Characteristic Fixed-Point Sets of Semifree Actions on Spheres

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Abstract

A group action is semifree if it is free away from its fixed-point set. P. A. Smith showed that when a finite group of order q acts semifreely on a sphere, the fixed set is a mod q homology sphere. Conversely, given a mod q homology sphere as a subset of a sphere, one may try to construct a group action on the sphere fixing the subset. The converse question was first systematically studied by Jones and then by many others. In this note, we give new numerical congruences satisfied by the homology of the fixed sets and give a definitive solution to the problem for characteristic fixed-point sets. © 1999 John Wiley & Sons, Inc.

Introduction

This paper is a contribution to the "Smith-Jones" philosophy that the theory of finite transformation groups is dominated by phenomena depending on homotopy theory only involving the primes dividing the order of the group. We are interested in measuring precisely the extent to which this philosophy is true, that is, the study of the obstructions which arise. The problem of producing group actions on homology-equivalent manifolds is called *homology propagation* and was studied in [3, 6, 12, 13, 14, 18, 22, 29] and elsewhere; see [30] for a survey. The complementary problem is more classical: replacing the fixed set of a G-action by a homology-equivalent manifold (this is called the *replacement problem*). This has not yet been systematically attacked; there is a large literature dealing with the special case of the sphere [1, 2, 7, 17, 24, 25, 31] and for homotopy-equivalent manifolds [8], with aspects of a general theory to appear in a sequel to that paper.

This paper will deal with a very simple type of fixed-point set, where the action is on the sphere and where the equivariant normal bundle of the fixed set is trivial. (The precise class of fixed sets is called characteristic; for the definition, see Section 1.) By restricting our attention to this special case, we are able to isolate precisely the obstruction that arises from numerical differences between the homology of the fixed set and its putative replacement. This method is ultimately an adaptation of our work on propagation [14] together with some additional work dealing with secondary invariants that arise from "relative propagation" problems.

Communications on Pure and Applied Mathematics, Vol. LII, 0935–0947 (1999)
 CCC 0010–3640/99/080935-13

All this done, one can consider what happens for a general semifree action. Here we analyze some specific examples of quaternionic group actions and show that the possible fixed sets of smooth actions are more restricted than had been believed [25]. In [31] the general PL, locally linear, fixed-point set theory is reduced to the case considered here. We speculate that the general study of semifree actions on spheres should involve the L-theory congruences given here. We believe our methods will ultimately generalize to compute the numerical obstructions to replacement in general (at least in the PL locally linear case).

In this paper we analyze only the case where the codimension of the fixed set is even and greater than 2. The case of codimension 2 involves knot theory, and the remaining case of an orientation-reversing involution was analyzed in the thesis of David Chase [9].

1 Characteristic Fixed-Point Sets

Throughout this paper G will denote a finite group.

A G-action on a set X is *semifree* if it is free off the fixed set; i.e., for all $x \in X - X^G$ and for all $g \in G - \{e\}$, gx does not equal x. A G-map $f : X \to Y$ is *isovariant* if $gf(x) = f(x)$ implies $gx = x$. A semifree G-action on a manifold M^n is *normally trivial* if the fixed set is a submanifold Σ^k that has a neighborhood G-homeomorphic to $\Sigma^k \times D^{n-k}$ for some linear G-action on D^{n-k} that is free off $\{0\} \in D^{n-k}$. For example, a semifree linear action on S^n is normally trivial. Note if G acts semifreely and normally trivially on M, then G acts freely and linearly on the sphere normal to the fixed set, which restricts the possible G that can arise (see [32]). For example, the 2-Sylow subgroup must be cyclic or generalized quaternionic, and the p-Sylow subgroup must be cyclic for p odd.

In general, a subset of a G-sphere is *characteristic* if it is the transverse inverse image of a invariant sphere under an isovariant map to a linear G-sphere. As such they have been useful in the study of G-spheres (see [6, 20]). In the case of a semifree action, we require that the fixed set be the transverse inverse image of the fixed sphere under an isovariant map to a linear G-sphere. In particular, this implies that the fixed set is a manifold with trivial equivariant normal bundle.

DEFINITION Let Σ be the fixed set of a semifree G-action $\Phi : G \times S^n \to S^n$. Then Σ is a *characteristic fixed-point set* if there is a semifree linear action $\Psi : G \times S^n \to S^n$ and an isovariant map $f : (S^n, \Phi) \to (S^n, \Psi)$ that is transverse to the fixed set of the linear action.

The action Φ is only assumed to be a topological action. A more intrinsic characterization of a characteristic fixed-point set is given by the next lemma.

LEMMA 1.1 *Let Σ^k be the fixed set of a semifree, normally trivial G-action on S^n. If the codimension $n - k$ is even and greater than 2, and if there is a map $g : S^n - \Sigma^k \to S^{n-k-1}$ that is an isomorphism on H_{n-k-1}, then Σ^k is a characteristic fixed-point set of the action.*

The last condition says that $S^n - \Sigma^k$ is *coreducible*. If Σ^k is any submanifold of S^n with trivial normal bundle, then Σ^k is its own Spanier-Whitehead dual, and so $S^m - \Sigma^k$ is coreducible for some m. Since one can stabilize an action by taking the join with a free action on the normal sphere, we have the following:

COROLLARY 1.2 *If Σ^k is the fixed set of a semifree, normally trivial G-action on S^n, then Σ^k is the characteristic fixed-point set of some semifree G-action on S^m for some m.*

PROOF OF LEMMA 1.1: This follows by the method of homotopy propagation (see [6]); we review the ideas here for the sake of the reader. Let P be a set of primes. A CW complex Y is P-local if $\pi_i Y \cong \pi_i Y \otimes \mathbf{Z}_{(P)}$ for all $i > 1$. A map $f : Y \to Z$ is a *P-local equivalence* if $f_* : \pi_1 Y \to \pi_1 Z$ is an isomorphism and if $f_* \otimes \mathrm{Id} : \pi_i Y \otimes \mathbf{Z}_{(P)} \to \pi_i Z \otimes \mathbf{Z}_{(P)}$ is an isomorphism for all $i > 1$. The fiberwise-localization functor of Bousfield-Kan [4] shows that for every CW complex X, there exists a P-local equivalence $f : X \to X_{(P)}$ where $X_{(P)}$ is a P-local space. This is unique up to homotopy. If P and Q are complementary sets of primes, then X is the homotopy pullback of

$$\begin{array}{ccc} & & X_{(P)} \\ & & \downarrow \\ X_{(Q)} & \longrightarrow & X_{(0)} \end{array}$$

where $X_{(0)}$ means invert all primes. If $\pi = \pi_1 Y$ is finite of order q, and π acts trivially on $H_*(\widetilde{X}; \mathbf{Z}[1/q])$, then $X_{[1/q]} \simeq \widetilde{X}_{[1/q]} \times B\pi$ (see [29, lemma 3.1]).

Now assume Σ^k is the fixed set of a semifree, normally trivial G-action Φ on S^n with the complement of the fixed set coreducible. Then G acts semifreely on D^{n-k} and freely on the boundary sphere. By adding on $k+1$ copies of the trivial representation and taking the boundary sphere, we get a semifree, linear action Ψ on S^n with fixed set S^k. Let $N(\Sigma^k)$ and $N(S^k)$ be closed G-tubular neighborhoods of the fixed set of the actions Φ and Ψ. Then $(S^n - \operatorname{int} N(S^k))/G$ is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & S^{n-k-1}_{[1/q]} \times BG \\ & & \downarrow \\ (S^{n-k-1}/G)_{(q)} & \longrightarrow & S^{n-k-1}_{(0)} \times BG. \end{array}$$

(Here q is the order of G.) Using that $S^n - \Sigma$ is coreducible and that the action on $H_*(S^n - \Sigma)$ is trivial (by Alexander duality), one obtains a map $(S^n - \operatorname{int} N(\Sigma^k))/G \to (S^n - \operatorname{int} N(S^k))/G$. (The map localized at q is the homotopy inverse of the localization of the inclusion of a normal sphere to Σ into the complement.) We then map $N(\Sigma^k) \to N(S^k)$ by taking the product of a degree 1 collapse map $\Sigma^k \to S^k$ with the identity map on D^{n-k}. The two maps

$$\partial N(\Sigma^k)/G \to \partial N(S^k)/G \hookrightarrow (S^n - \operatorname{int} N(S^k))/G$$

and

$$\partial N(\Sigma^k)/G \hookrightarrow (S^n - \operatorname{int} N(\Sigma^k))/G \to (S^n - \operatorname{int} N(S^k))/G$$

are homotopic by the pullback diagram. By pasting in the homotopy, we get the desired map to the linear action. □

DEFINITION If Σ^k is a rational homology sphere, define

$$\chi^{\text{tor}}(\Sigma) = \prod_{0<i<k} |H_i(\Sigma)|^{(-1)^i} \in \mathbf{Q}^\times$$

and

$$\chi^{\text{tor}}_{1/2}(\Sigma) = \prod_{0<i<k/2} |H_i(\Sigma)|^{(-1)^i} \in \mathbf{Q}^\times .$$

The main result of this paper is the following:

THEOREM 1.3 *Let G be a finite group that acts freely and linearly, preserving orientation on S^{n-k-1} with $n-k$ even and greater than 2. A submanifold Σ^k of S^n is a characteristic fixed-point set of some semifree G-action on S^n if and only if the following conditions are satisfied*:

1. *Σ^k is the transverse inverse image of S^k under a degree 1 self-map of S^n,*
2. *$H_*(\Sigma^k;\mathbf{Z}/|G|) \cong H_*(S^k;\mathbf{Z}/|G|)$,*
3. *$\tau(\chi^{\text{tor}}(\Sigma)) = 0 \in \widetilde{K}_0(\mathbf{Z}G)$, and*
4. *$\tau_n(\chi^{\text{tor}}(\Sigma)) = 0 \in L^h_n(\mathbf{Z}G)/C^h_n(\mathbf{Z}G)$, n odd, or $\tau_n(\chi^{\text{tor}}_{1/2}(\Sigma)) = 0 \in L^h_n(\mathbf{Z}G)/C^h_n(\mathbf{Z}G)$, n even.*

Remark. The case of $G = \mathbf{Z}_k$ for k odd was shown in [6]. When G has odd order, condition 4 is automatic.

The necessity of condition 2 follows easily from results of P. A. Smith [27]. Here $\tau : \mathbf{Z}/|G|^\times \to \widetilde{K}_0(\mathbf{Z}G)$ is the Swan map in K-theory defined by $\tau(\bar{a}) = \ker(\text{augmentation} : \mathbf{Z}G \to \mathbf{Z}/a)$. Condition 3 was first pointed out in the thesis of A. Assadi. For the precise definition of the Swan maps in L-theory τ_n, we refer the reader to [14], and here state the key property that for a degree 1 normal map $f : M^n \to X^n$ with $\pi_1 X = G$, where the surgery kernels have trivial G-action and are torsion prime to the order of G, the surgery obstruction $\theta(f) \in L^h_n(\mathbf{Z}G)$ equals $\tau_n(\prod_i |K_i(M)|^{(-1)^i})$ for n odd and $\tau_n(\prod_{i<n/2} |K_i(M)|^{(-1)^i})$ for n even. For $a \equiv 1 \pmod{4|G|}$, $\tau_n(a) = 0$, and for $n \equiv 2 \pmod 4$, τ_n vanishes. If $\tau = 0$, then for $n \equiv 0$, 1, or $2 \pmod 4$, τ_n is also zero.

Here $C^h_n(\mathbf{Z}G) \subset L^h_n(\mathbf{Z}G)$ is the group of surgery obstructions of degree 1 normal maps between closed manifolds, which is the sum of the images of the maps I_i and κ_i from [16]. Recall $C_n^h(\mathbf{Z}G)$ equals $i_*C_n^h(\mathbf{Z}G_2)$ where $i : G_2 \to G$ is the inclusion of a 2-Sylow subgroup. If a 2-group acts freely on a sphere, then it must be cyclic or generalized quaternionic. In either of these cases, $C_n^h(\mathbf{Z}G)$ is zero for $n \equiv 2 \pmod 4$ and is isomorphic to the image of $L_n^h(\mathbf{Z}) \cong \mathbf{Z}$ for $n \equiv 0 \pmod 4$, so it intersects the image of τ_n trivially. For $n \equiv 1 \pmod 4$ and G cyclic, $C_n^h(\mathbf{Z}G)$ is zero, while for G generalized quaternionic $C_n^h(\mathbf{Z}G) \cong \mathbf{Z}_2$, given by the image of $H^{n+1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}G))$, which also equals the image of τ_n. Finally $n \equiv 3 \pmod 4), C_n^h(\mathbf{Z}G)$ equals the image of $\kappa_1 : H_1(G; \mathbf{Z}_2) \to L_3^h(\mathbf{Z}G)$ (which is injective), and for G cyclic this equals the image of τ_n. In the cases of eventual interest, $C_n^h(\mathbf{Z}G)$ corresponds precisely to the change in surgery obstructions given by changing the normal invariant.

We examine the following example in detail: Let p be an odd prime and $Q_{4p} = \langle x, y : x^p = y^4 = 1,\ yxy^{-1} = x^{-1}\rangle$ denote the binary dihedral group. This is a subgroup of the unit quaternions, and hence it acts linearly and semifreely on S^n with fixed set S^k whenever 4 divides $n - k$. The Swan subgroup $T(Q_{4p})$ is zero [15], so if $n \equiv 0$, 1, or $2 \pmod 4$, then $\tau_n = 0$, but if $n \equiv 3 \pmod 4$, computations of Hambleton-Madsen [15] combined with [13] show that for $G = Q_{4p}$

$$\tau_n : (\mathbf{Z}_{(4p)})^\times \to L_n^h(\mathbf{Z}G)/C_n^h(\mathbf{Z}G)$$

is given by $\tau_n(a) = (a/p)$, the quadratic residue symbol.

COROLLARY 1.4 *Let Q_{4p} act on S^n, smoothly and semifreely, and with fixed set Σ^k. If $k \equiv 3 \pmod 8$ and if Σ is highly connected in the sense that $H_i(\Sigma) = 0$ for $0 < i < (k-3)/2$, then $\chi^{\mathrm{tor}}(\Sigma)$ is a quadratic residue modulo p.*

Remark. This contradicts the statement ([25, p. 273]), which gave certain sufficient conditions for the existence of a smooth semifree action, without including the numerical conditions coming from L-theory. (The statement there is correct for PL actions that are smooth in the complement of a point.) Our feeling that L-theoretic numerical conditions must play a role in any replacement problem lead us to this example.

PROOF OF COROLLARY 1.4: After suitable stabilization, we will produce a normally trivial action with fixed set Σ and coreducible complement. Then by Lemma 1.1 the action is characteristic, so by Theorem 1.3 and the computations of [15], the result follows.

Following [26], we note that if $E \to X$ is a G-vector bundle with G a finite group and the action on X trivial, then E splits as

$$E = \bigoplus_M M \otimes_{D_M} \mathrm{Hom}_{\mathbf{R}[G]}(M, E)$$

where the sum is over isomorphism classes of simple $\mathbf{R}[G]$-modules and $D_M = \mathrm{End}_{\mathbf{R}[G]}(M, M)$. The bundle $\mathrm{Hom}_{\mathbf{R}[G]}(M, E)$ admits the structure of an O, U, or Sp-bundle depending on whether $D_M = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

Let $G = Q_{4p}$ and E be the normal bundle of Σ. Since the action on a fiber E_x is free away from x, all modules in the decomposition of E_x as an $\mathbf{R}[G]$-module are symplectic with $D_M = \mathbf{H}$. By obstruction theory and Bott periodicity $\widetilde{KSp}(\Sigma)$ is zero (this accounts for the connectivity hypothesis on Σ), so that the bundles $\mathrm{Hom}_{\mathbf{R}[G]}(M, E)$ are stably trivial. To stabilize, we assume that the action on a fiber E_x is orthogonal (by averaging), and we replace S^n by $S^n * S(E_x)$ by taking the diagonal action on the join. This has the property of replacing E by

$$E = \bigoplus_M M \otimes_{D_M} (\mathrm{Hom}_{\mathbf{R}[G]}(M, E) \oplus \varepsilon(E, M))$$

where $\varepsilon(E, M)$ denotes $\dim_{D_M}(\mathrm{Hom}_{\mathbf{R}[G]}(M, E_x))$ copies of the trivial Sp-bundle over Σ. By repeating this process, all the bundles $\mathrm{Hom}_{\mathbf{R}[G]}(M, E)$ can be assumed to be trivial, and then the action is normally trivial.

Finally, since the normal bundle is now trivial, its Thom space is coreducible. Since Thom spaces and manifolds are Spanier-Whitehead dual, Σ is stably reducible, and so its complement in the sphere must be coreducible after further stabilization. □

2 Proof of Theorem 1.3

As we shall shortly see, the method of proof depends on the parity of the dimension. We first prove that if the conditions are satisfied, the action exists. Let $f : S^n \to S^n$ be a degree 1 map, transverse to S^k with $\Sigma^k = f^{-1}(S^k)$. Let $H : S^n \times I \to S^n$ be a homotopy with $H(x, 0) = x$ and $H(x, 1) = f(x)$. By homotoping the homotopy H, we may assume H is transverse to S^k. Then $H^{-1}(S^k)$ is a framed bordism between S^k and Σ^k. This gives a well-defined surgery obstruction in $L_{k+1}(\mathbf{Z}_{(q)})$, where q is the order of G. By the local Browder splitting theorem [6], this vanishes if and only if H can be further perturbed relative to $S^n \times \partial I$ so that $H_*(H^{-1}(S^k), S^k; \mathbf{Z}_q)$ is zero with $H^{-1}(S^k)$ simply connected. If k is even, the group $L_{k+1}(\mathbf{Z}_{(q)})$ vanishes, so this is always possible.

Next we take care of $k \equiv 1 \pmod 4$. The problem is that for q even, the local splitting invariant may be nonzero. Instead of the splitting problem, we consider the framed submanifold $H^{-1}(S^k)$ of $S^n \times I$. There is a local surgery obstruction in $L_{k+1}(\mathbf{Z}_{(q)}) \cong \mathbf{Z}_2$ to doing framed ambient surgery (cf. the proof of the local splitting theorem in [6]) to obtain a proper submanifold Z^{k+1} of $S^n \times I$ whose mod q homology is that of S^k and whose boundary is $S^k \cup \Sigma^k$. This obstruction can be corrected by taking the connect sum with the Kervaire manifold. This is the domain of the degree 1 normal map $K^{k+1} \to S^{k+1}$ with nontrivial Arf invariant. The punctured Kervaire manifold admits a smooth structure and embeds in (D^{k+3}, S^{k+2}), as can be seen by either constructing the Kervaire manifold by plumbing or as a Brieskorn variety. We do the connect sum inside of $S^n \times I$, killing the local obstruction.

The case where $k \equiv 3 \pmod 4$ we will have to take care of separately, but in all other cases the existence of the action will use the following lemma, which is a generalization of "extension across homology collars" (see [3, 6, 30]). Recall that a manifold triad $(Y; X_+, X_-)$ is a compact manifold Y with $\partial Y = X_+ \cup X_-$, where X_+ and X_- are compact manifolds with $\partial X_+ = \partial X_- = X_+ \cap X_-$.

LEMMA 2.1 *Let $(W; \partial_+, \partial_-)$ be a simply connected manifold triad. Let G be a finite group of order q that acts freely on ∂_+ and trivially on $H_*(\partial_+; \mathbf{Z}[1/q])$. Suppose $H_*(W, \partial_\pm; \mathbf{Z}_q) = 0$.*

1. *The action extends to a free action on W, trivial on $H_*(W; \mathbf{Z}[1/q])$ and $H_*(\partial_-; \mathbf{Z}[1/q])$ if and only if $\tau(\chi^{\mathrm{tor}}(W, \partial_+)) = 0 \in \widetilde{K}_0(\mathbf{Z}G)$.*
2. *Let $\partial : \widehat{H}^{\dim W}(\mathbf{Z}_2; \widetilde{K}_0(\mathbf{Z}G)) \to L^h_{\dim \partial W}(\mathbf{Z}G)$ be the boundary map in the Ranicki-Rothenberg L^h–L^p-exact sequence. If $\tau(\chi^{\mathrm{tor}}(W, \partial_+))$ is mapped to 0, then the action on ∂_+ extends to a free action on ∂W that is trivial on $H_*(\partial W; \mathbf{Z}[1/q])$.*

Addendum. The proof shows a bit more. In fact, the action in part 2 extends to an action on $W \times S^1$, but not on W if $\tau(\chi^{\mathrm{tor}}(W, \partial_+))$ is nonzero. Furthermore, by changing the normal invariant of the homotopy quotient, an action on ∂W exists under the hypothesis that $\partial[\tau(\chi^{\mathrm{tor}}(W, \partial_+)]$ is the surgery obstruction of a degree 1 normal map $f : (M, \partial M) \to (\partial_+/G, \partial(\partial_+/G))$ that is a homeomorphism on the boundary. If G acts freely on a sphere and if ∂_+/G maps to $K(G, 1)$ with a surjection on H_i for $i \leq 3$, we may replace this last condition by saying $\partial[\tau(\chi^{\mathrm{tor}}(W, \partial_+)] \in C^h_n(\mathbf{Z}G)$ (see [16]).

PROOF: If $(W; \partial_+, \partial_-)$ is a manifold triad as above, and G acts freely on ∂_+ and trivially on $H_*(\partial_+; \mathbf{Z}[1/q])$, then one can construct a Poincaré triad consisting of $(Y; \partial_+/G, X_-)$ by letting Y be the homotopy pullback of

$$\begin{array}{ccc} & & W_{[1/q]} \times BG \\ & & \downarrow \\ (\partial_+/G)_{(q)} & \longrightarrow & W_{(0)} \times BG \end{array}$$

and X_- be the homotopy pullback of

$$\begin{array}{ccc} & & (\partial_-)_{[1/q]} \times BG \\ & & \downarrow \\ (\partial_+/G)_{(q)} & \longrightarrow & (\partial_-)_{(0)} \times BG\,. \end{array}$$

The fact that this is a Poincaré triad, with normal invariant relative to the manifold structure on ∂_+/G, follows by checking one prime at a time, transfering to the cover away from q, and using the manifold structure at q. Since ∂_+/G is finite, the finiteness obstruction of Y is given by $\tau(\chi^{\mathrm{tor}}(W, \partial_+))$. Since $(Y, \partial_+/G)$ and (Y, X_-) are dual, we see that in any case X_- has the homotopy type of a finite complex. Part 1 follows from Wall's $\pi - \pi$ theorem [28, theorem 3.3]. Part 2 follows from an algebraic analogue due to Ranicki [23] by constructing a surgery problem over the triad, which leads to a Poincaré pair (D, C), where C is the surgery obstruction on the boundary (relative to ∂_+/G) and D is a projective coboundary, whose finiteness obstruction is that of $Y \operatorname{rel} \partial_+/G$. The precise statement here is that the relative algebraic bordism group $L_n^{h,p}(\mathbf{Z}G)$ is isomorphic to $\widehat{H}^n(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}G))$ via the map $[D, C] \to \chi(D)$. □

Suppose now that we have a framed proper submanifold Z^{k+1} of $S^n \times I$ that is a $\operatorname{mod} q$ homology sphere with boundary $S^k \cup \Sigma^k$. (We can always arrange this when $k \not\equiv 3 \pmod 4$.) The complement of a closed tubular neighborhood $N(Z^{k+1})$ gives a homology collar. Let $W = S^n \times I - \operatorname{int} N(Z^{k+1})$. Then the homology collar $(W; \partial_+, \partial_-)$ is given by

$$(W; (W \cap (S^n \times \{0\})) \cup \partial N(Z^{k+1}), W \cap (S^n \times \{1\}))\,.$$

Since G acts freely on S^{n-k-1}, there is a free, homologically trivial action on ∂_+. Let $\Omega^{k+1} = Z^{k+1} \cup_{S^k} D^{k+1}$. Then Ω is a $\mathbf{Z}_q$-homology disk with boundary Σ. By the addendum to extension across homology collars and Alexander duality, we see that if $\partial(\tau(\chi^{\text{tor}}(\Omega))$ is in the image of $C_n^h(\mathbf{Z}G)$, then Σ^k is a characteristic fixed-point set. To compute $\chi^{\text{tor}}(\Omega)$ we have the following lemma:

LEMMA 2.2 *Let (Ω^{k+1}, Σ^k) be a Poincaré pair with $\mathbf{Z}_q$-homology of (D^{k+1}, S^k) with q even. Then if k is odd, $\chi^{\text{tor}}(\Sigma) = \chi^{\text{tor}}(\Omega)^2$ and for $k \equiv 0 \pmod 4$, $a^2\chi^{\text{tor}}_{1/2}(\Sigma) = \chi^{\text{tor}}(\Omega)$ for some integer a.*

PROOF: We have an exact sequence

$$0 \to H_k(\Omega) \to H_k(\Omega, \Sigma) \to \cdots \to H_1(\Omega) \to H_1(\Omega, \Sigma) \to 0\,.$$

Thus $\chi^{\text{tor}}(\Sigma) = \chi^{\text{tor}}(\Omega)(\chi^{\text{tor}}(\Omega, \Sigma))^{-1}$, and for k odd $(\chi^{\text{tor}}(\Omega, \Sigma))^{-1} = \chi^{\text{tor}}(\Omega)$. For $k \equiv 0 \pmod 4$ we have

$$H_{k/2}(\Omega) \xrightarrow{h} H_{k/2}(\Omega, \Sigma) \to \cdots \to H_1(\Omega) \to H_1(\Omega, \Sigma) \to 0\,,$$

so $|\operatorname{im} h| \cdot \chi^{\text{tor}}_{1/2}(\Sigma) = \chi^{\text{tor}}(\Omega)$. Duality gives a nonsingular skew-symmetric linking form on image h, and hence $|\operatorname{im} h|$ is a square. □

We first complete the proof of Theorem 1.3 when $k \not\equiv 3 \pmod 4$. We have already seen that when $k \not\equiv 3 \pmod 4$ and Σ satisfies 1, 2, and 3, then Σ is a characteristic fixed-point set if and only if $\partial(\tau(\chi^{\text{tor}}(\Omega)) \in C_n^h(\mathbf{Z}G)$. (It is an if and only if because the homotopy type $\operatorname{rel} \partial$ of the complement is determined and a surgery problem is constructed by extension across homology collars.) Note that $n \equiv k \pmod 2$ and, if $n \not\equiv k \pmod 4$, then the period of G divided by 2 is odd. In this case, one can show that G is metacyclic and a product of a cyclic 2-group and a group of odd order. By induction theory, the Swan subgroup of such a group is odd torsion.

If $n \equiv 0 \pmod 4$, then $\tau_n(a) = \partial(\tau(a))$; in particular, $\tau_n(a^2) = 0$. Thus

$$\tau_n(\chi^{\text{tor}}_{1/2}(\Sigma)) = \partial(\tau(\Omega))$$

by Lemma 2.2 if $k \equiv 0 \pmod 4$ and because both sides are zero if $k \equiv 2 \pmod 4$.

If $n \equiv 1 \pmod 4$, then $\tau_n(a^2) = \partial(\tau(a))$. Thus if $k \equiv 1 \pmod 4$,

$$\tau_n(\chi^{\text{tor}}_{1/2}(\Sigma)) = \partial(\tau(\Omega))\,.$$

If $n \equiv 2 \pmod 4$, then $\tau_n = \partial \circ \tau = 0$.

If $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$, then the image of τ_n is contained in $C^h_n(\mathbf{Z}G)$ and the Swan subgroup is odd torsion, so the hypotheses are automatic.

Another approach is needed for $k \equiv 3 \pmod 4$. As usual, we have a candidate for the orbit space of the complement of the fixed set by letting X be the homotopy pullback of

$$\begin{array}{ccc} & & (S^n - \Sigma)_{[1/q]} \times BG \\ & & \big\downarrow \\ (S^{n-k-1}/G)_{(q)} & \longrightarrow & (S^n - \Sigma)_{(0)} \times BG\,. \end{array}$$

Since $\tau(\chi^{\text{tor}}(\Sigma)) = 0$, X can be taken to be a finite complex. Then $(X, \Sigma \times S^{n-k-1}/G)$ is a finite Poincaré pair. The Spivak bundle of X reduces to BTOP by lifting to the cover away from q and taking the lift of S^{n-k-1}/G at q. To get a lift relative to the given one on the boundary, one needs to check that $\Sigma^k \times S^{n-k-1}/G \to S^k \times S^{n-k-1}/G$ has trivial normal invariant at q, but this is automatic since k is odd. Thus there exists a degree 1 normal map $f : (M, \partial M) \to (X, \Sigma \times S^{n-k-1}/G)$ that is a homeomorphism on the boundary. If the order of G is odd or if $n \not\equiv k \pmod 4$, the L-group vanishes so we are done. For even-order groups with $n \equiv k \pmod 4$, the rest of the proof of Theorem 1.3 is similar to that of [14] except for the new ingredient given by the following:

PROPOSITION 2.3 *Suppose G acts freely on S^t with $t \equiv 3 \pmod 4$ and $q = |G|$ even. For any $\alpha \in L_0(\mathbf{Z}_{(q)})$, then $\alpha \otimes \sigma^*(S^t/G)$ is zero in $L^A_t(\mathbf{Z}_{(q)}G)$ where $A = \operatorname{im} K_1(\mathbf{Z}G)$.*

We postpone the proof of this proposition, which depends on the main result of [13], and complete the proof of the theorem assuming it. By the above proposition and the Ranicki product formula,

$$\Omega^{k+1} \times S^{n-k-1}/G \to D^{k+1} \times S^{n-k-1}/G$$

is normally bordant relative to the boundary to a $\mathbf{Z}_{(q)}G$-homology equivalence $h : N^n \to D^{k+1} \times S^{n-k-1}/G$, whose torsion lies in A. Recall the Bass localization sequence

$$K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}_{(q)}G) \to K_1(\mathbf{Z}G, S) \to K_0(\mathbf{Z}G) \to K_0(\mathbf{Z}_{(q)}G)\,,$$

where $K_1(\mathbf{Z}G, S)$ is the Grothendieck group of finitely generated $\mathbf{Z}G$-modules that are torsion prime to q. The homological interpretation of the statement that the torsion of h lies in A is that $\Delta(h) = \Sigma(-1)^i[H_i(h; \mathbf{Z}G)] = 0 \in K_1(\mathbf{Z}G, S)$ (see [13]). We now have a composite map of degree 1 normal maps

$$M \cup_\partial N \xrightarrow{f \cup \mathrm{Id}} X \cup_\partial N \xrightarrow{g \cup h} D^{k+1} \times S^{n-k-1}/G \cup_\partial D^{k+1} \times S^{n-k-1}/G$$

where g is given by a map on pullback diagrams (so g lifts to our original map on the complement of Σ). The double composite is a map between closed manifolds. By [10], the surgery semicharacteristic of the double composite with coefficients in $\mathbf{Z}_{(q)}G/\mathrm{rad}$ is zero, and this equals the surgery semicharacteristic of f. By [10], this is the obstruction to doing surgery to a $\mathbf{Z}_{(q)}G$-homology equivalence, so without loss of generality we may assume f is a such a homology equivalence. By [11], for $n \equiv 3 \,(\mathrm{mod}\, 4), L_n(\mathbf{Z}G) \to L_n^A(\mathbf{Z}_{(q)}G)$ is injective, and for a $\mathbf{Z}_{(q)}G$-homology equivalence f, the image of $\theta(f)$ is given by the image of $\Delta(f)$ in the Rothenberg exact sequence

$$\widehat{H}^{n+1}(\mathbf{Z}_2; K_1(\mathbf{Z}_{(q)}G)/A) \to L_n^A(\mathbf{Z}_{(q)}G) \to L_n(\mathbf{Z}_{(q)}G)\,.$$

Computing torsions in $K_1(\mathbf{Z}G, S)$, we see

$$\Delta(f) = \Delta(f \cup \mathrm{Id}) = \Delta(f \cup \mathrm{Id} \circ g \cup h) - \Delta(g \cup h)\,,$$

where the first term is the image of the closed manifold problems $C_n^h(\mathbf{Z}G)$ and the second equals $\Delta(g)$, which is given by $\chi^{\mathrm{tor}}(\Sigma)$. Since $C_n^h(\mathbf{Z}G)$ gives the variation in the normal invariant of f by [16], we are done.

To prove Proposition 2.3, we need a very specific set of generators for $L_0(\mathbf{Z}_{(q)})$.

LEMMA 2.4 (0-Dimensional Surgery) *For q even,* $L_0(\mathbf{Z}_{(q)})$ *is generated by* $\{\theta_d = \theta(d\, pts \to pt) \mid (d, q) = 1\}$.

Apropos of this lemma we give a quotation from [5, p. 151]:

> We do point out that for $d > 1$, the f-maps over a *point*, p, give rise to normal maps $d \cdot p \to p$, covered by the identity map on trivial normal bundles. It may not seem worthwhile, at first, to study these zero-dimensional normal maps. However, we will specifically want the product formula (5.2) to include zero-dimensional factors, and there is something mysteriously nontrivial about the Arf-invariant of $d \cdot p \to p$.

PROOF: We first review the computation of $L_0(\mathbf{Z}_{(q)})$ for q even. Localization exact sequences give a short exact sequence

$$0 \to L_0(\mathbf{Z}) \to L_0(\mathbf{Z}_{(q)}) \to \oplus L_0(\mathbf{F}_p) \to 0\,,$$

where the sum is over all primes not dividing q. This localization sequence is similar to the symmetric version in [21]; suitable modifications for the quadratic case are in [23]. Then $L_0(\mathbf{Z})$ is isomorphic to $8\mathbf{Z}$ by the signature. The map $L_0(\mathbf{Z}_{(q)}) \to L_0(\mathbf{F}_p)$ factors through $L_0(\mathbf{Q})$, with the technical advantage that all forms over a field of characteristic $\neq 2$ can be diagonalized, and that the symmetric and quadratic theories coincide. The map $L_0(\mathbf{Q}) \to L_0(\mathbf{F}_p)$ send a 1-by-1 form $\langle ap^i \rangle$ (with $(a,p)=1$) to zero if i is even and to $\langle a \rangle$ if i is odd.

To compute the surgery obstruction θ_d of $e_1 \amalg e_2 \amalg \cdots \amalg e_d \to e$, we take

$$\{e_1 - e_2, e_1 + e_2 - 2e_3, \ldots, e_1 + \cdots + e_{d-1} - (d-1)e_d\}$$

as basis for the surgery kernel tensored with $\mathbf{Q}$. The corresponding intersection matrix is

$$\begin{pmatrix} 1\cdot 2 & 0 & \cdots & 0 \\ 0 & 2\cdot 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d-1)\cdot d \end{pmatrix}$$

Consider the image of two adjacent entries under the map $L_0(\mathbf{Q}) \to L_0(\mathbf{F}_p)$ for p odd. Here

$$\begin{pmatrix} (ap^i - 1)ap^i & 0 \\ 0 & ap^i(ap^i+1) \end{pmatrix}$$

maps either to zero or $\left(\begin{smallmatrix} -a & 0 \\ 0 & a \end{smallmatrix}\right)$, which is in any case Witt-equivalent to zero. So the image of θ_d in $L_0(\mathbf{F}_p)$ is determined by the last entry and is only nonzero if $d = ap^i$ with i odd, in which case it maps to $\langle a \rangle$. If $\{p_1, p_2, p_3, \ldots\}$ is a list of all primes not dividing q, then by considering θ_{ap_i} with $(a, p_j) = 1$ for all $j < i$, one can show inductively that the image of the span of $\{\theta_d\}$ maps onto $\oplus_{j \le i} L_0(\mathbf{F}_{p_j})$, and hence onto all of $\oplus_j L_0(\mathbf{F}_{p_j})$.

Finally, we hit $L_0(\mathbf{Z})$. Using Dirichlet's theorem on primes in arithmetic progressions, we choose primes p, p', and p'' all relatively prime to q so that $p \equiv 3$ or $5 \pmod 8$, $p' \equiv 1 \pmod 4$, and $p'' \equiv 1 \pmod{p'}$, so that for all odd prime factors s of p^2-1, both $p'-1$ and $p''-1$ are relatively prime to s. Then we claim θ_{p^2} and $\theta_{p'p''} - \theta_{p'} - \theta_{p''}$ map to zero in $\oplus_j L_0(\mathbf{F}_{p_j})$. (Note here $(p''/p') = 1$ by choice and $(p'/p'') = 1$ by quadratic reciprocity.) The g.c.d. of their signatures is $\gcd(p^2-1, (p'-1)(p''-1)) = 8$, so these two forms generate $L_0(\mathbf{Z})$. □

PROOF OF PROPOSITION 2.3: By the lemma we need only analyze the surgery obstruction of the degree d map

$$S^t/G \amalg S^t/G \amalg \cdots \amalg S^t/G \to S^t/G$$

for $(d,q) = 1$. By the discussion in [15] or [5], this surgery obstruction equals the surgery obstruction of any degree d normal map modulo $C_n^h(\mathbf{Z}G)$. Let Π^t be a simply connected homology sphere with $\chi^{\mathrm{tor}}(\Pi^t) = d$. (This can be constructed as a Brieskorn variety or by surgery on a lens space.) By the main result of [13], the G action on the sphere propagates across the degree d map $\Pi^t \to S^t$, and the surgery obstruction of the degree d map $\Pi^t/G \to S^t/G$ is zero in $L_n^A(\mathbf{Z}_{(q)}G)$. □

Acknowledgments. Both authors are supported by NSF grants. The first author was supported by an Alexander von Humboldt–Stiftung Forschungsstipendium when this paper was written.

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Received October 1997.