z}} \mathbf{P}^{h_i}(n_i)$ with $n_i \leq n$ even and $h + 1 = \sum(h_i + 1)$. The restriction $\check{H}^n(X; \mathbf{Z}) \to \check{H}^n(F_i; \mathbf{Z})$ is an isomorphism if $n_i = n$ and $h_i \geq 1$, for some $i$.*

*Proof* We remark that in this generality the Universal Coefficient Theorem is not known to be valid and this is the reason that we shall not utilize rational coefficients anywhere in the proof. The universal coefficient formula for $\mathbf{Z}_p$ coefficients holds in general since it can be derived from the exact sequence associated with the coefficient sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0.$$

Note that by III.10.13, $\check{H}^*(F; \mathbf{Z})$ has finite type and, by III.10.7, $\check{H}^i(F; \mathbf{Z}) = 0$ for $i > nh$. Let $q$ be an odd prime so large that $\mathbf{Z}_q \subset \mathbf{S}^1$ is contained in no isotropy group other than $\mathbf{S}^1$ itself and also such that $\check{H}^*(F; \mathbf{Z})$ has no $q$-torsion. Then

$$F(\mathbf{Z}_q, X) = F = F(\mathbf{S}^1, X).$$

Now $\mathbf{Z}_q$ acts trivially on $\check{H}^*(X; \mathbf{Z})$ and the integral spectral sequence of $X_{\mathbf{Z}_q} \to B_{\mathbf{Z}_q}$ degenerates since all nonzero elements of $E_2$ have both degrees even. This implies that $F = F(\mathbf{Z}_q, X)$ is nonempty since $\check{H}^*(F_{\mathbf{Z}_q}; \mathbf{Z}) \approx \check{H}^*(X_{\mathbf{Z}_q}; \mathbf{Z}) \neq 0$ in high even degrees.

By 3.1 we conclude that $\check{H}^*(F; \mathbf{Z}_q)$ vanishes in odd degrees and $\operatorname{rk} \check{H}^*(F; \mathbf{Z}_q) = h + 1$. Since $F$ has no $q$-torsion, we have that

$$\operatorname{rk} \check{H}^*(F; \mathbf{Z}) = h + 1.$$

If, for any prime $p$, $F$ had $p$-torsion, then we would have

$$\operatorname{rk} \check{H}^*(F; \mathbf{Z}_p) > \operatorname{rk} \check{H}^*(F; \mathbf{Z}) = h + 1 = \operatorname{rk} \check{H}^*(X; \mathbf{Z}_p).$$

Since $F = F(\mathbf{Z}_{p^r}, X)$ for some $r$, this would contradict III.7.9 applied inductively. Thus $\check{H}^*(F; \mathbf{Z})$ is free abelian of rank $h + 1$.

Let $F_0$ be a component of $F$. An inductive application of 3.1 implies that $F_0 \sim_p \mathbf{P}^k(m)$ for any given prime $p$. Thus

$$\check{H}^i(F_0; \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{for } i = 0, m, 2m, \ldots, km, \\ 0 & \text{otherwise,} \end{cases}$$

and $m \leq n$ is even. If $b \in \check{H}^m(F_0; \mathbf{Z}) \approx \mathbf{Z}$ is a generator, we must show that $b^k \in \check{H}^{km}(F_0; \mathbf{Z})$ is also a generator. However, if $b^k$ were divisible by some prime $p$, then $F_0$ would not have the correct ring structure over $\mathbf{Z}_p$. Thus $F_0 \sim_{\mathbf{Z}} \mathbf{P}^k(m)$. It remains to show that if $m = n$ and $k \geq 1$, then

$$\mathbf{Z} \approx \check{H}^n(X; \mathbf{Z}) \to \check{H}^n(F_0; \mathbf{Z}) \approx \mathbf{Z}$$

in an isomorphism. However, this is true for $\mathbf{Z}_p$-coefficients for all primes $p$ by an inductive application of 3.1, and it clearly follows for integral coefficients. ∎

Note that, in particular, if $X \sim_{\mathbf{Z}} \mathbf{CP}^h$, then $F = \bigcup F_i$ with $F_i \sim_{\mathbf{Z}} \mathbf{CP}^{h_i}$, $h + 1 = \sum(h_i + 1)$, and such that $\check{H}^2(X; \mathbf{Z}) \to \check{H}^2(F_i; \mathbf{Z})$ is an isomorphism when $h_i \geq 1$.

**5.2. Theorem** *Suppose that $X \sim_{\mathbf{Z}} \mathbf{QP}^h$, $h \geq 2$, and that $\mathbf{S}^1$ acts on $X$ with finitely many orbit types and with $X$ and $X/\mathbf{S}^1$ finitistic. Then at most one component $F_0$ of $F$ has $F_0 \sim_{\mathbf{Z}} \mathbf{QP}^k$, $k \geq 1$, the remainder being of* **CP**-*type (or acyclic). If $F_0 \sim_{\mathbf{Z}} \mathbf{QP}^k$, $k \geq 1$, then $\check{H}^4(X; \mathbf{Z}) \to \check{H}^4(F_0; \mathbf{Z})$ is an isomorphism. If $F_1 \sim_{\mathbf{Z}} \mathbf{CP}^k$, $k \geq 2$, then the cokernel of the restriction $\mathbf{Z} \approx \check{H}^4(X; \mathbf{Z}) \to \check{H}^4(F_1; \mathbf{Z}) \approx \mathbf{Z}$ is finite of order prime to* 6.

*Proof* Since $F(\mathbf{Z}_{3^r}, X) = F$ for some $r$, the fact that there is at most one component of $F$ of **QP**-type follows from 3.7. The other statement about such components follows from 5.1. If $F_1 \sim_{\mathbf{Z}} \mathbf{CP}^k$, $k \geq 2$, then $\check{H}^4(X; \mathbf{Z}_3) \to \check{H}^4(F_1; \mathbf{Z}_3)$ is an isomorphism by 3.7 and this also follows for $\mathbf{Z}_2$ coefficients by an inductive argument on 3.1 and 3.2. Thus $\mathbf{Z} \approx \check{H}^4(X; \mathbf{Z}) \to \check{H}^4(F_1; \mathbf{Z}) \approx \mathbf{Z}$ is multiplication by some integer prime to 2 and to 3. ∎

*Remark* One would expect that $\check{H}^4(X; \mathbf{Z}) \to \check{H}^4(F_1; \mathbf{Z})$ is an isomorphism, but this is unknown. Theorem 5.1 and the first part of 5.2 are from

Bredon [10]. The case $X \sim_{\mathbf{Z}} \mathbf{CP}^h$ was proved earlier by Su [3] using a geometrical argument to reduce it to the case of actions on cohomology spheres.

Returning to the general situation of 5.1, let $a \in \check{H}^n(X; \mathbf{Z})$ be a given generator. From the spectral sequence of $X_{\mathbf{S}^1} \to B_{\mathbf{S}^1}$ over $\mathbf{Z}$ we derive the exact sequence

$$0 \to \check{H}^n(B_{\mathbf{S}^1}; \mathbf{Z}) \to \check{H}^n(X_{\mathbf{S}^1}; \mathbf{Z}) \to \check{H}^n(X; \mathbf{Z}) \to 0$$

since everything vanishes in odd degrees. Thus there is an element $\alpha \in \check{H}^n(X_{\mathbf{S}^1}; \mathbf{Z})$ restricting to $a \in \check{H}^n(X; \mathbf{Z})$, and $\alpha$ is uniquely determined up to addition of an integral multiple of $t^{n/2} = t^{n/2} \cdot 1 \in \check{H}^n(X_{\mathbf{S}^1}; \mathbf{Z})$. Taking $1, a, \ldots, a^h$ to $1, \alpha, \ldots, \alpha^h$ defines a cohomology extension of the fiber and, by the Leray–Hirsch Theorem 1.4, we conclude that $\check{H}^*(X_{\mathbf{S}^1}; \mathbf{Z})$ is the free $\check{H}^*(B_{\mathbf{S}^1}; \mathbf{Z}) \approx \mathbf{Z}[t]$ module with basis $1, \alpha, \alpha^2, \ldots, \alpha^h$.

Applying the homomorphism $j^*\colon \check{H}^*(X_{\mathbf{S}^1}; \mathbf{Z}) \to \check{H}^*(F_{\mathbf{S}^1}; \mathbf{Z})$ to $\alpha$ we may write

$$j^*(\alpha) = t^{n/2} \otimes c + (\star),$$

where $c \in \check{H}^0(F; \mathbf{Z})$ and $(\star)$ consists of terms of positive second degree. For a component $F_i$ of $F$ let $c_i \in \check{H}^0(F_i; \mathbf{Z})$ be the restriction of $c$, and note that $c_i$ can be regarded as an integer under the canonical isomorphism $\check{H}^0(F_i; \mathbf{Z}) \approx \mathbf{Z}$.

**5.3. Proposition** *Let $p$ be a prime. With the above notation, two components $F_i$ and $F_j$ of $F(\mathbf{S}^1, X)$ are contained in the same component of $F(\mathbf{Z}_p, X)$ iff* $c_i \equiv c_j \pmod p$.

***Proof*** The restriction of $\alpha$ to $(F_i)_{\mathbf{S}^1}$ is $t^{n/2} \otimes c_i = c_i t^{n/2} \otimes 1$. If $x \in F_i$, then it follows that the restriction of $\alpha$ to $x_{\mathbf{S}^1} = B_{\mathbf{S}^1}$ is $c_i t^{n/2}$. Since $E_{\mathbf{Z}_p}$ can be taken to coincide with $E_{\mathbf{S}^1}$ there is the commutative diagram of spaces

$$\begin{array}{ccc} B_{\mathbf{Z}_p} \approx x_{\mathbf{Z}_p} & \longrightarrow & x_{\mathbf{S}^1} \approx B_{\mathbf{S}^1} \\ \downarrow & & \downarrow \\ X_{\mathbf{Z}_p} & \longrightarrow & X_{\mathbf{S}^1} \end{array}$$

which, together with reduction mod $p$, induces

$$\begin{array}{ccc} \check{H}^*(X_{\mathbf{S}^1}; \mathbf{Z}) & \longrightarrow & \check{H}^*(X_{\mathbf{Z}_p}; \mathbf{Z}_p) \\ \downarrow & & \downarrow \\ \check{H}^*(B_{\mathbf{S}^1}; \mathbf{Z}) \approx \check{H}^*(x_{\mathbf{S}^1}; \mathbf{Z}) & \longrightarrow & \check{H}^*(x_{\mathbf{Z}_p}; \mathbf{Z}_p) \approx \check{H}^*(B_{\mathbf{Z}_p}; \mathbf{Z}_p). \end{array}$$

The bottom map takes $t$ to the corresponding generator of $\check{H}^2(B_{\mathbf{Z}_p}; \mathbf{Z}_p) \approx \mathbf{Z}_p$, which we have also denoted by $t$ for $p$ odd and by $t^2$ for $p = 2$. Let $\alpha' \in \check{H}^n(X_{\mathbf{Z}_p}; \mathbf{Z}_p)$ be the image of $\alpha$ and note that $\alpha'$ represents the generator $a' \in \check{H}^n(X; \mathbf{Z}_p)$ which is the mod $p$ reduction of $a$. The diagram shows that $\alpha'$ goes to $c_i t^{n/2}$ (mod $p$) in $\check{H}^*(B_{\mathbf{Z}_p}; \mathbf{Z}_p)$ (or to $c_i t^n$ when $p = 2$). However, the image of $\alpha'$ under

$$j^* : \check{H}^*(X_{\mathbf{Z}_p}; \mathbf{Z}_p) \to \check{H}^*(F(\mathbf{Z}_p, X)_{\mathbf{Z}_p}; \mathbf{Z}_p) \approx \check{H}^*(B_{\mathbf{Z}_p}; \mathbf{Z}_p) \otimes \check{H}^*(F(\mathbf{Z}_p, X); \mathbf{Z}_p)$$

is $t^{n/2} \otimes d + (\star)$, where $d \in \check{H}^0(F(\mathbf{Z}_p, X); \mathbf{Z}_p)$, and it follows that the component part of $d$ corresponding to the component of $F(\mathbf{Z}_p, X)$ containing $x$ is just $c_i$ (mod $p$). This shows that $c_i \equiv c_j$ (mod $p$) if $F_i$ and $F_j$ are contained in the same component of $F(\mathbf{Z}_p, X)$, and the converse follows from the fact that the component parts of $d$ are all distinct since $d$ generates the ring $\check{H}^0(F(\mathbf{Z}_p, X); \mathbf{Z}_p)$ multiplicatively. ∎

The following fact is due to W.-Y. Hsiang. It has obvious analogs for $\mathbf{Z}_p$-actions and toral actions, etc.

**5.4. Proposition** *With the above notation,*

$$j^* : \check{H}^*(X_{S^1}; \mathbf{Z}) \to \check{H}^*(F_{S^1}; \mathbf{Z})$$

*is a monomorphism and there is a ring isomorphism*

$$\check{H}^*(X_{S^1}; \mathbf{Z}) \approx \mathbf{Z}[\alpha, t]/\prod(\alpha - c_i t)^{h_i+1}.$$

***Proof*** For rational coefficients $j^*$ is a monomorphism by 1.5 and this would imply the same fact over $\mathbf{Z}$ if the Universal Coefficient Theorem were valid. For the general case, one needs a slightly more sophisticated argument. For a given element of $\check{H}^*(X_{S^1}; \mathbf{Z})$ one can pick a prime $p$ so large that the image of this element is nonzero in $\check{H}^*(X_{\mathbf{Z}_p}; \mathbf{Z}_p)$. We can also take $\mathbf{Z}_p$ such that $F(\mathbf{Z}_p, X) = F(S^1, X)$ so that the result follows from 1.5, for this $\mathbf{Z}_p$-action, and an obvious commutative diagram.

For the second part of the proposition, let $j_i$ be the inclusion $(F_i)_{S^1} \subset X_{S^1}$. Then we have

$$j_i^*(\alpha - c_i t) = \sum_{k \geq 1} A_k \otimes b_i^k,$$

where $A_k \in \check{H}^*(B_{S^1}; \mathbf{Z})$ and $b_i \in \check{H}^{n_i}(F_i; \mathbf{Z})$ is a generator. Thus

$$j_i^*\left((\alpha - c_i t)^{h_i+1}\right) = \left(j_i^*(\alpha - c_i t)\right)^{h_i+1} = 0$$

so that $\prod(\alpha - c_i t)^{h_i+1}$ restricts trivially to all components of $F_{S^1}$; that is, it is killed by $j^*$. Thus

$$\prod(\alpha - c_i t)^{h_i+1} = 0$$

since $j^*$ is a monomorphism. This is a monic polynomial in $\alpha$ of degree $h + 1 = \sum(h_i + 1)$ and hence it expresses $\alpha^{h+1}$ in terms of $1, \alpha, \ldots, \alpha^h$ and $t$ and completely determines the ring structure of $\check{H}^*(X_{S^1}; \mathbf{Z})$. ▮

Let us denote by $\eta^k$ the representation $z \mapsto z^k$ of $\mathbf{S}^1$, considered as a real 2-dimensional representation. Thus $\eta^k = t^k + t^{-k}$ in the notation of Chapter VI, Section 10.

For the remainder of this section we shall assume that $X \sim_{\mathbf{Z}} \mathbf{CP}^h$ and that $X$ is a closed manifold with a locally smooth $\mathbf{S}^1$-action. Let $F_0$ be a component of $F$ and let $x \in F_0$. There is a representation $\nu$ in the normal plane at $x$ to $F_0$ and we can write

$$\nu = \sum \eta^{m_i},$$

where the $m_i$ are nonzero integers. For a smooth action, W.-Y. Hsiang has claimed to prove that the set of integers $\{m_i\}$ coincides, up to signs, with the set of integers $\{c_i - c_0\}$ with *multiplicities* $h_i + 1$, where the $c_i$ are as above. Subsequently, T. Petrie [3] has found a counterexample to this. However, we shall prove some partial results in this direction for locally smooth actions. Stronger results for the smooth case can be found in Petrie [3].

**5.5. Theorem** *For a locally smooth* $\mathbf{S}^1$*-action on a closed manifold* $X \sim_{\mathbf{Z}} \mathbf{CP}^h$ *and with the above notation, we have*

$$\prod m_i = \pm \prod_{i \neq 0} (c_0 - c_i)^{h_i+1},$$

*where* $\sum \eta^{m_i}$ *is the normal representation about* $F_0$.

*Proof* We write $G = \mathbf{S}^1$. Consider the element

$$\beta = (\alpha - c_0 t)^{h_0} \prod_{i \neq 0} (\alpha - c_i t)^{h_i+1}$$

in $\check{H}^{2h}(X_G; \mathbf{Z})$. Let $\mathbf{D}^{2h}$ be an invariant $2h$-disk in $X$ about a point $x$ of $F_0$ and let $U$ be its interior. Then $X - U \sim_{\mathbf{Z}} \mathbf{CP}^{h-1}$, $F_0 - U \sim_{\mathbf{Z}} \mathbf{CP}^{h_0-1}$, and $F_i = F_i - U \sim_{\mathbf{Z}} \mathbf{CP}^{h_i}$. Thus, as in the proof of 5.4, $\beta$ goes to zero in $\check{H}^*((F - U)_G; \mathbf{Z})$ and hence must go to zero in $\check{H}^*((X - U)_G; \mathbf{Z})$. Thus

$\beta$ pulls back to an element $\tau \in \check{H}^{2h}(\mathbf{D}_G^{2h}, \mathbf{S}_G^{2h-1}; \mathbf{Z})$ under

$$\check{H}^{2h}(\mathbf{D}_G^{2h}, \mathbf{S}_G^{2h-1}) \approx \check{H}^{2h}(X_G, (X - U)_G) \to \check{H}^{2h}(X_G).$$

Inclusion of a fiber gives the diagram

$$\begin{array}{ccc} \check{H}^{2h}(\mathbf{D}_G^{2h}, \mathbf{S}_G^{2h-1}) & \longrightarrow & \check{H}^{2h}(X_G) \\ \downarrow \approx & & \downarrow i^* \\ \check{H}^{2h}(\mathbf{D}^{2h}, \mathbf{S}^{2h-1}) & \xrightarrow{\approx} & \check{H}^{2h}(X) \end{array}$$

where the vertical isomorphism follows from the triviality of the spectral sequence of $(\mathbf{D}_G^{2h}, \mathbf{S}_G^{2h-1}) \to B_G$. Now $i^*(\beta) = a^h$, which is a generator, and hence $\tau$ is a generator of $\check{H}^{2h}(\mathbf{D}_G^{2h}, \mathbf{S}_G^{2h-1})$.

Let $F \cap \mathbf{D}^{2h} = \mathbf{D}^{2h_0}$. Restriction to $F_G$ gives the diagram

$$\begin{array}{ccccc} \check{H}^{2h}(\mathbf{D}_G^{2h}, \mathbf{S}_G^{2h-1}) & \longrightarrow & \check{H}^{2h}(\mathbf{D}_G^{2h_0}, \mathbf{S}_G^{2h_0-1}) & \approx & \check{H}^{2(h-h_0)}(B_G) \otimes \check{H}^{2h_0}(\mathbf{D}^{2h_0}, \mathbf{S}^{2h_0-1}) \\ \downarrow & & \downarrow & & \\ \check{H}^{2h}(X_G) & \xrightarrow{j_0^*} & \check{H}^{2h}((F_0)_G) & \subset & \check{H}^*(B_G) \otimes \check{H}^*(F_0). \end{array}$$

Now $j_0^*(\alpha) = c_0 t \otimes 1 + 1 \otimes b_0$, where $b_0 \in \check{H}^2(F_0; \mathbf{Z})$ is a generator, and $b_0 = 0$ if $h_0 = 0$. From the definition of $\beta$ we calculate

$$\begin{aligned} j_0^*(\beta) &= (1 \otimes b_0)^{h_0} \prod_{i \neq 0} [(c_0 - c_i)t \otimes 1 + 1 \otimes b_0]^{h_i+1} \\ &= \prod_{i \neq 0} (c_0 - c_i)^{h_i+1} t^{h-h_0} \otimes b_0^{h_0}. \end{aligned}$$

It follows that the generator $\tau$ goes to $\prod_{i \neq 0} (c_0 - c_i)^{h_i+1}$ in $\check{H}^{2(h-h_0)}(B_G) \otimes \check{H}^{2h_0}(\mathbf{D}^{2h_0}, \mathbf{S}^{2h_0-1}) \approx \mathbf{Z}$. Now $(\mathbf{D}^{2h}, \mathbf{S}^{2h-1})$ can be considered as the product of copies of $(\mathbf{D}^2, \mathbf{S}^1)$ with the representations $\eta^{m_i}$, or the trivial representation. By a Künneth Theorem argument, we see that it suffices to show that, for the representation $\eta^m$, $m \neq 0$, the generator of $\check{H}^2(\mathbf{D}^2{}_G, \mathbf{S}^1{}_G)$ goes to $\pm m$ in $\check{H}^2(B_G) \otimes \check{H}^0(\{0\}) \approx \mathbf{Z}$. (A trivial factor in the representation clearly only has the effect of raising the dimension of the fixed set.) Since $\{0\}$ is an equivariant deformation retract of $\mathbf{D}^2$ the homomorphism in question is just $\varphi$ of the following exact sequence

$$\check{H}^2(\mathbf{D}^2{}_G, \mathbf{S}^1{}_G; \mathbf{Z}) \xrightarrow{\varphi} \check{H}^2(\mathbf{D}^2{}_G; \mathbf{Z}) \xrightarrow{\psi} \check{H}^2(\mathbf{S}^1{}_G; \mathbf{Z}) \to 0.$$

However, with $H = G/\mathbf{Z}_m \approx \mathbf{S}^1$, we have

$$\mathbf{S}^1{}_G = \mathbf{S}^1 \times_G E_G \approx \mathbf{S}^1 \times_H B_{\mathbf{Z}_m} \approx B_{\mathbf{Z}_m}.$$

Thus $\check{H}^2(\mathbf{S}^1{}_G; \mathbf{Z}) \approx \check{H}^2(B_{\mathbf{Z}_m}; \mathbf{Z}) \approx \mathbf{Z}_m$ and the result follows. ∎

**5.6. Proposition** *For a locally smooth* $\mathbf{S}^1$*-action on a closed manifold* $X \sim_{\mathbf{Z}} \mathbf{CP}^h$ *and with the above notation, suppose that* $\{x\} = F_0$ *is an isolated fixed point and that the tangential representation at* $x$ *is* $\sum \eta^{m_i}$. *Then each* $m_i$ *divides some* $c_j - c_0$.

*Proof* In fact we shall prove more. Let $m \in \mathbf{Z}$ be one of the $m_i$ and let $N$ be the component of $F(\mathbf{Z}_m, X)$ containing $x$. Then $N$ is a manifold of positive dimension on which $\mathbf{S}^1$ acts (with $\mathbf{Z}_m$ acting trivially). By IV.2.3 applied to a large $\mathbf{Z}_p$ subgroup of $\mathbf{S}^1$, this $\mathbf{S}^1$-action must have more than one stationary point and hence $N$ must touch another component, say $F_1$, of $F = F(\mathbf{S}^1, X)$. (We remark that this is the only use we will make of the assumption that $F_0 = \{x\}$ is isolated.) Let $\varphi\colon \mathbf{I} \to N$ be an arc between $x$ and some point $y \in F_1 \cap N$. The map $\varphi$ extends uniquely to an equivariant map $\mathbf{I} \times \mathbf{S}^1 \to X$, where $G = \mathbf{S}^1$ acts on $\mathbf{S}^1$ via the representation $\eta^m$. Collapsing the ends of $\mathbf{I} \times \mathbf{S}^1$ yields an equivariant map

$$\psi\colon \mathbf{S}^2 \to X,$$

where $\mathbf{S}^1$ acts on $\mathbf{S}^2$ via the suspension of $\eta^m$ and where $\psi(u) = x$ and $\psi(v) = y$, $u$ and $v$ being the poles of $\mathbf{S}^2$.

Now we may normalize $\alpha$, and hence the $c_i$, by taking $\alpha \in \check{H}^2(X_G, x_G)$ and hence $c_0 = 0$. Write $\psi^*(a) = da'$, where $a$ generates $\check{H}^2(X, x)$ and $a'$ generates $\check{H}^2(\mathbf{S}^2, u)$ and $d \in \mathbf{Z}$ is the "degree" of $\psi$. We claim that $c_1 = \pm dm$ (i.e., $c_1 - c_0 = \pm dm$, in the unnormalized version).

To see this, consider the diagram

$$\begin{array}{ccc} \check{H}^2(X_G, x_G) & \xrightarrow{\psi_G^*} & \check{H}^2(\mathbf{S}^2{}_G, u_G) \\ \downarrow{\scriptstyle j_1^*} & & \downarrow{\scriptstyle j_1^*} \\ \check{H}^2((F_1)_G) & \xrightarrow{(\psi^G)^*} & \check{H}^2(B_G) \otimes \check{H}^0(v) \approx \check{H}^2(B_G). \end{array}$$

By definition of $c_1$ we have

$$j_1^*(\alpha) = c_1 t \otimes 1 + (\star)$$

which goes to $c_1 t$ in $\check{H}^2(B_G)$. However, $\psi_G^*(\alpha)$ is the unique element representing $\psi^*(a) = da'$ in $\check{H}^2(\mathbf{S}^2, u)$. Thus $\psi_G^*(\alpha) = d\alpha'$ where $\alpha'$ represents $a'$. From 5.5 applied to the action on $\mathbf{S}^2$ we have that $j_1^*(\alpha') = \pm mt$ in $\check{H}^2(B_G)$, so that

$$j_1^*\psi_G^*(\alpha) = \pm dmt \in \check{H}^2(B_G).$$

Thus $c_1 = \pm dm$ by the commutativity of the diagram. ∎

*Remark* It is of interest to consider the example of an $S^1$-action on $S^2 \times D^2$ constructed in Section 4 in the light of 5.5 and 5.6 even though this manifold is not closed. In that example, it can be seen just as in the proof of 5.6 that $c_1 - c_0 = \pm 6$, and the representation of $S^1$ at either fixed point is $\eta^2 \oplus \eta^3$. Guided by that example, Petrie [3] has constructed a smooth $S^1$-action on $CP^3$ with exactly four fixed points $x_i$, $0 \le i \le 3$, with $c_0 = 0$, $c_1 = 7$, $c_2 = 6$, $c_3 = 1$ and such that the representations about $x_0$ and $x_1$ are $\eta^2 \oplus \eta^3 \oplus \eta^7$ and those about $x_2$ and $x_3$ are $\eta^2 \oplus \eta^3 \oplus \eta^5$.

## 6. ACTIONS ON POINCARÉ DUALITY SPACES

Let $K$ be a field. If $A$ is a closed subspace of a space $X$ then $(X, A)$ is said to be a **Poincaré duality pair over $K$ of formal dimension $n$** if the following conditions are satisfied:

(i) $\check{H}^*(X; K)$ is finitely generated.
(ii) $\check{H}^i(X, A; K) = 0$ for $i > n$ and $\check{H}^n(X, A; K) \approx K$.
(iii) For all $i$ the cup product pairing

$$\check{H}^i(X; K) \otimes \check{H}^{n-i}(X, A; K) \to \check{H}^n(X, A; K) \approx K$$

is nonsingular (i.e., it is a duality pairing).

For an action of $Z_p$ on a finitistic Poincaré duality space $X$ (i.e., $A = \varnothing$) over $Z_p$ it was conjectured by Su [4] that each component of the fixed set $F$ is also a Poincaré duality space over $Z_p$. For locally smooth (or, in fact arbitrary) actions on a *closed* manifold $M^n$ this conjecture is obvious and uninteresting since $F$ must then be a closed manifold. However, even for smooth actions on spaces like $M^n \times D^r$, the conjecture is highly nontrivial and is of fundamental interest.

The conjecture was proved in Bredon [10, 21] under the assumption that $X$ is totally nonhomologous to zero in $X_G$. Here we shall present the proof of the more general relative version of this result. Despite the seeming simplicity of this result, it is a powerful tool and is the strongest known general fact about the cohomology ring structure of a fixed point set. No further progress has been made on the general conjecture and there is, in fact, very little evidence for it, mainly because of the very great difficulty in proving anything when $X$ is not totally nonhomologous to zero in $X_G$.

**6.1. Theorem** *Let $(X, A)$ be a finitistic Poincaré duality pair over $\mathbf{Z}_p$ of formal dimension $n$, where $p$ is prime. Suppose that $G = \mathbf{Z}_p$ acts on $X$ with $A$ invariant and such that $X$ is totally nonhomologous to zero in $X_G$ over $\mathbf{Z}_p$. Then, for each component $F_0$ of $F = X^G$, $(F_0, F_0 \cap A)$ is a Poincaré duality pair over $\mathbf{Z}_p$ of formal dimension $r \leq n$. If $p \neq 2$ then $n - r$ is even. If $r = n$ then $F = F_0$ is connected and the restrictions $\check{H}^*(X; \mathbf{Z}_p) \to \check{H}^*(F; \mathbf{Z}_p)$ and $\check{H}^*(X, A; \mathbf{Z}_p) \to \check{H}^*(F, F \cap A; \mathbf{Z}_p)$ are isomorphisms. The analogous statements hold for $\mathbf{S}^1$ actions over the rationals ($p = 0$) if $X$ and $X/\mathbf{S}^1$ are both finitistic.*

*Proof* We shall give the proof for odd primes $p$ since this is the most difficult case. The proof for $p = 0,2$ is essentially obtained by simply deleting $s$ from all the formulas. (The proof for $p = 2$ in the absolute case $A = \varnothing$ is given explicitly in Bredon [21].) Cohomology coefficients will always be in $\mathbf{Z}_p$. First, we need the following fact:

**6.2. Lemma** *$(X, A)$ is totally nonhomologous to zero in $(X_G, A_G)$.*

*Proof* Because of the naturality of the duality isomorphism $\check{H}^i(X, A) \approx \operatorname{Hom}(\check{H}^{n-i}(X), \mathbf{Z}_p)$, $G$ must act trivially on $\check{H}^*(X, A)$. In the spectral sequence of $(X_G, A_G) \to B_G$ suppose we know that all differentials vanish before the $k$-th and that $d_k$ vanishes on terms of fiber degree greater than $i$. Let $0 \neq a \in \check{H}^i(X, A)$ so that we may regard $1 \otimes a$ as an element of $E_k^{0,i} \approx E_2^{0,i} \approx \check{H}^0(B_G) \otimes \check{H}^i(X, A)$. Then $d_k(1 \otimes a) = C \otimes c$ where $C \in \check{H}^k(B_G)$ and $c \in \check{H}^{i-k+1}(X, A)$. If $C$ and $c$ are nonzero then there is an element $b \in \check{H}^{n-i+k-1}(X)$ (of positive degree since $i \leq n$) with $cb \neq 0$. Then

$$0 = d_k(1 \otimes ab) = \big(d_k(1 \otimes a)\big)(1 \otimes b) = C \otimes cb \neq 0$$

by the inductive assumption and since $X$ is totally nonhomologous to zero in $X_G$. This contradiction shows that $d_k(1 \otimes a) = 0$ and it follows that $d_k = 0$. ∎

Returning to the proof of 6.1, let $x \in F_0$ and regard $\check{H}^*(F_0, x)$ as an internal direct summand of $\check{H}^*(F, x)$, and similarly for $\check{H}^*(F_0, F_0 \cap A)$. Note that, via $\check{H}^*(X, x) \to \check{H}^*(X)$, there is a cup product $\check{H}^*(X, x) \otimes \check{H}^*(X, A) \to \check{H}^*(X, A)$ even though $x$ need not be in $A$, and similarly with other pairs. Many of our cup products will be implicitly of this type.

Let $r$ be the maximum degree for which $\check{H}^r(F_0, F_0 \cap A) \neq 0$. We must prove that $\check{H}^r(F_0, F_0 \cap A)$ has rank one and that if $0 \neq b \in \check{H}^i(F_0, F_0 \cap A)$ for $i < r$ then there is an element $c \in \check{H}^{r-i}(F_0, x)$ with $0 \neq bc \in \check{H}^r(F_0, F_0 \cap A)$. [This will suffice, since it implies that the duality homomorphism $\check{H}^i(F_0, F_0 \cap A) \to \operatorname{Hom}(\check{H}^{r-i}(F_0), \mathbf{Z}_p)$ is a monomorphism, and hence that $\operatorname{rk} \check{H}^*(F_0, F_0 \cap A) \leq \operatorname{rk} \check{H}^*(F_0)$; which is also true when $\check{H}^*(F_0, F_0 \cap A) = 0$. But

$$\operatorname{rk} \check{H}^*(F, F \cap A) = \operatorname{rk} \check{H}^*(X, A) = \operatorname{rk} \check{H}^*(X) = \operatorname{rk} \check{H}^*(F)$$

by 6.2 and 1.6, and this implies, in turn, that the duality homomorphism above is an isomorphism.] For the last fact, it suffices to find $c \in \check{H}^j(F_0, x)$ with $bc \neq 0$ for *some* $j$ (necessarily $j > 0$), since repeated application of this eventually gives us a product in $\check{H}^r(F_0, F_0 \cap A)$.

Let $\varphi$: $\check{H}^*(X, A) \to \check{H}^*(X_G, A_G)$ be a cohomology extension of the fiber. Denote the restriction to a typical fiber by

$$i^*\colon\ \check{H}^*(X_G, A_G) \to \check{H}^*(X, A),$$

so that $i^*\varphi = 1$. Note that since the composition $X \to X_G \to B_G$ factors through a point, we have that $i^*(s\alpha) = 0$ and $i^*(t\alpha) = 0$ for all $\alpha \in \check{H}^*(X_G, A_G)$. We shall also use $i^*$ for the restriction $\check{H}^*(X_G, x_G) \to \check{H}^*(X, x)$. It will be convenient to isolate the following fact:

**6.3. Lemma** *Let $s\sum t^i\varphi(a_i) \in \check{H}^*(X_G, A_G)$ be a homogeneous element with $a_0 \neq 0$ and* $\deg a_0 < n$. *Then there is an element $\alpha \in \check{H}^*(X_G, x_G)$ with $0 \neq a = i^*(\alpha) \in \check{H}^*(X, x)$, necessarily of positive degree, such that*

$$0 \neq s\alpha \textstyle\sum t^i\varphi(a_i) \in \check{H}^*(X_G, A_G).$$

***Proof*** Since $\deg a_0 < n$ there is an $a \in \check{H}^*(X, x)$ with $aa_0 \neq 0$. Let $\alpha \in \check{H}^*(X_G, x_G)$ be an arbitrary representative of $a$. Then

$$i^*\big(\alpha \textstyle\sum t^i\varphi(a_i)\big) = i^*(\alpha)a_0 = aa_0 \neq 0.$$

But $s\alpha \sum t^i\varphi(a_i)$ then represents $s \otimes aa_0 \neq 0$ in $E_2^{1,*} = E_\infty^{1,*}$ of the spectral sequence of $(X_G, A_G) \to B_G$ and thus cannot be zero. ∎

Returning to the proof of the theorem, suppose that $0 \neq b \in \check{H}^*(F_0, F_0 \cap A) \subset \check{H}^*(F, F \cap A)$. Since $j^*$ is onto in high degrees, we can find an integer $k$ and elements $a_i, a_i'$ in $\check{H}^*(X, A)$ with

$$t^k \otimes b = j^*[\textstyle\sum t^i\big(\varphi(a_i) + s\varphi(a_i')\big)].$$

Multiplying by $s$ gives

$$st^k \otimes b = j^*\big(s \textstyle\sum t^i \varphi(a_i)\big). \tag{1}$$

Since multiplication by $t$ is a monomorphism, powers of $t$ can be canceled from such equations and thus we may assume that

$$a_0 \neq 0.$$

Suppose that $\deg a_0 < n$ in (1). Then by 6.3 there is an $\alpha \in \check{H}^*(X_G, x_G)$ with $s\alpha \sum t^i \varphi(a_i) \neq 0$. Put

$$j^*(\alpha) = \textstyle\sum(t^i \otimes b_i + st^i \otimes b_i'),$$

where the $b_i$ and $b_i'$ are in $\check{H}^*(F, x)$. Since $j^*$ is a monomorphism we have

$$\begin{aligned} 0 \neq j^*\big(\alpha s \textstyle\sum t^i \varphi(a_i)\big) &= \textstyle\sum (t^i \otimes b_i + st^i \otimes b_i')(st^k \otimes b) \\ &= \textstyle\sum \pm(st^{k+i} \otimes b_i b) \end{aligned}$$

and hence

$$b_i b \neq 0 \qquad \text{for some } i.$$

Now this conclusion is impossible if $\deg b = r$. Thus

$$\deg b = r \Rightarrow \deg a_0 = n \qquad \text{in (1).} \tag{2}$$

(Note that when $p \neq 2$, $\deg t = 2$ so that, from (1), we have that $n - r = 2k$ is even.)

Suppose that $\operatorname{rk} \check{H}^r(F_0, F_0 \cap A) > 1$ and let $b$ and $b'$ be independent elements of $\check{H}^r(F_0, F_0 \cap A)$. By (1) and (2) we can write

$$\begin{aligned} st^k \otimes b &= j^*\big(s \textstyle\sum t^i \varphi(a_i)\big) \\ st^k \otimes b' &= j^*\big(s \textstyle\sum t^i \varphi(a_i')\big) \end{aligned}$$

where $a_0 \neq 0$ and $a_0' \neq 0$ have degree $n$. Since $\check{H}^n(X, A)$ has rank one we can multiply $b'$ by a scalar to make $a_0' = a_0$. Then

$$st^k \otimes (b - b') = j^*\big(st\varphi(a_1 - a_1') + st^2\varphi(a_2 - a_2') + \cdots\big)$$

where $\deg(a_i - a_i') = n - 2i < n$. But this, upon canceling some power of $t$, contradicts (2). Thus we have proved that $\check{H}^r(F_0, F_0 \cap A)$ has rank one.

Let us now fix

$$0 \neq b \in \check{H}^r(F_0, F_0 \cap A)$$

and let $b' \in \check{H}^q(F_0, F_0 \cap A)$ for some $q < r$. As in (1) we can find $a_i'$ such that

$$st^m \otimes b' = j^* \left(s \sum t^i \varphi(a_i')\right), \qquad a_0' \neq 0.$$

If $\deg a_0' < n$ then, as shown above, there exists a $b'' \in \check{H}^*(F_0, x)$ with $b'b'' \neq 0$, our desired conclusion. Thus assume that $\deg a_0' = n$. By multiplying $b'$ by a nonzero scalar we can assume that $a_0' = a_0$. Thus

$$0 \neq st^k \otimes b - st^m \otimes b' = j^*\left(s \sum t^i \varphi(a_i'')\right)$$

where each nonzero $a_i'' = a_i - a_i'$ has degree *less* than $n$, since $a_0'' = 0$. By 6.3 there is an $\alpha \in \check{H}^*(X_G, x_G)$ with $\alpha s \sum t^i \varphi(a_i'') \neq 0$. Putting

$$j^*(\alpha) = \sum (t^i \otimes b_i + st^i \otimes b_i'),$$

where the $b_i$ and $b_i'$ are in $\check{H}^*(F, x)$, we calculate

$$\begin{aligned} 0 \neq j^*\left(\alpha s \sum t^i \varphi(a_i'')\right) &= \sum (t^i \otimes b_i + st^i \otimes b_i')(st^k \otimes b - st^m \otimes b') \\ &= \sum (\pm\, st^{m+i} \otimes b_i b') \end{aligned}$$

since $b \in \check{H}^r(F_0, F_0 \cap A)$ is killed by the cup product with any element of $\check{H}^*(F, x)$. Thus $b_i b' \neq 0$ for some $i$ which completes the proof that $(F_0, F_0 \cap A)$ is a Poincaré duality pair over $\mathbf{Z}_p$.

Now suppose that $n = r$ and let $0 \neq b \in \check{H}^n(F_0, F_0 \cap A)$. By the derivation of (1) and (2) we can write

$$1 \otimes b = j^*\left(\varphi(a_0) + s\varphi(a_0') + t\varphi(a_1) + st\varphi(a_1') + \cdots\right)$$

with $a_0 \neq 0$. By commutativity of the diagram

$$\begin{array}{ccc} \check{H}^*(X_G, A_G) & \xrightarrow{j^*} & \check{H}^*(F_G, (F \cap A)_G) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow \\ \check{H}^*(X, A) & \longrightarrow & \check{H}^*(F_0, F_0 \cap A) \end{array}$$

and since $i^*\left(\varphi(a_0) + s\varphi(a_0') + \cdots\right) = i^*\left(\varphi(a_0)\right) = a_0$ we see that $a_0 \in \check{H}^n(X, A)$ restricts to $0 \neq b \in \check{H}^n(F_0, F_0 \cap A)$. Since $(X, A)$ is a Poincaré duality pair it follows that the restrictions $\check{H}^*(X) \to \check{H}^*(F_0)$ and $\check{H}^*(X, A) \to \check{H}^*(F_0, F_0 \cap A)$ are monomorphisms. Since

$$\begin{aligned} \operatorname{rk} \check{H}^*(F_0, F_0 \cap A) = \operatorname{rk} \check{H}^*(F_0) &\leq \operatorname{rk} \check{H}^*(F) \\ &= \operatorname{rk} \check{H}^*(X) \\ &= \operatorname{rk} \check{H}^*(X, A), \end{aligned}$$

they must be isomorphisms. Then rk $\check{H}^*(F_0) = \text{rk}\, \check{H}^*(X) = \text{rk}\, \check{H}^*(F)$, and we conclude that $F_0 = F$. ∎

## 7. A THEOREM ON INVOLUTIONS

In this section we shall prove a simple but powerful general theorem about involutions and shall give some of its consequences. There are analogous results for $\mathbf{Z}_p$-actions, but they are not as useful and interesting as the case $p = 2$.

Let $X$ be a paracompact space and let $T$ be the involution $(x, y) \mapsto (y, x)$ on $X \times X$. We recall the construction of the Steenrod squares given in Steenrod and Epstein [1]. First a natural map

$$P\colon \check{H}^n(X; \mathbf{Z}_2) \to \check{H}^{2n}((X \times X)_G; \mathbf{Z}_2)$$

(not a homomorphism) is defined, such that

$$i^*P(a) = a \times a \in \check{H}^{2n}(X \times X; \mathbf{Z}_2),$$

where $i\colon X \times X \to (X \times X)_G$ is the inclusion of a typical fiber, and $G = \mathbf{Z}_2$ with generator $T$. Then $Sq^k(a) \in \check{H}^{n+k}(X; \mathbf{Z}_2)$ is defined by the equation

$$j^*P(a) = \sum t^{n-k} \otimes Sq^k(a) \in \check{H}^*(B_G; \mathbf{Z}_2) \otimes \check{H}^*(\Delta; \mathbf{Z}_2),$$

where $\Delta \subset X \times X$ is the diagonal (the fixed set of $T$ on $X \times X$) and $j$: $B_G \times \Delta = \Delta_G \to (X \times X)_G$ is the inclusion.

In Steenrod and Epstein [1], $P$ is defined for regular cell complexes, but it extends by naturality to Čech cohomology of paracompact spaces. Since this extension may not be immediately clear to the reader, we shall recall the construction of $P$ and show how to extend it to Čech theory at the end of this section. For now, we assume this to be done.

Now suppose that $X$ has an involution itself, which will also be denoted by $T$. Define a map

$$h\colon X \to X \times X$$

by $h(x) = (x, Tx)$, and note that $h$ is equivariant. This induces a map

$$h_G\colon X_G \to (X \times X)_G$$

of bundles over $B_G$, and the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & X_G & \xleftarrow{j} & F_G = B_G \times F \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_G} & & \downarrow{\scriptstyle 1\times k} \\ X \times X & \xrightarrow{i} & (X \times X)_G & \xleftarrow{j} & \Delta_G = B_G \times \Delta \end{array}$$

commutes, where $F$ is the fixed set of $T$ on $X$ and $k\colon F \to X \approx \Delta$ is the inclusion.

We now define a natural map

$$Q\colon \check{H}^n(X; \mathbf{Z}_2) \to \check{H}^{2n}(X_G; \mathbf{Z}_2)$$

by

$$Q(a) = h_G{}^*P(a).$$

If $d\colon X \to X \times X$ is the diagonal map, then $h = (1 \times T)d$. Thus we have

$$i^*Q(a) = i^*h_G{}^*P(a) = h^*i^*P(a) = h^*(a \times a) = d^*(a \times T^*a) = a \cdot T^*a.$$

We also have

$$\begin{aligned} j^*Q(a) = j^*h_G{}^*P(a) &= (1 \otimes k^*)j^*P(a) \\ &= (1 \otimes k^*) \sum t^{n-i} \otimes Sq^i(a) \\ &= \sum t^{n-i} \otimes k^*\bigl(Sq^i(a)\bigr) = \sum t^{n-i} \otimes Sq^i(a \mid F). \end{aligned}$$

Summing up these facts, we have the following result.

**7.1. Theorem** *For involutions $T$ on paracompact spaces $X$, there is a natural map*

$$Q\colon \check{H}^n(X; \mathbf{Z}_2) \to \check{H}^{2n}(X_G; \mathbf{Z}_2)$$

*(not a homomorphism) such that*

$$i^*Q(a) = a \cdot T^*a \in \check{H}^{2n}(X; \mathbf{Z}_2)$$

*and*

$$j^*Q(a) = \sum t^{n-i} \otimes Sq^i(a \mid F) \in \check{H}^*(B_G; \mathbf{Z}_2) \otimes \check{H}^*(F; \mathbf{Z}_2). \quad \blacksquare$$

**7.2. Corollary** *The element $a \cdot T^*a \in \check{H}^{2n}(X; \mathbf{Z}_2)^G = \check{H}^0\bigl(B_G; \check{H}^{2n}(X; \mathbf{Z}_2)\bigr) = E_2^{0,2n}$ is a permanent cocycle in the spectral sequence of $X_G \to B_G$, for any $a \in \check{H}^n(X; \mathbf{Z}_2)$.*

*Proof* It is the image of $Q(a)$ under the edge homomorphism. $\blacksquare$

Note that if $G$ acts trivially on $\check{H}^{2n}(X; \mathbf{Z}_2)$, then we usually denote this element by $1 \otimes a \cdot T^*a \in \check{H}^0(B_G) \otimes \check{H}^{2n}(X)$.

**7.3. Corollary** *Suppose that $G = \mathbf{Z}_2$ acts on the finitistic space $X$ and that $X$ is totally nonhomologous to zero in $X_G$. Then any class $a \in \check{H}^n(X; \mathbf{Z}_2)$ with $a^2 \neq 0$ restricts nontrivially to $F$.*

*Proof* In this case $T^* = 1$ so that $i^*Q(a) = a^2 \neq 0$. Since $j^*$ is a monomorphism we have $0 \neq j^*Q(a) = \sum t^{n-i} \otimes Sq^i(a \mid F)$ and thus $a \mid F \neq 0$. ∎

The following theorem gives a sufficient condition for the existence of fixed points, and also a lower bound on the dimension of $F$.

**7.4. Theorem** *Let $T$ be an involution on the finitistic space $X$ and suppose that $\check{H}^i(X; \mathbf{Z}_2) = 0$ for $i > 2n$ and that $T^*$ is the identity on $\check{H}^{2n}(X; \mathbf{Z}_2)$ [e.g., $\check{H}^{2n}(X; \mathbf{Z}_2) \approx \mathbf{Z}_2$]. Suppose that $a \in \check{H}^n(X; \mathbf{Z}_2)$ is an element such that $a \cdot T^*a \neq 0$. Then the fixed point set $F$ of $T$ on $X$ is nonempty and $a \mid F \in \check{H}^n(F; \mathbf{Z}_2)$ is nonzero.*

*Proof* By 7.2, $1 \otimes a \cdot T^*a$ is a permanent cocycle which represents $Q(a) \in \check{H}^{2n}(X_G; \mathbf{Z}_2)$. It follows that

$$t \otimes a \cdot T^*a \in \check{H}^1(B_G) \otimes \check{H}^{2n}(X; \mathbf{Z}_2) = E_2^{1,2n}$$

is also a permanent cocycle and hence survives to $E_\infty^{1,2n}$. However, $tQ(a) \in \check{H}^{2n+1}(X_G; \mathbf{Z}_2)$ represents this element and hence $tQ(a) \neq 0$. Since $j^*$ is an isomorphism in degrees above $2n$, by 1.5, we have $tj^*(Q(a)) = j^*(tQ(a)) \neq 0$ and hence $j^*(Q(a)) \neq 0$. By 7.1 this implies that $a \mid F \neq 0$. In particular $F \neq \varnothing$. ∎

**7.5. Corollary** *Let $T$ be an involution on the finitistic space $X \sim_2 \mathbf{S}^n \times \mathbf{S}^n$ and suppose that $T^* \neq 1$ on $\check{H}^n(X; \mathbf{Z}_2)$. Then $F$ is nonempty and $F \sim_2 \mathbf{S}^n$.*

*Proof* Let $a$, $b$ generate $\check{H}^n(X; \mathbf{Z}_2)$ with $a^2 = 0 = b^2$ and $ab \neq 0$. Since $T^* \neq 1$ it cannot preserve both $a$ and $b$ and thus we may assume that $T^*a \neq a$. Then $T^*a = b$ or $T^*a = a + b$ and, in both cases, $a \cdot T^*a \neq 0$. Then $a \mid F \neq 0$ by 7.4. Since $\check{H}^n(F; \mathbf{Z}_2) \neq 0 \neq \check{H}^0(F; \mathbf{Z}_2)$ we have $2 \leq \operatorname{rk} H^*(F; \mathbf{Z}_2) < 4$ by 1.6, and $\operatorname{rk} \check{H}^*(F; \mathbf{Z}_2) \equiv \chi(F) \equiv \chi(X) \equiv 0 \pmod 2$ by III.7.10. Thus $F \sim_2 \mathbf{S}^n$. ∎

**7.6. Corollary** *Let $T$ be an involution on the finitistic Poincaré duality space $X$ over $\mathbf{Z}_2$. Suppose that the euler characteristic $\chi(X)$ is odd. Then $F \neq \emptyset$ and the restriction $\check{H}^n(X; \mathbf{Z}_2) \to \check{H}^n(F; \mathbf{Z}_2)$ is nontrivial, where $2n$ is the formal dimension of $X$ (necessarily even).*

*Proof* Consider the bilinear form

$$\langle a, b\rangle = a \cdot T^*b \in \check{H}^{2n}(X; \mathbf{Z}_2) \approx \mathbf{Z}_2$$

on $\check{H}^n(X; \mathbf{Z}_2)$, which is nonsingular. It is symmetric since $b \cdot T^*a = T^*(b \cdot T^*a) = T^*b \cdot a = a \cdot T^*b$. Since $\chi(X)$ is odd it follows that rk $\check{H}^n(X; \mathbf{Z}_2)$ is odd.

Suppose that $\langle a, a\rangle = 0$ for all $a$. Then, with respect to a given basis, this form is represented by a symmetric matrix $B$ over $\mathbf{Z}_2$ with zero diagonal and with $0 \neq \det B \in \mathbf{Z}_2$. Clearly $B$ is the reduction modulo 2 of an integral *skew-symmetric* matrix $A$ (with entries 0, 1, and $-1$). Since $A$ has an odd number of rows we have $\det A = -\det(-A) = -\det A' = -\det A$, so that $\det A = 0$. However, $\det B$ is the reduction modulo 2 of $\det A$ and hence must be zero.

This contradiction shows that $0 \neq \langle a, a\rangle = a \cdot T^*a$ for some $a$, and 7.4 applies to give the desired result. ∎

*Remark* Corollary 7.6 was first proved for smooth actions on compact manifolds by Conner and Floyd [8, p. 71] using cobordism methods. The general case first appeared in Bredon [21].

As promised, we shall now indicate the construction of the map $P$. With $G = \mathbf{Z}_2$ let the universal space $E = E_G$ be given a triangulation as a regular $G$-complex (e.g., an infinite join of hexagons) and let $W$ denote the simplicial chain complex $W = C_*(E)$. Let $\varepsilon$: $W \to \mathbf{Z}$ be the augmentation and note that $\varepsilon \circ T = \varepsilon$.

Let $K$ be a simplicial complex and give $|K| \times |K| \times E$ the product cell structure, which we denote by $K \times K \times E$. Let $G$ act by $(x, y, e) \mapsto (y, x, Te)$ on $K \times K \times E$. Since $E$ is regular, the quotient $(|K| \times |K| \times E)/G$ has an induced cell structure which we denote by $(K \times K)_G$. Note that the cellular chain complex of $K \times K \times E$ can be identified with

$$C_*(K) \otimes C_*(K) \otimes W.$$

Moreover, the cochains on this which are invariant under $T$ can clearly be identified with the cellular cochains of $(K \times K)_G$.

For a cochain $u\colon C_n(K) \to \mathbf{Z}_2$ define $P(u) = u \otimes u \otimes \varepsilon$, that is $P(u)(c_1 \otimes c_2 \otimes w) = u(c_1)u(c_2)\varepsilon(w)$. Since $P(u) \circ T = P(u)$, $P(u)$ can be regarded as a cellular cochain

$$P(u) \in C^{2n}\big((K \times K)_G; \mathbf{Z}_2\big).$$

In Steenrod and Epstein [1, p. 100] it is shown that $P\colon C^n(K; \mathbf{Z}_2) \to C^{2n}\big((K \times K)_G; \mathbf{Z}_2\big)$ induces a map (clearly natural with respect to simplicial maps)

$$P\colon H^n(K; \mathbf{Z}_2) \to H^{2n}\big((K \times K)_G; \mathbf{Z}_2\big)$$

and we shall not repeat the proof of this here. (The finiteness condition on $K$ (loc. cit.) is not needed for this.) Moreover, the Steenrod squares for $K$ are defined from this as indicated at the beginning of this section.

Now suppose that $X$ is a paracompact space and let $\mathscr{U} = \{U\}$ be a locally finite covering of $X$. Let $\mathscr{V} = \{V\}$ be a locally finite refinement of $\mathscr{U}$ and $p\colon \mathscr{V} \to \mathscr{U}$ a refinement projection with induced simplicial map $\bar{p}\colon |\mathbf{K}(\mathscr{V})| \to |\mathbf{K}(\mathscr{U})|$. Let $f = \{f_U\}$ and $g = \{g_V\}$ be partitions of unity subordinate to $\mathscr{U}$ and $\mathscr{V}$, respectively, and let $\bar{f}\colon X \to |\mathbf{K}(\mathscr{U})|$ and $\bar{g}\colon X \to |\mathbf{K}(\mathscr{V})|$ be the induced maps. Now $\bar{p}\bar{g}(x) = \sum g_V(x)p(V)$ and $\bar{f}(x) = \sum f_U(x)U$ both lie in the simplex $(U_0, U_1, \ldots, U_k)$ of $\mathbf{K}(\mathscr{U})$, where the $U_i$ are those elements of $\mathscr{U}$ containing $x$. Thus the line segment joining $\bar{p}\bar{g}(x)$ to $\bar{f}(x)$ lies in $|\mathbf{K}(\mathscr{U})|$ and provides a canonical homotopy between the maps $\bar{f}$ and $\bar{p}\bar{g}$. Consequently $\bar{f} \times \bar{f}$ is equivariantly homotopic to $\bar{p}\bar{g} \times \bar{p}\bar{g}$ and thus $(\bar{f} \times \bar{f})_G$ is homotopic to

$$(\bar{p}\bar{g} \times \bar{p}\bar{g})_G = (\bar{p} \times \bar{p})_G \circ (\bar{g} \times \bar{g})_G : (X \times X)_G \to \big(\mathbf{K}(\mathscr{U}) \times \mathbf{K}(\mathscr{U})\big)_G.$$

It follows from this and the naturality of $P$ with respect to simplicial maps that the diagram

$$\begin{array}{ccccccc}
 & & H^n(\mathbf{K}(\mathscr{U})) & \xrightarrow{P} & H^{2n}((\mathbf{K}(\mathscr{U}) \times \mathbf{K}(\mathscr{U}))_G) & & \\
 & \swarrow^{\bar{f}^*} & \downarrow{\scriptstyle \bar{p}^*} & & \downarrow{\scriptstyle (\bar{p}\times\bar{p})_G^*} & \searrow^{(\bar{f}\times\bar{f})_G^*} & \\
\check{H}^n(X) & & & & & & \check{H}^{2n}((X \times X)_G) \\
 & \nwarrow_{\bar{g}^*} & & & & \nearrow_{(\bar{g}\times\bar{g})_G^*} & \\
 & & H^n(\mathbf{K}(\mathscr{V})) & \xrightarrow{P} & H^{2n}((\mathbf{K}(\mathscr{V}) \times \mathbf{K}(\mathscr{V}))_G) & &
\end{array}$$

commutes. Thus we may pass to the direct limit and obtain the maps

$$\check{H}^n(X) \xleftarrow{\approx} \varinjlim H^n\big(\mathbf{K}(\mathscr{U})\big) \to \varinjlim H^{2n}\big((\mathbf{K}(\mathscr{U}) \times \mathbf{K}(\mathscr{U}))_G\big) \to \check{H}^{2n}\big((X \times X)_G\big)$$

whose composition is defined to be $P$ for Čech theory. (The last map is not generally an isomorphism, but this is immaterial.) That the desired properties of $P$ carry over to Čech theory is an elementary exercise which we shall omit.

## 8. INVOLUTIONS ON $\mathbf{S}^n \times \mathbf{S}^m$

In this section and the next we shall study $\mathbf{Z}_p$-actions on $\mathbf{S}^n \times \mathbf{S}^m$, or on spaces with such a cohomology ring. Since it seems desirable to separate the case $p = 2$ from the case of odd $p$, we shall restrict our attention to involutions in this section.

**8.1. Theorem** *Suppose that $G = \mathbf{Z}_2$ acts on the finitistic space $X \sim_2 \mathbf{S}^n \times \mathbf{S}^m$, $n \leq m$. If $F \neq \emptyset$, then one of the following possibilities must occur:*

(1) $F \sim_2 \mathbf{S}^q \times \mathbf{S}^r$, $\quad 0 \leq q \leq n, \quad 0 \leq r \leq m$.
(2) $F \sim_2 \mathbf{P}^2(q) \# -\mathbf{P}^2(q)$, $\quad n \geq q = 1, 2, 4,$ *or* $8$.
(3) $F \sim_2 \mathbf{P}^3(q)$, $\quad n > q$.
(4) $F \sim_2$ point $+ \mathbf{P}^2(q)$, $\quad n > q$.
(5) $F \sim_2 \mathbf{S}^q + \mathbf{S}^r$, $\quad 0 \leq q \leq m, \quad 0 \leq r \leq m$.
(6) $F \sim_2 \mathbf{S}^q$.

***Proof*** Since $\chi(F) \equiv \chi(X) \equiv 0 \pmod 2$ and $\operatorname{rk} \check{H}^*(F; \mathbf{Z}_2) \leq 4$ either $F \sim_2 \mathbf{S}^q$ or $X$ is totally nonhomologous to zero in $X_G$. In the latter case the components of $F$ are Poincaré duality spaces over $\mathbf{Z}_2$ by 6.1. Cases (1)–(5) simply list all possibilities for the cohomology ring of $F$ consistent with this and with the inequalities III.7.9. (The inequalities $n > q$ in (3) and (4) and those in (5) need more argument. They follow easily from the spectral sequence of $X_G \to B_G$. For example, the inequalities in (5) result from the fact that the image under $j^*$ of representatives of ring generators for $X$ must involve ring generators for $F$. Also see the proof of 8.3.) ∎

We remark that case (2) means that $\check{H}^*(F; \mathbf{Z}_2)$ is generated by classes $u$ and $v$ in degree $q$ with $uv = 0$ and $u^2 = -v^2 \neq 0$. (Of course the sign is immaterial over $\mathbf{Z}_2$.)

Of course this result is just the surface information available and it is not of interest for smooth actions on an actual product of spheres. It was first proved by Su [4], before the proof of the Poincaré Duality Theorem 6.1, and hence required the explicit investigation of several cases.

Probably one's first impression with the list in 8.1 is that most of the cases cannot occur. However, it turns out that there are examples of all cases and, except for case (4), there are examples on actual products of spheres; see Su [4] and Bredon [21]. The remainder of this section will be devoted to the discussion of the more interesting examples (some of which are new) and to obtaining further results about such actions having a deeper nature than 8.1. We will be concerned mainly with the case of smooth actions on $\mathbf{S}^n \times \mathbf{S}^m$, or, more generally, on manifolds like $\mathbf{S}^n \times \mathbf{S}^m \times \mathbf{R}^k$.

We begin by constructing examples of case (2). For $q = 1, 2, 4$, or $8$ let $\eta$ be the Hopf $q$-plane bundle over $\mathbf{S}^q$ and let $\eta^*$ be its inverse (the bundle whose characteristic class in $\pi_{q-1}(\mathbf{O}(q))$ is the negative of that for $\eta$). Then $\eta \oplus \eta^*$ is trivial. Let $\varepsilon$ be a trivial line bundle and $\mu$ a trivial $k$-plane bundle, $k \geq 0$. On the $2q + k + 1$ plane bundle $\eta \oplus \varepsilon \oplus \eta^* \oplus \mu$ take the involution which is trivial on $\eta \oplus \varepsilon$ and is $-1$ in the fibers of $\eta^* \oplus \mu$. Then the restriction of this involution to the sphere bundle is an involution on $\mathbf{S}^q \times \mathbf{S}^{2q+k}$ with fixed set the sphere bundle $F = S(\eta \oplus \varepsilon)$ which is the connected sum $F = \mathbf{P}^2(q) \# -\mathbf{P}^2(q)$, where $\mathbf{P}^2(q) = \mathbf{RP}^2, \mathbf{CP}^2, \mathbf{QP}^2$ or **Cay** $\mathbf{P}^2$ according as $q = 1, 2, 4$, or $8$. Note that $F$ is the Klein bottle when $q = 1$.

Starting with this example we now give a further construction to yield such examples in other dimensions. Suppose we have, for example, an involution $T$ on $\mathbf{S}^n \times \mathbf{S}^m$ and on $\mathbf{S}^n$ with the projection $\mathbf{S}^n \times \mathbf{S}^m \to \mathbf{S}^n$ equivariant and with $T$ linear in the fibers. Then let $Y$ and $Y^*$ be two copies of $\mathbf{D}^{n+1} \times \mathbf{S}^m$ and let

$$X = Y \cup_T Y^*.$$

If $y^*$ denotes the element of $Y^*$ corresponding to $y \in Y$, then $y \leftrightarrow y^*$ defines an involution on $X$ extending $T$ on $\mathbf{S}^n \times \mathbf{S}^m \subset X$ and having the *same* fixed set $F$. Now $X$ is an $\mathbf{S}^m$-bundle over $\mathbf{S}^{n+1}$ and the projection $X \to \mathbf{S}^{n+1}$ is equivariant. If this is a trivial bundle, so that $X \approx \mathbf{S}^{n+1} \times \mathbf{S}^m$, then we can proceed one more step with the construction.

For the given involution on $\mathbf{S}^2 \times \mathbf{S}^{k+4}$ with $F = \mathbf{CP}^2 \# -\mathbf{CP}^2$, the next step $X \to \mathbf{S}^3$ is a trivial bundle since $\pi_2(\mathbf{O}(k+5)) = 0$. Thus we have an involution on $\mathbf{S}^3 \times \mathbf{S}^{k+4}$ with $F = \mathbf{CP}^2 \# -\mathbf{CP}^2$. Repeating the construction then gives an $\mathbf{S}^{k+4}$-bundle $X$ over $\mathbf{S}^4$. Thus $X \sim_{\mathbf{Z}} \mathbf{S}^4 \times \mathbf{S}^{k+4}$, with $F = \mathbf{CP}^2 \# -\mathbf{CP}^2$. The obstruction to trivializing this bundle, and hence to proceeding with the construction, is in $\pi_3(\mathbf{O}(k+5)) \neq 0$. It turns out, in fact, that this obstruction is nonzero (see 8.2) and, moreover, that $X$ cannot even have the homotopy type of $\mathbf{S}^4 \times \mathbf{S}^{k+4}$.

Starting with the involution on $\mathbf{S}^4 \times \mathbf{S}^{8+k}$ with $F = \mathbf{QP}^2 \# -\mathbf{QP}^2$ we can construct, in the same way, involutions on $\mathbf{S}^n \times \mathbf{S}^{8+k}$ with $F = \mathbf{QP}^2 \# -\mathbf{QP}^2$

for $n = 4, 5, 6$, and 7. Also one obtains such an involution on some $\mathbf{S}^{8+k}$-bundle over $\mathbf{S}^8$.

Another class of examples are the involutions on $X = S(\eta \oplus \varepsilon \oplus \mu) \sim_{\mathbf{Z}} \mathbf{S}^q \times \mathbf{S}^{q+k}$, $k > 0$, fixing $S(\eta \oplus \varepsilon) = \mathbf{P}^2(q) \# -\mathbf{P}^2(q)$. These spaces $X$ are *not* of the homotopy type of $\mathbf{S}^q \times \mathbf{S}^{q+k}$ since it can be seen that $Sq^q$: $H^{q+k}(X; \mathbf{Z}_2) \to H^{2q+k}(X; \mathbf{Z}_2)$ is nonzero.

The following theorem greatly clarifies the situation of these examples, and shows that the given examples cover all possibilities when $q = 1, 2$, or 4.

**8.2. Theorem** *Suppose that $\mathbf{Z}_2$ acts on the finitistic space $X \sim_2 \mathbf{S}^n \times \mathbf{S}^m$, $n \le m$, and assume that $F \sim_2 \mathbf{P}^2(q) \# -\mathbf{P}^2(q)$ (so that $q = 1, 2, 4$, or 8). If $Sq^n$: $\check{H}^m(X; \mathbf{Z}_2) \to \check{H}^{m+n}(X; \mathbf{Z}_2)$ is trivial, then $q \le n < 2q \le m$, and if, on the other hand, it is nontrivial, then $n < m$ and $n = q$ or $n = 2q$.*

*Proof* If $Sq^n$ is nontrivial, then $n < m$ since otherwise $X$ would not have the cohomology ring of $\mathbf{S}^n \times \mathbf{S}^m$. Let $x \in F$ and let $\alpha \in \check{H}^n(X_G, x_G)$ and $\beta \in \check{H}^m(X_G, x_G)$ represent generators $a$ and $b$ of $\check{H}^*(X, x)$, so that $\check{H}^*(X_G, x_G)$ is the free $\check{H}^*(B_G) = \mathbf{Z}_2[t]$ module with basis $\alpha$, $\beta$, and $\alpha\beta$. We may write

$$j^*(\alpha) = At^{n-q} \otimes u + Bt^{n-2q} \otimes w,$$
$$j^*(\beta) = Ct^{m-q} \otimes v + Dt^{m-2q} \otimes w,$$

where $A, B, C, D \in \mathbf{Z}_2$, $u, v \in \check{H}^q(F, x)$, and $0 \neq w \in \check{H}^{2q}(F, x)$. Since $j^*$ is onto in high degrees, $Au$ and $Cv$ must span $\check{H}^q(F, x)$ and hence are independent. Thus $A = 1 = C$ and $q \le n \le m$. Since $j^*(\alpha\beta) = t^{m+n-2q} \otimes uv$ and since $\alpha\beta$ is *independent* of $\alpha$ and $\beta$ over $\check{H}^*(B_G)$ we see that $uv \neq 0$ and hence $w = uv$.

Since $a^2 = 0$ and $2n \le n + m$, $\alpha^2$ depends on $\alpha$ and $\beta$ over $\check{H}^*(B_G)$. However, $j^*(\alpha^2) = t^{2n-2q} \otimes u^2$, and this implies that $u^2 = 0$. If $v^2$ were zero, then $F$ would have the wrong cohomology ring. Thus $v^2 = w$.

We may assume that $Sq^q = 0$ on $\check{H}^m(X; \mathbf{Z}_2)$, for otherwise $n$ would be $q$ and there would be nothing to prove. Thus $Sq^q(\beta)$ must depend on $\alpha$ and $\beta$ over $\check{H}^*(B_G)$. However,

$$j^*(Sq^q\beta) = Sq^q(j^*(\beta)) = \binom{m-q}{q} t^m \otimes v + (1 + D\binom{m-2q}{q})t^{m-q} \otimes w.$$

Since this does not involve $u$, $Sq^q(\beta)$ must depend on $\beta$ alone, but this then implies that $D \neq 0$ and consequently also that $m \ge 2q$. We must also have that $B = 0$ for otherwise $\beta$ could be replaced by $\beta - t^{m-n}\alpha$ and the new $D$ would be zero. We now have that

$$j^*(\alpha) = t^{n-q} \otimes u,$$
$$j^*(\beta) = t^{m-q} \otimes v + t^{m-2q} \otimes w$$

with $u^2 = 0$, $uv = w = v^2$. If $q < n < 2q$, then $Sq^n = 0$ on $\check{H}^m(X)$ since $Sq^n$ is decomposable when $n$ is not a power of 2. Thus it remains to show that $n \leq 2q$ and that $Sq^n \neq 0$ on $X$ when $n = 2q$. Thus suppose that $n > 2q$ or that $n = 2q$ but $Sq^{2q}$: $\check{H}^m(X) \to \check{H}^{m+n}(X)$ is trivial. Then $Sq^{2q}(\beta)$ depends on $\alpha$ and $\beta$ over $\check{H}^*(B_G)$. However,

$$j^*(Sq^{2q}\beta) = Sq^{2q}(j^*(\beta)) = \binom{m-q}{2q} t^{m+q} \otimes v + \left[\binom{m-q}{q} + \binom{m-2q}{2q}\right] t^m \otimes w.$$

As before this does not involve $u$ so that $Sq^{2q}(\beta)$ must depend on $\beta$ alone. But then this implies that

$$\binom{m-q}{2q} \equiv \binom{m-q}{q} + \binom{m-2q}{2q} \pmod 2.$$

It is known (see Steenrod and Epstein [1, p. 5]) that $\binom{k}{2^i} \equiv 1 \pmod 2$ iff $2^i$ occurs in the dyadic expansion of $k$. A straightforward check of the four possibilities for the terms $q$ and $2q$ in the dyadic expansion of $m - 2q$ shows that the above congruence is never satisfied. ▮

*Remark* For $q = 8$ the case $n = 2q$ is impossible since $Sq^{16}$: $\check{H}^m(X; \mathbf{Z}_2) \to \check{H}^{m+16}(X; \mathbf{Z}_2)$ must be trivial by a result of Adams [1] because $X$ has zero cohomology between these two dimensions. It follows that the construction given above 8.2, starting with an involution of $\mathbf{S}^8 \times \mathbf{S}^{16+k}$ with $F = \mathbf{Cay\,P}^2 \# -\mathbf{Cay\,P}^2$, must be obstructed before it reaches $\mathbf{S}^{15} \times \mathbf{S}^{16+k}$. Thus this construction must end with such an involution on $\mathbf{S}^n \times \mathbf{S}^{16+k}$ for $n = 8$, 9, or 11, the next stage giving a nontrivial bundle over $\mathbf{S}^{n+1}$. We conjecture that only the case $n = 8$ can actually be realized.

We now turn to case (3) of 8.1, giving examples only; also see Exercise 5. If $\tau$ is the tangent bundle of $\mathbf{S}^n$ and $\varepsilon$ is a trivial $k + 1$ plane bundle, $k \geq 0$, then $\tau \oplus \varepsilon$ is trivial. Take the standard involution on $\mathbf{S}^n$ fixing $\mathbf{S}^2$ and take the differential of this on $\tau$. Add the involution $-1$ on $\varepsilon$. This gives an involution on $\mathbf{S}^n \times \mathbf{S}^{n+k} = S(\tau \oplus \varepsilon)$ with $F$ the tangent $\mathbf{S}^1$-bundle to $\mathbf{S}^2$; hence $F = \mathbf{RP}^3$. This example is due to J. C. Su. Using the fact that $\mathbf{CP}^3$

is an $\mathbf{S}^2$-bundle over $\mathbf{S}^4$, Su [8] gave a similar example on $\mathbf{S}^4 \times \mathbf{S}^m$, $m \geq 5$, with $F = \mathbf{CP}^3$; also see Section 9. For $m$ large, the construction above 8.2 applies to give involutions on $\mathbf{S}^n \times \mathbf{S}^m$ with $F = \mathbf{CP}^3$ for $n = 4, 5, 6$, and 7. There is an obstruction to doing this for $n = 8$, but it can be shown to be trivial, and hence there is also such an example on $\mathbf{S}^8 \times \mathbf{S}^m$. It seems likely that such examples exist for all $n \geq 4$.

There are no known examples for case (4) of 8.1 on $\mathbf{S}^n \times \mathbf{S}^m$, but there are two examples on manifolds with the correct cohomology ring. The involution $A \mapsto A^{-1}$ on $\mathrm{SU}(3) \sim_{\mathbf{Z}} \mathbf{S}^3 \times \mathbf{S}^5$ has $F =$ point $+ \mathbf{CP}^2$ and the canonical involution on the symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3) \sim_2 \mathbf{S}^2 \times \mathbf{S}^3$ has $F =$ point $+ \mathbf{RP}^2$. The following theorem narrows considerably the possibilities for such involutions on actual products of spheres.

**8.3. Theorem** *Let $\mathbf{Z}_2$ act smoothly on $\mathbf{S}^n \times \mathbf{S}^m$ with $n \leq m$ and assume that $F \sim_2 (\text{point} + \mathbf{P}^2(q))$. Then the only possible values of $n$, $m$, and $q$ are $q = 2$ and $(n, m) = (3, 4)$, or $q = 4$ and $(n, m) = (5, 8), (6, 8), (6, 9), (7, 8), (7, 9)$, or $(7, 10)$. In particular, $q \neq 1$ and $q \neq 8$.*

*Proof* To apply the cohomological method it is convenient to remove the isolated fixed point and thereby obtain an involution on a space $X \simeq \mathbf{S}^n \vee \mathbf{S}^m$ with $F \sim_2 \mathbf{P}^2(q)$. Let $x \in F$ and let $\alpha \in \check{H}^n(X_G, x_G)$ and $\beta \in \check{H}^m(X_G, x_G)$ represent generators $a$ and $b$ of $\check{H}^*(X, x)$. Let $u \in \check{H}^q(F, x)$ be a generator. Then

$$j^*(\alpha) = At^{n-q} \otimes u + Bt^{n-2q} \otimes u^2,$$
$$j^*(\beta) = Ct^{m-q} \otimes u + Dt^{m-2q} \otimes u^2$$

must be independent over $\check{H}^*(B_G)$ so that $AD - BC \neq 0$ in $\mathbf{Z}_2$. In particular, we must have $n \geq q$. If $n = q$, then we must have $j^*(\alpha) = 1 \otimes u$ which means that $a$ restricts to $u$, contrary to the fact that $a^2 = 0$ while $u^2 \neq 0$. Thus $q < n \leq m$.

Now we apply VI.11.6 to the original smooth action on $\mathbf{S}^n \times \mathbf{S}^m$ to conclude that $2q$ (the dimensional difference of the components of $F$) is divisible by $2^{\varphi(n)}$. This implies that for $q = 1, 2, 4, 8$ we must have $n \leq 1$, 3, 7, 8, respectively. This contradicts the previous fact that $q < n$ if $q = 1$ or 8. Also, for $q = 2$ we must have $n = 3$ and for $q = 4$ we must have $n = 5$, 6 or 7.

Since we now know that $n < 2q$ we must have $m \geq 2q$ in order that $j^*(\alpha)$ and $j^*(\beta)$ be independent. It remains to show that $m - n < q$. Suppose on the contrary that $m - n \geq q$ and note that $Sq^q\colon \check{H}^n(X, x)$

$\to \check{H}^{n+q}(X, x)$ is trivial even when $m = n + q$. This implies that $Sq^q(\alpha)$ depends on $\alpha$ over $\check{H}^*(B_G)$. However, $j^*(\alpha) = t^{n-q} \otimes u$, since $n < 2q$, so that

$$j^*(Sq^q\alpha) = Sq^q(j^*(\alpha)) = \binom{n-q}{q} t^n \otimes u + t^{n-q} \otimes u^2$$

is independent of $j^*(\alpha)$; a contradiction. ∎

We now turn to case (5) of 8.1. Since $\mathbf{S}^q + \mathbf{S}^q \approx \mathbf{S}^0 \times \mathbf{S}^q$ the case $q = r$ is not interesting. For examples, recall that the proof of VI.11.6 contains a construction of an equivariant map $\varphi: \mathbf{S}^n \to \mathbf{O}(m+1)$ for $m$ large, where the generator $T$ of $\mathbf{Z}_2$ acts linearly on $\mathbf{S}^n$ with two fixed points $w_0$ and $w_1$, and acts on $\mathbf{O}(m+1)$ by $T(A) = A^{-1}$. The construction gives such maps with

$$\varphi(w_0) = \begin{bmatrix} I_{q+1} & 0 \\ 0 & -I_{m-q} \end{bmatrix}, \qquad \varphi(w_1) = \begin{bmatrix} I_{r+1} & 0 \\ 0 & -I_{m-r} \end{bmatrix}$$

for any $-1 \le q, r \le m$ such that $r-q$ is divisible by $2^{\Phi(n)}$. By the Smoothing Theorem VI.4.2 we can assume that $\varphi$ is smooth (it can be taken to be constant near $w_0$ and $w_1$). Then the smooth involution $T(x, y) = (Tx, \varphi(x) \cdot y)$ on $\mathbf{S}^n \times \mathbf{S}^m$ has $F = \mathbf{S}^q + \mathbf{S}^r$. Note that for $r = -1$ this gives examples of case (6) of 8.1. For $q, r \neq -1$ these examples of case (5) of 8.1 are best possible by VI.11.6, which we restate in this case as follows.

**8.4. Theorem** *Let $\mathbf{Z}_2$ act smoothly on $\mathbf{S}^n \times \mathbf{S}^m$ with $F \sim_2 \mathbf{S}^q + \mathbf{S}^r$, $q \ge 0$, $r \ge 0$ and $n \le m$. Then $r-q$ is divisible by $2^{\Phi(n)}$.* ∎

***Remark*** A somewhat weaker result is known for the nonsmooth case and for actions on more general spaces such as $\mathbf{S}^n \times \mathbf{S}^m \times \mathbf{R}^k$; see Bredon [21].

Case (6) of 8.1 can occur in two distinct ways. First, we can have $n = m$ with $\mathbf{Z}_2$ acting nontrivially on $\check{H}^n(X; \mathbf{Z}_2)$. In this case $q = n$ by 7.5. Second, $\mathbf{Z}_2$ can act trivially on $\check{H}^*(X; \mathbf{Z}_2)$, but the spectral sequence has a nonzero differential. This is the case with the examples above (where $F = \mathbf{S}^q + \mathbf{S}^r$ with $r = -1$). One would then expect that $q + 1$ would be divisible by some high power of 2 (depending on $n$), perhaps even by $2^{\Phi(n)}$. With some further hypotheses, such a theorem (probably not the best possible) is proved in Bredon [21]. The proof involves a detailed study of the spectral sequences of $X_G$, $(X_G, F_G)$, and $X_G - F_G$ and their relationships with one another. It is the only nontrivial proof (of this type of result) we know of in the case of a nontrivial spectral sequence, but we shall not give it here.

## 9. $\mathbf{Z}_p$-ACTIONS ON $\mathbf{S}^n \times \mathbf{S}^m$

The basic information for $\mathbf{Z}_p$-actions on $\mathbf{S}^n \times \mathbf{S}^m$ with $p$ odd is given by the following theorem. The condition of finite generation is for convenience only. The theorem can be proved without it, at the cost of considerably more argument.

**9.1. Theorem** *Let $p$ be an odd prime and suppose that $G = \mathbf{Z}_p$ acts on the finitistic space $X \sim_p \mathbf{S}^n \times \mathbf{S}^m$, $n \leq m$. Assume that $\check{H}^*(X; \mathbf{Z})$ is finitely generated. If $F \neq \varnothing$, then one of the following possibilities must occur:*

(1) $F \sim_p \mathbf{S}^q \times \mathbf{S}^r$, *$n - q \geq 0$ even, $m - r \geq 0$ even.*
(2) $F \sim_p \mathbf{P}^3(q)$, *$n$, $m$, $q$ even, $q < n$.*
(3) $F \sim_p (\text{point} + \mathbf{P}^2(q))$, *$n$, $m$, $q$ even, $q < n$.*
(4) $F \sim_p \mathbf{S}^q + \mathbf{S}^r$, *either $n$, $m$, $q$, $r$ all even and $m \geq q$, $m \geq r$; or $n$ even, $m \geq q$, $r$ all odd; or $m$ even, $n \geq q$, $r$ all odd.*
(5) $F \sim_p \mathbf{S}^q$, *$n$ even, $m$ odd, $q$ odd.*
(6) $F \sim_p (\text{point} + \mathbf{S}^q)$, *$n = m$ odd, $q$ even, $q < n$, and $p = 3$.*
(7) $F \sim_p \mathbf{P}^2(q)$, *$n = m$ odd, $q$ even, $q < n$, and $p = 3$.*

*Moreover, if $X = \mathbf{S}^n \times \mathbf{S}^m$, then $n = m = 1$, 3, or 7 in cases (6) and (7).*

*Proof* If $X$ is totally nonhomologous to zero in $X_G$ mod $p$, then by 2.1 and 6.1 (and a little extra work to prove the inequalities in (4) and the fact that $q < n$ in (3); see the proof of 8.3) we must have one of the cases (1)–(4), or one other possibility: For some $q$ we may have rk $\check{H}^q(F; \mathbf{Z}_p) = 2$ and rk $\check{H}^{2q}(F; \mathbf{Z}_p) = 1$, and we must prove that the ring structure is that of case (1) with $r = q$. In this case a proof similar to the first part of that of 8.2 shows easily that $\check{H}^q(F; \mathbf{Z}_p)$ has a basis $u$, $v$ with $u^2 = 0$, $uv \neq 0$ and $v^2 = Auv$ for some $A \in \mathbf{Z}_p$ with $A = 0$ unless $q$ is even. Putting $w = v - \frac{1}{2} Au$ gives $u^2 = 0 = w^2$ and $uw \neq 0$, which shows that $F \sim_p \mathbf{S}^q \times \mathbf{S}^q$.

Suppose now that $X$ is not totally nonhomologous to zero in $X_G$, so that rk $\check{H}^*(F; \mathbf{Z}_p) \leq 3$. First, assume that $G$ acts trivially on $\check{H}^*(X; \mathbf{Z}_p)$; e.g., $n \neq m$. Then some differential in the spectral sequence of $(X_G, x_G) \to B_G$ must be nontrivial and the multiplicative properties of the spectral sequence show that the first such differential must be

$$d = d_{m-n+1}\colon E_{m-n+1}^{0,m} \to E_{m-n+1}^{m-n+1,n}$$

which becomes, under $E_2^{*,*} \approx E_{m-n+1}^{*,*}$,

$$d\colon \check{H}^0(B_G) \otimes \check{H}^m(X, x) \to \check{H}^{m-n+1}(B_G) \otimes \check{H}^n(X, x).$$

Let $d(1 \otimes b) = A \otimes a$ for $a, b \neq 0$ and $0 \neq A \in \check{H}^{m-n+1}(B_G; \mathbf{Z}_p)$. Then $m$ must be odd, for if $m$ is even, then

$$0 = d(1 \otimes b^2) = 2(1 \otimes b)(A \otimes a) = 2A \otimes ab \neq 0.$$

Since $\check{H}^*(X; \mathbf{Z})$ was assumed to be finitely generated, it follows easily that this differential is induced from that in the *integral* spectral sequence. Since $\check{H}^i(B_G; \check{H}^n(X; \mathbf{Z})) = 0$ for $i$ odd, it follows that $n$ must be even. Thus, for the $\mathbf{Z}_p$-spectral sequence $d(1 \otimes b) = t^r \otimes a$, where $r = (m - n + 1)/2$, and hence $d(t^i \otimes b) = t^{r+i} \otimes a$ and $d(st^i \otimes b) = -st^{r+i} \otimes a$. This kills, in high degrees, two rows of the spectral sequence, so that the remaining row $E_2^{*,n+m}$ survives to $E_\infty^{*,n+m}$. It follows that rk $\check{H}^*(F; x; \mathbf{Z}_p) = 1$ and hence $F \sim_p S^q$ for some $q$. Since $\chi(F) \equiv \chi(X) = 0 \pmod p$ we must have that $q$ is odd, and we are in case (5).

Now suppose that $G$ acts nontrivially on $\check{H}^*(X; \mathbf{Z}_p)$. Then $n = m$. If $T = 1 - g^*$, where $g$ generates $G$, then $T^p = 0$ and hence ker $T \neq 0$. Thus, for an appropriate basis $a, b$ of $\check{H}^n(X; \mathbf{Z}_p)$, we have

$$g^*(a) = a,$$
$$g^*(b) = b - a$$

(since $\det(g^*) = (\det(g^*))^p = \det(g^p)^* = 1$ and $g^* \neq I$). Now $g^*$ is the identity on $\check{H}^{2n}(X; \mathbf{Z}_p)$ so that $ab = g^*(ab) = a(b - a) = ab - a^2$, whence $a^2 = 0$. If $n$ is even, then

$$b^2 = g^*(b^2) = (b - a)^2 = b^2 - 2ab + a^2 = b^2 - 2ab,$$

whence $ab = 0$. However, this violates duality, and hence $n$ must be *odd* and $ab \neq 0$. Note that on $\check{H}^n(X; \mathbf{Z}_p)$ we also have

$$N = 1 + g^* + g^{*2} + \cdots + g^{*(p-1)} = 0$$

by an easy calculation.

Let $x \in F$ and consider the mod $p$ spectral sequence of $(X_G, x_G) \to B_G$. Since $N = 0$ and Im $T =$ ker $T \approx \mathbf{Z}_p$, generated by $a$, we have

$$\check{H}^{2i}(B_G; \check{H}^n(X, x)) = \ker T = \check{H}^n(X, x)^G \approx \mathbf{Z}_p$$

generated by $a$, and

$$\check{H}^{2i+1}(B_G; \check{H}^n(X, x)) = \frac{\check{H}^n(X, x)}{\operatorname{Im} T} = \frac{\check{H}^n(X, x)}{\check{H}^n(X, x)^G} \approx \mathbf{Z}_p$$

with a generator represented by $b$. Since $c = ab$ generates $\check{H}^{2n}(X, x)$, it represents a generator of $\check{H}^i(B_G; \check{H}^{2n}(X, x))$ for all $i$. Now by the computation of products in $\check{H}^*(B_G; \cdot) \approx H^*(G; \cdot)$ given in Cartan and Eilenberg [1, p. 252] we see that multiplication by $t \in H^2(B_G; \mathbf{Z}_p)$ is an isomorphism.

Let $a_0$ denote the element of $E_2^{0,n} \approx \check{H}^n(X, x)^G$ represented by $a$, let $b_0 \in E_2^{1,n} \approx \check{H}^n(X, x)/\check{H}^n(X, x)^G$ be the generator represented by $b$, and let $c_0 \in E_2^{0,2n} \approx \check{H}^{2n}(X, x)$ be represented by $c = ab$. Thus $t^i a_0$ generates $E_2^{2i,n}$, $t^i b_0$ generates $E_2^{2i+1,n}$, $t^i c_0$ generates $E_2^{2i,2n}$, and $st^i c_0$ generates $E_2^{2i+1,2n}$. Computation of products in $E_2$ shows that $a_0 b_0 = sc_0$. (One can also see that $sa_0 = 0$, that $sb_0 = ta_0$ for $p = 3$ and $sb_0 = 0$ for $p > 3$, but we do not need these facts.)

Consider the differential

$$d_{n+1}\colon E_2^{0,2n} = E_{n+1}^{0,2n} \to E_{n+1}^{n+1,n} = E_2^{n+1,n}$$

which has the form

$$d_{n+1}(c_0) = At^{(n+1)/2}\, a_0$$

for some $A \in \mathbf{Z}_p$. Then

$$0 = d_{n+1}(c_0 b_0) = At^{(n+1)/2}\, a_0 b_0,$$

which shows that $A = 0$ and consequently that the spectral sequence degenerates. It follows that $\operatorname{rk} \check{H}^*(F; \mathbf{Z}_p) = 3$. Since $\chi(F) \equiv \chi(X) = 0 \pmod p$ we must have $\chi(F) = 3$ and $p = 3$. Let $q$ and $r$ be the dimensions, both even and possibly equal, for which $\check{H}^*(F, x; \mathbf{Z}_3) \neq 0$. Since multiplication by $t$ is clearly a monomorphism in $\check{H}^*(X_G, x_G)$ it follows as in the proof of 1.5 that $j^*$ is a monomorphism.

There is the exact sequence

$$0 \to E_2^{i,n} \to \check{H}^{i+n}(X_G, x_G) \to E_2^{i-n,2n} \to 0$$

and we define elements $\alpha \in \check{H}^n(X_G, x_G)$ and $\beta \in \check{H}^{n+1}(X_G, x_G)$ to be the images of $a_0$ and $b_0$, and take a representative $\gamma \in \check{H}^{2n}(X_G, x_G)$ for $c_0$. Then a $\mathbf{Z}_3$-basis for $H^*(X_G, x_G)$ is given by $\{t^i\alpha, t^i\beta, t^i\gamma, st^i\gamma\}$. Moreover $\alpha\beta = s\gamma + Kt^{(n+1)/2}\alpha$ where $K \in \mathbf{Z}_3$. (Also $s\alpha = 0$ and $s\beta = t\alpha$, facts which we will not use.)

It is clear that $j^*(\alpha)$ and $j^*(\beta)$ must involve ring generators for $\check{H}^*(F, x)$ [whence $q < n$ in cases (6) and (7)]. If $q = 0$ or $r = 0$, then we are in case (6). Thus suppose that $0 < q \leq r$ and let $u \in \check{H}^q(F, x)$ and $v \in \check{H}^r(F, x)$

be generators. Put

$$j^*(\alpha) = Ast^{q'} \otimes u + Bst^{r'} \otimes v,$$
$$j^*(\beta) = Ct^{q'+1} \otimes u + Dt^{r'+1} \otimes v,$$

where $2q' = n - q - 1$, $2r' = n - r - 1$, and $A, B, C, D \in \mathbf{Z}_3$. Since $q \leq r$ we have $uv = 0 = v^2$. Since $\alpha\beta \neq 0$ we have $0 \neq j^*(\alpha\beta) = ACst^{n-q} \otimes u^2$ and hence $u^2 \neq 0$ and $A, C \neq 0$. This implies case (7). It is of interest to note that if $\beta_3$ denotes the mod 3 Bockstein, then $j^*(\beta_3(\alpha)) = \beta_3(j^*(\alpha)) \neq 0$ and hence $\beta_3(\alpha) = \pm\beta$. (Actually, $\beta_3(\alpha) = \beta$ for our choice of generators.)

For the last statement, let $a, b \in H^n(\mathbf{S}^n \times \mathbf{S}^n; \mathbf{Z})$ be the standard basis elements and put

$$g^*(a) = Aa + Bb,$$
$$g^*(b) = Ca + Db$$

so that $AD - BC = 1$. Since the eigenvalues of $g^*$ are cube roots of unity, we see that the trace $A + D = -1$. The map

$$\mathbf{S}^n \times \mathbf{S}^n \xrightarrow{g} \mathbf{S}^n \times \mathbf{S}^n \xrightarrow{\text{proj}_1} \mathbf{S}^n$$

has bidegree $(A, B)$ and hence the Hopf invariant of the induced map

$$\mathbf{S}^{2n+1} = \mathbf{S}^n \star \mathbf{S}^n \to S\mathbf{S}^n = \mathbf{S}^{n+1}$$

is $AB$. The other projection gives a map of Hopf invariant $CD$. Thus $AB$ and $CD$ are even for $n \neq 1, 3, 7$. This is clearly inconsistent with $AD - BC = 1$ and $A + D = -1$. ▮

*Remark* Suppose that $X$ is finitistic and that $\check{H}^*(X; \mathbf{Z}_3) \approx \check{H}^*(\mathbf{S}^n \times \mathbf{S}^n; \mathbf{Z}_3)$ *additively* for $n$ odd. Let $\mathbf{Z}_3$ act on $X$ and nontrivially on $\check{H}^n(X; \mathbf{Z}_3)$, and suppose that $F \sim_3 \mathbf{P}^2(q)$ with $2q > n$. Then, in the above proof, we still have $j^*(\alpha) = st^{q'} \otimes u$ for an appropriate generator $u \in \check{H}^q(F, x)$. Taking $\beta$ as the Bockstein of $\alpha$, as we may, $j^*(\beta) = t^{q'+1} \otimes u$. Thus $j^*(\alpha\beta) = st^{n-q} \otimes u^2$, which is independent of $j^*(\alpha)$ and $j^*(\beta)$. This implies that $\alpha\beta$ represents a *nontrivial* element in $E_\infty^{1,2n} = E_2^{1,2n} \approx \check{H}^{2n}(X, x; \mathbf{Z}_3)$. Since this element is just $ab$, we have that $ab \neq 0$. Thus we conclude that $X$ has the mod 3 cohomology *ring* of $\mathbf{S}^n \times \mathbf{S}^n$ after all. These remarks are similar to the proof of a general theorem about circle actions in Bredon [23], and they will be important for one of our examples below.

For an example of case (2) of 9.1 note that $\mathbf{CP}^3$ is the unit sphere bundle in a 3-plane bundle $\eta$ over $\mathbf{S}^4$. Let $\mu$ be a stable real vector bundle of even dimension over $\mathbf{S}^4$ with $\eta \oplus \mu$ trivial. Since the decomplexification $\widetilde{KU}(\mathbf{S}^4) \to \widetilde{KO}(\mathbf{S}^4)$ is an isomorphism, $\mu$ has a complex structure. Thus there is a semifree $\mathbf{S}^1$-action, and hence a $\mathbf{Z}_p$-action, on $\mathbf{S}^4 \times \mathbf{S}^m = S(\eta \oplus \mu)$ with $F = S(\eta) = \mathbf{CP}^3$. It suffices for $m$ to be even and $m \geq 8$. By VI.11.1 such an $\mathbf{S}^1$-bundle $\eta \oplus \mu$ corresponds to an equivariant map $\mathbf{S}^4 \to \mathbf{Map}(\mathbf{S}^1, e; \mathbf{O}(m+1), I) = \Omega\mathbf{O}(m+1)$, with the trivial action on $\mathbf{S}^4$. Since $\pi_4(\Omega\mathbf{O}(m+1)) = \pi_5(\mathbf{O}(m+1)) = 0$, this extends to an equivariant map $\mathbf{S}^6 = \mathbf{S}^1 \star \mathbf{S}^4 \to \Omega\mathbf{O}(m+1)$. Thus $\mathbf{S}^1$ also acts semifreely on $\mathbf{S}^6 \times \mathbf{S}^m$, $m \geq 8$ even, with $F = \mathbf{CP}^3$.

We know of no examples for case (3) of 9.1. When $p = 3$ we have the following theorem.

**9.2. Theorem** *There does not exist a smooth $\mathbf{Z}_3$-action on $\mathbf{S}^n \times \mathbf{S}^m$ with $F \sim_3$ point $+ \mathbf{P}^2(q)$.*

***Proof*** Recall that $n$, $m$, and $q$ are even with $q < n \leq m$. By VI.11.5 we must have that $q$ is divisible by $3^{[n/4]}$ and a short calculation shows this to be impossible. ∎

Examples of case (4) of 9.1 can be found in Chapter VI, Exercise 8. Examples of cases (4) and (5) can also be constructed by an obstruction killing argument using VI.11.1; see the proof of VI.11.6.

For examples of case (6) let $n = 1, 3$, or $7$ and regard $\mathbf{S}^n \times \mathbf{S}^n$ as the subspace of $\mathbf{S}^n \times \mathbf{S}^n \times \mathbf{S}^n$ consisting of those triples $(x, y, z)$ with $(xy)z = 1$. In the case $n = 7$ of the Cayley numbers, recall that any two elements lie in an associative subalgebra (see Curtis [1]) and, since $z = (xy)^{-1}$, we have that $(xy)z = x(yz)$ and $z(xy) = 1$ for such triples. The permutation $(x, y, z) \mapsto (z, x, y)$ defines a smooth $\mathbf{Z}_3$-action on $\mathbf{S}^n \times \mathbf{S}^n$ with $F \approx \{x \in \mathbf{S}^n \mid x^3 = 1\} \approx$ point $+ \mathbf{S}^{n-1}$. For another description of this example, see Su [4].

**9.3. Theorem** *Let $\mathbf{Z}_3$ act smoothly on $\mathbf{S}^n \times \mathbf{S}^n$ with $F \sim_3$ point $+ \mathbf{S}^q$. Then $q = n - 1$.*

***Proof*** Recall that $n = 1, 3$, or $7$. Since $q$ is even by 9.1 we have $q = 0$ when $n = 1$. Also $q = 0$ or $q = 2$ when $n = 3$. When $n = 7$, $q$ is divisible by 3 according to VI.11.5 and hence $q$ is either 0 or 6. Thus we must eliminate the possibility that $q = 0$ ($F$ equals three points) when $n = 3$ or $n = 7$. By removing disks about the three fixed points we see that three times the standard $\mathbf{Z}_3$-action on $\mathbf{S}^5$ (for $n = 3$) or $\mathbf{S}^{13}$ (for $n = 7$) would bound a *free* orientable $\mathbf{Z}_3$-action. That is, this standard action would have order 3 in the reduced bordism group $\tilde{\Omega}_*(\mathbf{Z}_3)$. However, the order of this element is known to be $3^2$ for $n = 3$ and $3^4$ for $n = 7$; see Conner and Floyd [8, p. 94]. ∎

Finally, we turn to case (7) of 9.1. This case has been overlooked in the literature. To construct an example, let $\mathbf{Z}_3$ act smoothly on $\mathbf{S}^3 \times \mathbf{S}^3$ with $F = \text{point} + \mathbf{S}^2$, as constructed above. Remove the isolated fixed point, or an open disk about it, to obtain an action of $\mathbf{Z}_3$ on a space $Y$ of the homotopy type of $\mathbf{S}^3 \vee \mathbf{S}^3$ and with fixed set $\mathbf{S}^2$. Let $\varphi\colon \mathbf{S}^3 \to \mathbf{S}^2 \subset Y$ be the Hopf map. We let $\mathbf{Z}_3$ act on $\mathbf{S}^5 = \mathbf{S}^1 * \mathbf{S}^3$, fixing $\mathbf{S}^3$, and attempt to extend $\varphi$ to an equivariant map $\psi\colon \mathbf{S}^5 \to Y$. Since $\mathbf{S}^2$ is homotopically trivial in $Y$ we see that $0 = [\varphi] \in \pi_3(Y)$ and hence $\varphi$ can be extended equivariantly to $\mathbf{Z}_3 * \mathbf{S}^3 \subset \mathbf{S}^1 * \mathbf{S}^3 = \mathbf{S}^5$. The only obstruction to extending $\varphi$ to $\mathbf{S}^5$ is then in $H^{\text{odd}}(\mathbf{Z}_3; \pi_4(Y))$. (The theory for this is in Bredon [14, 17], but in this simple case the extension can easily be constructed ad hoc.) Since $\pi_4(Y) \approx \pi_4(\mathbf{S}^3 \vee \mathbf{S}^3) \approx \pi_4(\mathbf{S}^3 \times \mathbf{S}^3) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$, this obstruction group vanishes and the extension exists. Now put

$$X = Y \cup_\psi \mathbf{D}^6$$

which has a $\mathbf{Z}_3$-action with $F = \mathbf{S}^2 \cup_\varphi \mathbf{D}^4 = \mathbf{CP}^2$. By the remark below the proof of 9.1 it follows that $X \sim_3 \mathbf{S}^3 \times \mathbf{S}^3$. Let us continue to improve this example. Note that $\pi_5(Y) \approx \pi_5(\mathbf{S}^3 \vee \mathbf{S}^3) \approx \mathbf{Z} \oplus \pi_5(\mathbf{S}^3 \times \mathbf{S}^3) \approx \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and, moreover, the free summand can be taken to be generated by the attaching map for the top cell of $\mathbf{S}^3 \times \mathbf{S}^3$. Denote this element by $\sigma$. This summand clearly consists of the invariants of the $\mathbf{Z}_3$-action on $\pi_3(Y)$ and $[\psi]$ is indeed invariant since $g \circ \psi = \psi \circ g \simeq \psi$ for $g \in \mathbf{Z}_3$. Now $\psi$ can be equivariantly modified so that $[\psi]$ changes by three times any invariant element. By changing orientation on $\mathbf{S}^5$ we can also change the sign of $[\psi]$. Thus we may assume that $[\psi] = \sigma$ or $[\psi] = 0$. However, $[\psi] = 0$ is impossible since then $X \simeq \mathbf{S}^3 \vee \mathbf{S}^3 \vee \mathbf{S}^6$ does *not* have the correct cohomology ring. Thus $[\psi]$ can be taken to be $\sigma$ and then $X$ has the *homotopy type* of $\mathbf{S}^3 \times \mathbf{S}^3$. We remark that $\mathbf{CP}^2$ cannot be embedded in a closed manifold homotopy equivalent to $\mathbf{S}^3 \times \mathbf{S}^3$, so that such an example cannot be smoothed. It is also probable that there is no *topological* action on $\mathbf{S}^3 \times \mathbf{S}^3$ homotopy equivalent to this example.

A similar example arises from the $\mathbf{Z}_3$-action on $\mathbf{S}^7 \times \mathbf{S}^7$ with $F = \{x\} + \mathbf{S}^6$. Here we map $\varphi: \mathbf{S}^{11} \to \mathbf{S}^6$ with Hopf invariant 2 and extend it equivariantly to $\mathbf{S}^{13} = \mathbf{S}^1 \star \mathbf{S}^{11} \to Y = \mathbf{S}^7 \times \mathbf{S}^7 - \{x\}$. (Here the extension is clear since $\pi_{12}(\mathbf{S}^7 \vee \mathbf{S}^7) \approx \pi_{12}(\mathbf{S}^7 \times \mathbf{S}^7) = 0$.) Again the extension $\psi$: $\mathbf{S}^{13} \to Y$ can be chosen so that $X = Y \cup_\psi \mathbf{D}^{14}$ has the homotopy type of $\mathbf{S}^7 \times \mathbf{S}^7$. Moreover $F = \mathbf{S}^6 \cup_\varphi \mathbf{D}^{12} \sim_3 \mathbf{P}^2(6)$.

***Remarks*** Most of 9.1 is due to Su [4], but, because of a sign error, case (7) was thought to have been ruled out.

Theorem 9.3 was conjectured in Bredon [21] and proved in Bredon [22]. It is clear that the analog of 9.1 holds for $\mathbf{S}^1$-actions over the rationals, with cases (6) and (7) not occuring.

## 10. CIRCLE ACTIONS ON A PRODUCT OF ODD-DIMENSIONAL SPHERES

Suppose that $X \sim_{\mathbf{Q}} \mathbf{S}^{n_1} \times \mathbf{S}^{n_2} \times \cdots \times \mathbf{S}^{n_k}$, where the $n_i$ are *odd* and that $\mathbf{S}^1$ acts on $X$ with finitely many orbit types and with $X$ and $X/\mathbf{S}^1$ finitistic. There are the obvious product actions on products of spheres. Other interesting examples are the actions by conjugation of circle subgroups of a connected Lie group $L$ on the underlying space of $L$. In these cases, the fixed set $F$ also has the rational cohomology ring of a product of $k$ odd-dimensional spheres. It is natural to ask whether this holds generally; see Bredon [22, 23]. A counterexample will be given below, but it is quite easy to prove this if we also assume that $X$ is totally nonhomologous to zero in $X_G$ over $\mathbf{Q}$. Note that this is the case when $n_1 \leq n_2 \leq \cdots \leq n_k$ and $n_1 + n_2 > n_k$ since then $\check{H}^*(X; \mathbf{Q})$ is generated by transgressive classes which must transgress to zero when $F \neq \emptyset$.

**10.1. Theorem** *Let $\mathbf{S}^1$ act on $X \sim_{\mathbf{Q}} \mathbf{S}^{n_1} \times \mathbf{S}^{n_2} \times \cdots \times \mathbf{S}^{n_k}$ with all $n_i$ odd, with finitely many orbit types and with $X$ and $X/\mathbf{S}^1$ finitistic. Suppose that $X$ is totally nonhomologous to zero in $X_G$ over the rationals. Then $F \sim_{\mathbf{Q}} \mathbf{S}^{r_1} \times \mathbf{S}^{r_2} \times \cdots \times \mathbf{S}^{r_k}$ with $r_i \leq n_i$ odd.*

***Proof*** Let the classes $a_1, \ldots, a_k$ be an exterior basis for $\check{H}^*(X; \mathbf{Q})$, with $n_i = \deg a_i$. Let $\alpha_i = \varphi(a_i)$, where $\varphi$ is a cohomology extension of the fiber, and put

$$j^*(\alpha_i) = \sum_q t^q \otimes b_{i,q},$$

where $b_{i,q} \in \check{H}^{n_i-2q}(F; \mathbf{Q})$. Since $\alpha_1\alpha_2\cdots\alpha_k$ represents $a_1a_2\cdots a_k \neq 0$ we have

$$0 \neq j^*(\alpha_1\alpha_2\cdots\alpha_k) = \left(\sum_q t^q \otimes b_{1,q}\right)\cdots\left(\sum_q t^q \otimes b_{k,q}\right)$$

and hence *some* product

$$b_{1,q_1}b_{2,q_2}\cdots b_{k,q_k}$$

is *nonzero*. Thus $b_{1,q_1}, \ldots, b_{k,q_k}$ generate an exterior algebra, which must equal $\check{H}^*(F; \mathbf{Q})$ since both have rank equal to $2^k = \operatorname{rk} \check{H}^*(X; \mathbf{Q})$. Thus $F \sim_{\mathbf{Q}} \mathbf{S}^{r_1} \times \cdots \times \mathbf{S}^{r_k}$, where $r_i = \deg b_{i,q_i} = n_i - 2q_i$. ∎

Under appropriate hypotheses, one can also prove such a result over the integers. The compactness condition below is included to ensure that the Universal Coefficient Theorem is valid over the rationals. Other conditions which would ensure this could be substituted; e.g., locally smooth actions on open or closed manifolds. Also note that the condition on fixed sets of the $\mathbf{Z}_{p^i}$ holds for locally smooth actions on compact manifolds.

**10.2. Theorem** *Let $G = \mathbf{S}^1$ act on the compact space $X \sim_{\mathbf{Z}} \mathbf{S}^{n_1} \times \mathbf{S}^{n_2} \times \cdots \times \mathbf{S}^{n_k}$ with all $n_i$ odd and with finitely many orbit types. Assume that $X$ is totally nonhomologous to zero in $X_G$ over $\mathbf{Q}$. Also assume that for each prime $p$ and integer $i$, $\check{H}^*(F(\mathbf{Z}_{p^i}, X); \mathbf{Z})$ has finite type. Then $F \sim_{\mathbf{Z}} \mathbf{S}^{r_1} \times \mathbf{S}^{r_2} \times \cdots \times \mathbf{S}^{r_k}$ with $r_i \leq n_i$ odd.*

*Proof* For a given prime $p$, consider $F_i = F(\mathbf{Z}_{p^i}, X)$. We have

$$\operatorname{rk} \check{H}^*(X; \mathbf{Z}_p) \geq \operatorname{rk} \check{H}^*(F_i; \mathbf{Z}_p).$$

Since $F_i$ has finite type we also have

$$\operatorname{rk} \check{H}^*(F_i; \mathbf{Z}_p) \geq \operatorname{rk} \check{H}^*(F_i; \mathbf{Q})$$

with equality iff $F_i$ has no $p$-torsion. However, $\mathbf{S}^1$ acts on $F_i$ with fixed set $F = F(\mathbf{S}^1, X)$ and hence

$$\operatorname{rk} \check{H}^*(F_i; \mathbf{Q}) \geq \operatorname{rk} \check{H}^*(F; \mathbf{Q}) = \operatorname{rk} \check{H}^*(X; \mathbf{Q}) = \operatorname{rk} \check{H}^*(X; \mathbf{Z}_p).$$

Thus these are all *equalities* and $F_i$ has no $p$-torsion. It follows that the mod $p$ reduction $\check{H}^*(F_i; \mathbf{Z}) \to \check{H}^*(F_i; \mathbf{Z}_p)$ is onto, and also that the squares of odd-dimensional classes in $\check{H}^*(F_i; \mathbf{Z}_p)$ are zero (which is trivial for $p \neq 2$).

Now we claim that $F_i$ has the mod $p$ cohomology ring of a product of $k$ odd-dimensional spheres. We prove this by induction on $i$. Thus let $Y = F_i$ and consider the $\mathbf{Z}_p$-action on $Y$ with fixed set $K = F_{i+1}$. Since this extends to a circle group action, the action on $\check{H}^*(Y; \mathbf{Z})$ is trivial. Consider the spectral sequences $E_r^{*,*}(\mathbf{Z}_p)$ and $E_r^{*,*}(\mathbf{Z})$ of $Y_{\mathbf{Z}_p} \to B_{\mathbf{Z}_p}$ with respect to $\mathbf{Z}_p$ and $\mathbf{Z}$ coefficients, respectively. Since $\operatorname{rk} \check{H}^*(Y; \mathbf{Z}_p) = \operatorname{rk} \check{H}^*(K; \mathbf{Z}_p)$, as seen above, $E_r^{*,*}(\mathbf{Z}_p)$ degenerates. Since $E_2^{a,b}(\mathbf{Z}) \to E_2^{a,b}(\mathbf{Z}_p)$ is a monomorphism for $a > 0$, it follows that $E_r^{*,*}(\mathbf{Z})$ also degenerates. In particular, the edge homorphism $\check{H}^*(Y_{\mathbf{Z}_p}; \mathbf{Z}) \to E_2^{0,*}(\mathbf{Z}) \approx \check{H}^*(Y; \mathbf{Z})$ is onto and hence

$$\check{H}^*(Y_{\mathbf{Z}_p}; \mathbf{Z}) \to \check{H}^*(Y; \mathbf{Z}_p)$$

is onto. Now we try to repeat, for $\mathbf{Z}_p$-coefficients, the proof of 10.1. By the above remarks, the $\alpha_i \in \check{H}^*(Y_{\mathbf{Z}_p}; \mathbf{Z}_p)$ can be chosen to come from $\check{H}^*(Y_{\mathbf{Z}_p}; \mathbf{Z})$. Hence $j^*(\alpha_i)$ is in the image of $\check{H}^*(B_{\mathbf{Z}_p}; \mathbf{Z}) \otimes \check{H}^*(K; \mathbf{Z}) \approx \check{H}^*(K_{\mathbf{Z}_p}; \mathbf{Z}) \to \check{H}^*(K_{\mathbf{Z}_p}; \mathbf{Z}_p) \approx \check{H}^*(B_{\mathbf{Z}_p}; \mathbf{Z}_p) \otimes \check{H}^*(K; \mathbf{Z}_p)$ (since $K$ has no $p$-torsion). Thus $j^*(\alpha_i)$ *involves only odd-dimensional classes* in $\check{H}^*(K; \mathbf{Z}_p)$. From this it is clear that the proof of 10.1 works to prove our claim. Since $F = F(\mathbf{S}^1, X) = F(\mathbf{Z}_{p^i}, X)$ for each $p$ and for sufficiently large $i$, we conclude that $F$ has no torsion and

$$F \sim_p \mathbf{S}^{r_1} \times \mathbf{S}^{r_2} \times \cdots \times \mathbf{S}^{r_k}$$

for each prime $p$ (with the $r_i$ possibly depending on $p$).

Let us choose classes $u_1, \ldots, u_k \in \check{H}^*(F; \mathbf{Z})$ which induce an exterior basis for $\check{H}^*(F; \mathbf{Q})$. There is a unique minimal *direct summand* $A$ of $\check{H}^*(F; \mathbf{Z})$ containing the $u_i$ and it is clear that, by changing the $u_i$, we may take the $u_i$ to be a $\mathbf{Z}$ *basis* of $A$. Since the mod $p$ reductions of the $u_i$ are then $\mathbf{Z}_p$-independent, they form an *exterior basis* of $\check{H}^*(F; \mathbf{Z}_p) \approx \check{H}^*(F; \mathbf{Z}) \otimes \mathbf{Z}_p$. Let $\Lambda$ be the exterior algebra on the $u_i$ and consider the canonical homomorphism $\Lambda \to H = \check{H}^*(F; \mathbf{Z})$, which is a monomorphism since $\Lambda \otimes \mathbf{Q} \to H \otimes \mathbf{Q}$ is an isomorphism. We shall regard $\Lambda \to H$ as inclusion. For each prime $p$ consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \xrightarrow{p} & \Lambda & \longrightarrow & \Lambda \otimes \mathbf{Z}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \approx & & \\
0 & \longrightarrow & H & \xrightarrow{p} & H & \longrightarrow & H \otimes \mathbf{Z}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H/\Lambda & \xrightarrow{p} & H/\Lambda & & & &
\end{array}$$

with exact rows and columns. This shows that $H/\Lambda$ is $p$-divisible for all $p$. Since $H/\Lambda$ is finitely generated, this implies that $H/\Lambda = 0$ so that $\Lambda = H = \check{H}^*(F; \mathbf{Z})$. ∎

**Example** Let $\tau$ be the tangent bundle of $\mathbf{S}^8$ and let $\varepsilon$ be the trivial 2-plane bundle on $\mathbf{S}^8$. Let $\mathbf{S}^1$ act trivially on $\tau$ and nontrivially on $\varepsilon$ over the trivial action on $\mathbf{S}^8$. Map $\varphi: \mathbf{S}^3 \times \mathbf{S}^5 \to \mathbf{S}^8$ with degree 1, let $\eta = \varphi^*(\tau \oplus \varepsilon)$ be the pull-back of $\tau \oplus \varepsilon$, with its $\mathbf{S}^1$-action, and let $X = S(\eta)$ be the total space of the unit sphere bundle of $\eta$. Since $\tau \oplus \varepsilon$ is trivial, $X \approx \mathbf{S}^3 \times \mathbf{S}^5 \times \mathbf{S}^9$. The fixed set $F$ of $\mathbf{S}^1$ on $X$ is the 7-sphere bundle $S(\varphi^*\tau)$ over $\mathbf{S}^3 \times \mathbf{S}^5$ and it is easily seen from the spectral sequence of this bundle that $H^i(F; \mathbf{Q}) \approx \mathbf{Q}$ for $i = 0, 3, 5, 10, 12, 15$ and it is trivial otherwise. Only the products *into* $H^{15}(F; \mathbf{Q})$ are nonzero. This example shows that the condition in 10.1, that $X$ be totally nonhomologous to zero in $X_G$, cannot be dropped. We remark that $\pi_i(F) \otimes \mathbf{Q} \approx \mathbf{Q}$ for $i = 3, 5$, and 7 and is zero otherwise, and that the Whitehead product $(\pi_3(F) \otimes \mathbf{Q}) \times (\pi_5(F) \otimes \mathbf{Q}) \to \pi_7(F) \otimes \mathbf{Q}$ is nontrivial. This answers a question in Bredon [23].

## 11. AN APPLICATION TO EQUIVARIANT MAPS

Suppose that we are given linear involutions on $\mathbf{S}^n$ and on $\mathbf{S}^m$ and that we have an equivariant map $\varphi: \mathbf{S}^m \to \mathbf{S}^n$. Then there is an induced involution on the mapping cone $X = \mathbf{S}^n \cup_\varphi \mathbf{D}^{m+1}$ whose fixed point set is the mapping cone of the restriction of $\varphi$ to the fixed sets in $\mathbf{S}^m$ and $\mathbf{S}^n$. This motivates the study of involutions on such spaces.

**11.1. Theorem** *Suppose that $T$ is an involution on the finitistic space $X$ with fixed set $F$. Let $x \in F$ and assume that*

$$\check{H}^i(X, x; \mathbf{Z}_2) \approx \begin{cases} \mathbf{Z}_2 & \text{for } i = n,\ n+k, \\ 0 & \text{otherwise,} \end{cases}$$

*and that*

$$Sq^k: \check{H}^n(X, x; \mathbf{Z}_2) \to \check{H}^{n+k}(X, x; \mathbf{Z}_2)$$

*is nontrivial. Suppose that*

$$\check{H}^i(F, x; \mathbf{Z}_2) \approx \begin{cases} \mathbf{Z}_2 & \text{for } i = q,\ q+r, \\ 0 & \text{otherwise,} \end{cases}$$

*and that $k \neq r$. Then $q + r \leq n$. If $r < k$ then $2r = k$ and $Sq^r$: $\check{H}^q(F, x; \mathbf{Z}_2)$*

$\to \check{H}^{q+r}(F, x; \mathbf{Z}_2)$ *is nontrivial. If* $r > k$, *then* $r = k + 2ki$ *for some integer* $i > 0$. (*The possibility that* $k > 0$ *but* $r = 0$ *and* $\check{H}^q(F, x; \mathbf{Z}_2) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *is also excluded.*)

***Proof*** From the structure of the Steenrod algebra, we know that $k$ is a power of 2, and in fact that $k = 1, 2, 4$, or 8. Let $a \in \check{H}^n(X, x)$ and $b \in \check{H}^{n+k}(X, x)$ be generators, so that $b = Sq^k(a)$. Let $u \in \check{H}^q(F, x)$ and $v \in \check{H}^{q+r}(F, x)$ be generators. Let $\alpha \in \check{H}^n(X_G, x_G)$ represent $a$ and put $\beta = Sq^k(\alpha) \in \check{H}^{n+k}(X_G, x_G)$ which represents $b$. By the Leray–Hirsch Theorem 1.4, $\check{H}^*(X_G, x_G)$ is the free $\check{H}^*(B_G) = \mathbf{Z}_2[t]$ module with basis $\alpha$ and $\beta$.

First assume either that $r > k$ or that $r \le k$ but $Sq^r(u) = 0$. Then $Sq^i(u) = 0$ for $0 < i \le k$. Then note that $j^*(\alpha)$ can be *neither* $t^{n-q} \otimes u$ nor $t^{n-q-r} \otimes v$ for, if it were, then $j^*(\beta) = j^*(Sq^k(\alpha)) = Sq^k(j^*(\alpha))$ would depend on $j^*(\alpha)$ over $\check{H}^*(B_G)$. Thus

$$j^*(\alpha) = t^{n-q} \otimes u + t^{n-q-r} \otimes v$$

and, in particular, $n \ge q + r$. Now for $i < k$, $Sq^i(\alpha)$ is either $t^i\alpha$ or 0. However

$$j^*(Sq^i(\alpha)) = Sq^i(j^*(\alpha)) = \binom{n-q}{i} t^{n-q+i} \otimes u + \binom{n-q-r}{i} t^{n-q-r+i} \otimes v.$$

It follows that

$$\binom{n-q}{i} \equiv \binom{n-q-r}{i} \pmod 2$$

for all $0 < i < k$, and that [since $\beta = Sq^k(\alpha)$]

$$\binom{n-q}{k} \not\equiv \binom{n-q-r}{k} \pmod 2.$$

Putting $k = 2^m$, this means that the dyadic expansions of $n - q$ and $n - q - r$ are equal below the $m$th term and the $m$th terms are different. Thus $r = k +$ (higher powers of 2), that is, $r = k + 2ki$ for some $i > 0$, since $r \ne k$.

If $r \ge k$, then we are done. If $r < k$, then the above contradiction shows that $Sq^r(u) = v$. Thus $r$ is a power of 2 and we must show that $2r = k$ and that $n \ge q + r$. Now $j^*(\alpha) \ne t^{n-q} \otimes u$ since otherwise $Sq^r(\alpha)$ would be independent of $\alpha$, contrary to the fact that $r < k$. Also $j^*(\alpha) \ne t^{n-q-r} \otimes v$ since otherwise $\beta = Sq^k(\alpha)$ would depend on $\alpha$ over $\check{H}^*(B_G)$. (This argument also shows that $r \ne 0$.) Thus we have again that

$$j^*(\alpha) = t^{n-q} \otimes u + t^{n-q-r} \otimes v$$

and hence that $n \geq q + r$. Suppose that $2r < k$. Then $Sq^{2r}(\alpha)$ is either 0 or $t^{2r}\alpha$. However

$$j^*(Sq^{2r}(\alpha)) = Sq^{2r}(j^*(\alpha))$$
$$= \binom{n-q}{2r} t^{n-q+2r} \otimes u + \left[\binom{n-q}{r} + \binom{n-q-r}{2r}\right] t^{n-q+r} \otimes v,$$

and the fact that

$$\binom{n-q}{2r} \not\equiv \binom{n-q}{r} + \binom{n-q-r}{2r} \pmod 2$$

for $r$ a power of 2, provides a contradiction; compare the proof of 8.2. ∎

We now turn to the applications of this result to equivariant maps between spheres. For any $n$ let us denote by

$$T_u = \begin{bmatrix} -I_u & 0 \\ 0 & I_{n+1-u} \end{bmatrix}$$

the reflection in the first $u$ coordinates of $\mathbf{S}^n \subset \mathbf{R}^{n+1}$. Thus $F(T_u, \mathbf{S}^n) \approx \mathbf{S}^{n-u}$.

**11.2. Theorem** *Let $\varphi$: $(\mathbf{S}^{m+1}, T_u) \to (\mathbf{S}^m, T_w)$ be equivariant and suppose that $0 \neq [\varphi] \in \pi_1{}^s = \lim \pi_{n+1}(\mathbf{S}^n)$. Then either $u - w \equiv 0 \pmod 4$ or $u - w = 1$. In the latter case the restriction $\varphi^G$: $\mathbf{S}^{m-w} \to \mathbf{S}^{m-w}$, to the fixed point sets, has degree congruent to 2 modulo 4. Moreover, if $u \neq w$, then $u \geq 2$.*

*Proof* Let $y \in \mathbf{S}^{m+1}$ be a fixed point and put $x = \varphi(y)$. Since $\check{H}^*(\mathbf{S}^m{}_G, x_G; \mathbf{Z}_2)$ is generated over $\check{H}^*(B_G; \mathbf{Z}_2)$ by an element of degree $m$ it follows that the map

$$\varphi_G{}^*: \check{H}^*(\mathbf{S}^m{}_G, x_G) \to \check{H}^*(\mathbf{S}^{m+1}_G, y_G)$$

is trivial. Since $j^*$ is an isomorphism in high degrees, the map $(\varphi^G)^*$: $\check{H}^*(\mathbf{S}^{m-w}, x; \mathbf{Z}_2) \to \check{H}^*(\mathbf{S}^{m-u+1}, y; \mathbf{Z}_2)$ is trivial. That is, $\deg \varphi^G$ is even when $u - w = 1$.

If $X = \mathbf{S}^m \cup_\varphi \mathbf{D}^{m+2}$ is the mapping cone of $\varphi$, then the fixed set $F = X^G = \mathbf{S}^{m-w} \cup_{\varphi^G} \mathbf{D}^{m-u+2}$. Since $0 \neq [\varphi] \in \pi_1{}^s \approx \mathbf{Z}_2$, it follows that $Sq^2: \check{H}^m(X; \mathbf{Z}_2) \to \check{H}^{m+2}(X; \mathbf{Z}_2)$ is an isomorphism. Thus (even if $u - w = 1$) the involution on $X$ satisfies the hypotheses of 11.1 with $n = m$, $k = 2$ and $r = |u - w - 2|$. Thus there are the two possibilities: $r \equiv 2 \pmod 4$

(which includes the case $r = k$), and $r = 1$ with $Sq^1 \neq 0$ on $F$. The case $r \equiv 2 \pmod 4$ implies that $u - w \equiv 0 \pmod 4$. Thus suppose that $r = 1$; that is, $u - w = 1$ or $u - w = 3$.

If $u - w = 3$, then $\varphi^G\colon \mathbf{S}^{m-w-2} \to \mathbf{S}^{m-w}$ is inessential, contrary to the fact that $Sq^1 \neq 0$ on $F$. Thus $u - w = 1$ and then $Sq^1 \neq 0$ is equivalent to $\varphi^G\colon \mathbf{S}^{m-w} \to \mathbf{S}^{m-w}$ having degree congruent to 2 modulo 4. (This follows immediately from the fact that $\check{H}^i(F, x; \mathbf{Z}) \approx \mathbf{Z}_d$ for $i = m - w + 1$ and is 0 otherwise, where $d = \deg \varphi^G$, and the fact that $Sq^1$ is the mod 2 Bockstein.) The last statement of the theorem follows from the inequality $q + r \leq n$ when $k \neq r$ of 11.1. ▮

In exactly the same way, the cases $k = 4$ and $k = 8$ of 11.1 yield the next two results. The reader may verify that the case $k = 1$ yields nothing of interest.

**11.3. Theorem** *Let $\varphi\colon (\mathbf{S}^{m+3}, T_u) \to (\mathbf{S}^m, T_w)$ be equivariant and suppose that $[\varphi] \in \pi_3{}^s \approx \mathbf{Z}_{24}$ is an odd multiple of a generator. Then either $u - w \equiv 0 \pmod 8$ or $u - w = 2$. In the latter case, $0 \neq [\varphi^G] \in \pi_1{}^s \approx \mathbf{Z}_2$. Moreover, if $u \neq w$, then $u \geq 4$.* ▮

**11.4. Theorem** *Let $\varphi\colon (\mathbf{S}^{m+7}, T_u) \to (\mathbf{S}^m, T_w)$ be equivariant and suppose that $[\varphi] \in \pi_7{}^s \approx \mathbf{Z}_{240}$ is an odd multiple of a generator. Then either $u - w \equiv 0 \pmod{16}$ or $u - w = 4$. In the latter case $[\varphi^G] \in \pi_3{}^s \approx \mathbf{Z}_{24}$ is an odd multiple of a generator. Moreover, if $u \neq w$, then $u \geq 8$.* ▮

*Remark* Theorem 11.1 was proved in Bredon [10]. The present application to equivariant maps is from Bredon [21], where 11.2 was inadvertently misstated. These results can be combined with homotopy methods to yield a great deal of detailed information about equivariant maps between spheres with linear involutions. Reports on some of this information may be found in Bredon [15, 20].

## EXERCISES FOR CHAPTER VII

**1.** If $\mathbf{Z}_5$ acts on $\mathbf{Cay\,P}^2$, show that the fixed point set has at most two components.

**2.** Suppose that $\mathbf{Z}_3$ acts on the finitistic space $X \sim_3 \mathbf{QP}^2$ with $F \sim_3 \mathbf{CP}^2$. Suppose that $P_3{}^1 = 0$ on $X$. Show that $\check{H}^4(X; \mathbf{Z}_3) \to \check{H}^4(F; \mathbf{Z}_3)$ is trivial.

**3.** Give an example of a semifree $S^1$-action on a space $X$ whose integral cohomology ring is that of the one-point union $S^4 \vee S^6$ and with fixed point set $F = CP^2$.

**4.** For $p$ an odd prime, suppose that $Z_p$ acts on the finitistic space $X \sim_p QP^h$ and that there are exactly $(p + 1)/2$ components $F_i$ of $F$. Show that some component $F_0 \sim_p QP^k$ for some $k \geq 0$.

**5.** Let $Z_2$ act on the finitistic space $X \sim_2 S^n \times S^m$, $n \leq m$, and assume that $F \sim_2 P^3(q)$. Prove the assertion in 8.1 that $n > q$. If $n < 2q$, then show that $2q \leq m < 3q$.

**6.** Suppose that $X$ is finitistic and has the mod 2 cohomology ring of a product of $2k$ copies of $S^n$. Suppose that $G = Z_2$ acts on $X$ and that $\operatorname{rk} \check{H}^n(X; Z_2)^G = k$. Show that $X^G$ is nonempty and that it has the mod 2 cohomology ring of a product of $k$ copies of $S^n$.

**7.** For $r = 1, 3$, or $7$ let $\varphi$: $(S^{m+r}, T_u) \to (S^m, T_w)$ be equivariant, where $T_v$ is as in Section 11. Suppose that $[\varphi] \in \pi_r^s$ is an odd multiple of a generator and that $u \leq r$ (so that $u = w$ by the results of Section 11). Show that $[\varphi^G] \in \pi_r^s$ is also an odd multiple of a generator.

**8.** For an odd prime $p$ let $X \sim_p L^{2n+1}$ mean that $\check{H}^*(X; Z_p) \approx Z_p[a, b]/(a^2, b^{n+1})$ as a ring, where $\deg a = 1$ and $b = \beta_p(a)$, where $\beta_p$ is the mod $p$ Bockstein. If $Z_p$ acts on the finitistic space $X \sim_p L^{2n+1}$, show that $F$ is either empty or is the union of at most $p$ components $F_i \sim_p L^{2k_i+1}$ with $\sum(k_i + 1) = n + 1$.

**9.** For an odd prime $p$ let $X \sim_p L^{2n}$ mean that $\check{H}^*(X; Z_p) \approx Z_p[a, b]/(a^2, b^{n+1}, ab^n)$ as a ring, where $\deg a = 1$ and $b = \beta_p(a)$. If $Z_p$ acts on the finitistic space $X \sim_p L^{2n}$, show that the fixed point set $F$ is nonempty and is the union of at most $p$ components $F_i$ with $F_1 \sim_p L^{2k_1}$ and $F_i \sim_p L^{2k_i+1}$ for $i \neq 1$ and with $\sum(k_i + 1) = n + 1$.

**10.** Suppose that $Z_3$ acts on the finitistic space $X \sim_3 \mathbf{Cay}\, P^2$ and that $F$ is disconnected. Show that either $F \sim_3$ (three points) or $F \sim_3$ (point + $S^6$).

**11.** Suppose that $Z_2$ acts on the finitistic space $X$ with $\check{H}^*(X; Z_2) \approx H^*(S^n \vee S^{n+k}; Z_2)$ and with $\check{H}^*(F; Z_2) \approx H^*(S^q \vee S^{q+r}; Z_2)$ as graded groups (additively only). Assume that $q \leq n < q + r \leq n + k$. If $r < k$, show that $Sq^r = 0$ on $F$. If $r = k$ show that $Sq^r = 0$ on $F$ iff $Sq^k = 0$ on $X$.

**12.** Construct a compact space $X \sim_{\mathbf{Z}} \mathbf{S}^3 \vee \mathbf{S}^5$ having an involution with $F \approx \mathbf{CP}^2$. Show that such a space $X$ cannot have the homotopy type of $\mathbf{S}^3 \vee \mathbf{S}^5$.

**13.** If $T$ is an involution on $\mathbf{S}^3 \times \mathbf{S}^5$ with $F \approx \mathbf{RP}^3$, show that $T$ acts nontrivially on $H^3(\mathbf{S}^3 \times \mathbf{S}^5; \mathbf{Z})$.

**14.** Let $\mathbf{S}^1 = \mathrm{U}(1) \subset \mathrm{U}(3)$ act by conjugation on $X = \mathrm{SU}(3) \sim_{\mathbf{Z}} \mathbf{S}^3 \times \mathbf{S}^5$. Show that this is a semifree action with $F \approx \mathbf{S}^1 \times \mathbf{S}^3$. Let $\alpha \in \check{H}^3(X_{\mathbf{S}^1}; \mathbf{Z})$ denote the (unique up to sign) representative of a generator of $\check{H}^3(X; \mathbf{Z}) \approx \mathbf{Z}$. For an appropriate choice of generators $u \in \check{H}^1(F; \mathbf{Z})$ and $v \in H^3(F; \mathbf{Z})$ show that $j^*(\alpha) = 1 \otimes v + t \otimes u$.

**15.** Show that $\mathbf{RP}^n$ does not support a nontrivial semifree $\mathbf{S}^1$-action.

**16.** Let $T$ be a free involution on a closed, oriented $4n$-manifold $M^{4n}$. Consider the intersection form $\langle x, y \rangle = x \cdot Ty$ on $H_{2n}(M^{4n}; \mathbf{Z})$. Show that this bilinear form is symmetric and even (i.e., $\langle x, x \rangle$ is even for all $x$).

**17.** Define an action of $\mathbf{S}^1$ on $\mathbf{S}^3 \times \mathbf{S}^3$ by putting $z(p, q) = (p, pzp^{-1}qz^{-1})$ and show that the fixed set $F \approx \mathbf{S}^1 \times \mathbf{S}^3$. Show that this action extends to $\mathbf{S}^5 \times \mathbf{S}^3$, freely outside $\mathbf{S}^3 \times \mathbf{S}^3$ and hence with the same fixed set $F$. (Note that the original action purposefully has a kernel $\mathbf{Z}_2$.) For $x \in F$ let $\alpha \in \check{H}^3((\mathbf{S}^5 \times \mathbf{S}^3)_G, x_G; \mathbf{Z})$, $u \in \check{H}^1(F, x; \mathbf{Z})$ and $v \in \check{H}^3(F, x; \mathbf{Z})$ be generators, which are unique up to sign. Show that, for an appropriate choice of these generators, we have

$$j^*(\alpha) = 1 \otimes v + 2t \otimes u$$

in $\check{H}^*(B_G; \mathbf{Z}) \otimes \check{H}^*(F, x; \mathbf{Z})$. (Note, in particular, that this implies that the action is not equivalent to an obvious product action on $\mathbf{S}^5 \times \mathbf{S}^3$.) For any choice of $\beta \in \check{H}^5((\mathbf{S}^5 \times \mathbf{S}^3)_G, x_G; \mathbf{Z})$ representing a generator of $\check{H}^5(\mathbf{S}^5 \times \mathbf{S}^3; \mathbf{Z})$ show that

$$j^*(\beta) = \pm\big(mt \otimes v + 2(m - 1)t^2 \otimes u\big),$$

where $m$ is an arbitrary integer depending on the choice of $\beta$. In particular, $\beta$ can be chosen so that

$$j^*(\beta) = t \otimes v.$$

For $n \geq 2$, show how to construct similar actions on $\mathbf{S}^{2n+1} \times \mathbf{S}^3$ with

$F \approx \mathbf{S}^1 \times \mathbf{S}^3$, which are homologically distinct (even over the rationals) from the products of $\mathbf{S}^1$-actions on $\mathbf{S}^{2n+1}$ and on $\mathbf{S}^3$.

Show, however, that our original action on $\mathbf{S}^3 \times \mathbf{S}^3$ is equivalent to a product action. Why does this not contradict the statements about the (arbitrary) extended action to $\mathbf{S}^5 \times \mathbf{S}^3$?

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