

THE GROMOV-LAWSON-ROSENBERG CONJECTURE FOR COCOMPACT FUCHSIAN GROUPS

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ABSTRACT. We prove the Gromov-Lawson-Rosenberg conjecture for cocompact Fuchsian groups, thereby giving necessary and sufficient conditions for a closed spin manifold of dimension greater than four with fundamental group cocompact Fuchsian to admit a metric of positive scalar curvature.

Given a smooth closed manifold M^n , it is a long-standing question to determine whether or not M admits a Riemannian metric of positive scalar curvature. Work of Gromov-Lawson and Schoen-Yau shows that if N^n admits positive scalar curvature and M is obtained from N by k -surgeries of codimension $n - k \geq 3$, then M admits positive scalar curvature as well. In the case when M is spin, this surgery result implies the following.

Bordism Theorem ([10], [27]). *Let M^n be a closed spin manifold, $n \geq 5$, $G = \pi_1(M)$, and suppose $u : M \rightarrow BG$ induces the identity on the fundamental group. If there is a positively scalar curved spin manifold N^n and a map $v : N \rightarrow BG$ such that $[M, u] = [N, v] \in \Omega_n^{\text{Spin}}(BG)$, then M admits a metric of positive scalar curvature.*

On the other hand, the work of Lichnerowicz gives an obstruction to manifolds admitting positive scalar curvature. Using the Weitzenböck formula for the Dirac operator and the Atiyah-Singer index theorem, he proves in [18] that if M^{4k} is a closed spin manifold with positive scalar curvature, then $\hat{A}(M)$, the A -hat genus of M , vanishes. Generalizations of the Dirac operator and its index by Hitchin [12] and Mišćenko-Fomenko [20] provide obstructions as well, taking final form in the following theorem of Rosenberg [23].

Obstruction Theorem ([23], [29]). *Let M^n be a closed spin manifold, and let $u : M \rightarrow BG$ be a continuous map for some discrete group G . If M admits a metric of positive scalar curvature, then $\alpha[M, u] = 0$ in $KO_n(C_r^*G)$, where $\alpha : \Omega_n^{\text{Spin}}(BG) \rightarrow KO_n(C_r^*G)$ is the index of the Dirac operator.*

Here C_r^*G is the reduced real C^* -algebra of G , a suitable completion of the group algebra $\mathbb{R}G$. In dimensions $4k$ with $G = 1$, the index $\alpha[M, u]$ agrees with $\hat{A}(M)$ up to a constant factor ([17], p. 149). The above theorem motivates the following conjecture.

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The Gromov-Lawson-Rosenberg (GLR) Conjecture for G . *Let M^n be a closed spin manifold, $n \geq 5$, $G = \pi_1(M)$, and suppose $u : M \rightarrow BG$ induces the identity on the fundamental group. Then M admits positive scalar curvature if and only if $\alpha[M, u]$ is zero in $KO_n(C_r^*G)$.*

The conjecture has been verified in the cases when G is trivial [28], $\mathbb{Z}/2$ [24], \mathbb{Z}/n for n odd ([16], [22]), has periodic cohomology ([3], [16]), or is one of certain elementary abelian groups [2]. Infinite groups for which the conjecture is proven are free and free abelian groups [25], certain other torsion-free groups [13], and some crystallographic groups with prime torsion [7]. However, in [26], Schick shows the conjecture is false for $\Gamma = \mathbb{Z}^4 \oplus \mathbb{Z}/3$, and other counterexamples have since been constructed in [13]. In [25], Rosenberg and Stolz introduced a “stable” version: that the conjecture holds after crossing M with sufficiently many copies of the Bott manifold B^8 .

In this paper we prove the original (unstable) conjecture when G is a cocompact Fuchsian group Γ . Fuchsian groups are the discrete subgroups of $PSL_2(\mathbb{R})$, which can be identified with the orientation-preserving isometries of the hyperbolic plane \mathbb{H}^2 , and as such they have rich geometric properties (see [15] as a reference). Each cocompact Fuchsian group is classified by its signature, $(g; \nu_1, \nu_2, \dots, \nu_r)$, where g is the genus of the compact orbit space \mathbb{H}^2/Γ and \mathbb{Z}/ν_j is the stabilizer of a lift \tilde{v}_j of a singular point $v_j \in \mathbb{H}^2/\Gamma$ of the branched covering $\mathbb{H}^2 \rightarrow \mathbb{H}^2/\Gamma$. The group Γ has r conjugacy classes of maximal finite subgroups, all of which are cyclic. The integers ν_i indicate their order. Furthermore, nearly all signatures $(g; \nu_1, \nu_2, \dots, \nu_r)$ are realizable: if $\nu_j \geq 2, g \geq 0, r \geq 0$ and $2g - 2 + \sum(1 - 1/\nu_j) > 0$, then there exists a Fuchsian group Γ with this signature. Generators and relations for Γ were found by Poincaré [21] (a more modern reference is [31]),

$$\Gamma = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid [a_1, b_1] \cdots [a_g, b_g] = c_1 \cdots c_r, c_1^{\nu_1} = \cdots = c_r^{\nu_r} = 1 \rangle.$$

We can now formally state our

Main Theorem. *The Gromov-Lawson-Rosenberg conjecture is true for all cocompact Fuchsian groups.*

Following the general line of proof in [7], our strategy is to use the validity of the Baum-Connes conjecture for the group Γ and the p -chain spectral sequence to understand the map α by studying how it factors through connective and periodic real K -theory, and then to exploit the fact that the Gromov-Lawson-Rosenberg conjecture is true for each finite subgroup of Γ .

1. ASSORTED DEFINITIONS

A real C^* -algebra is a real Banach $*$ -algebra \mathcal{A} which is $*$ -isomorphic to a norm-closed subalgebra of the bounded operators $\mathcal{B}(H)$ on a real Hilbert space H . An example of such is given by the reduced C^* -algebra C_r^*G of a discrete group G , which is defined to be the closure of $\mathbb{R}G$ in $\mathcal{B}(\ell^2(G))$, where $\mathbb{R}G$ acts on $\ell^2(G)$ by multiplication.

For a C^* -algebra \mathcal{A} with unit, and for $n > 0$, define $KO_n(\mathcal{A}) = \pi_{n-1}(GL(\mathcal{A}))$. There is the Bott map, defined using Clifford algebras, $GL(\mathcal{A}) \rightarrow \Omega^8(GL(\mathcal{A}))$, which, according to Bott periodicity [32], is a homotopy equivalence. Then for any n , define $KO_n(\mathcal{A}) = \pi_{n+8k-1}(GL(\mathcal{A}))$, where $n + 8k > 0$. One can show $KO_0(\mathcal{A})$ is the Grothendieck group of isomorphism classes of finitely generated projective

\mathcal{A} -modules, in other words, the algebraic K_0 of the ring \mathcal{A} . For more details on C^* -algebras and their K -theory we refer the reader to [17].

The above theory is sufficient for analysis, but to compute using algebraic topology one needs spectra. By a spectrum \mathbf{K} , we mean a sequence of based spaces $\{K_n\}_{n \geq 0}$ and based maps $\sigma_n : SK_n \rightarrow K_{n+1}$, where S denotes suspension. A map of spectra $f : \mathbf{K} \rightarrow \mathbf{K}'$ is a sequence of based maps $f_n : K_n \rightarrow K'_n$ commuting with the structure maps, i.e. $f'_{n+1} \circ \sigma_n = \sigma'_n \circ S f_n$. For a real C^* -algebra with unit \mathcal{A} , define the Bott spectrum $\mathbf{KO}(\mathcal{A})$ to have as its n -th space $\Omega^i(GL(\mathcal{A}))$ where $i \in \{0, 1, 2, \dots, 7\}$ and $i + n + 1 \equiv 0 \pmod{8}$. Most of the structure maps $S\Omega^i(GL(\mathcal{A})) \rightarrow \Omega^{i-1}(GL(\mathcal{A}))$ are given as the adjoint of the identity map on $\Omega^i(GL(\mathcal{A}))$, while one-eighth of the maps are given by the adjoint of the Bott map. Then $\pi_n(\mathbf{KO}(\mathcal{A})) = KO_n(\mathcal{A})$ for all $n \in \mathbb{Z}$. We will write \mathbf{KO} for $\mathbf{KO}(\mathbb{R})$.

However, the above spectra are still insufficient for our purposes. A preliminary problem is that the fundamental group depends on a choice of base point, but a much more severe problem is that $(X, x_0) \mapsto C_r^*(\pi_1(X, x_0))$ is not a functor from the category of based spaces, since the reduced C^* -algebra of a group is not functorial for group homomorphisms with infinite kernel. This obstructs both a definition of the assembly map in terms of algebraic topology and an axiomatic characterization of the assembly map, both of which are necessary in this paper.

These base point and functorial problems can be avoided by the use of C^* -categories, which is the point of view of [5]. In this paper we will need a composite of three functors

$$\text{Or}(G) \rightarrow \text{GROUPOIDS}^{\text{proper}} \rightarrow C^*\text{-CATEGORIES} \rightarrow \text{SPECTRA}$$

which we describe from right to left. A C^* -category \mathcal{A} is a category, so that for any two objects c, d , the morphism set $\text{mor}_{\mathcal{A}}(c, d)$ is a Banach space with an involution $*$: $\text{mor}_{\mathcal{A}}(c, d) \rightarrow \text{mor}_{\mathcal{A}}(d, c)$ satisfying the expected properties (e.g. bilinearity with respect to composition), and so that for all $f \in \text{mor}_{\mathcal{A}}(c, d)$, $\|f^* \circ f\| = \|f\|^2$ and so that $f^* \circ f$ has non-negative spectrum (in the sense of analysis). A C^* -category is a topological category: the morphisms sets are topological spaces, although in this case the objects are discrete. A C^* -category with exactly one object is a C^* -algebra. The category $C^*\text{-CATEGORIES}$ is the category of all small (i.e. the objects form a set) C^* -categories. The functor $F : C^*\text{-CATEGORIES} \rightarrow \text{SPECTRA}$ is to have two properties: when the C^* -category has a single object, one gets the Bott spectrum and an equivalence of C^* -categories gives a homotopy equivalence of spectra. Unfortunately, the construction of F in [5] was flawed by failing to heed the warning of [30]; the problem can be repaired without a great deal of difficulty using Segal's Γ -spaces. Alternatively, such a functor has been constructed in [19].

A (discrete) groupoid \mathcal{G} is a category all of whose morphisms are isomorphisms; a groupoid with a single object is a group. Associated to a groupoid \mathcal{G} is a C^* -category $C_r^*(\mathcal{G})$, whose objects are the same as those of \mathcal{G} , and whose morphisms are given by the closure of the linear span of $\text{mor}_{\mathcal{G}}(c, d) \subset \mathcal{B}(\ell^2(\text{mor}_{\mathcal{G}}(c, c)), \ell^2(\text{mor}_{\mathcal{G}}(c, d)))$. A morphism of groupoids is a functor; a morphism $F : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is proper if for all objects c and d of \mathcal{G}_1 , there is an $n = n(c, d)$ so that for all $f \in \text{mor}_{\mathcal{G}_2}(F(c), F(d))$, $F^{-1}(f) \subset \text{mor}_{\mathcal{G}_1}(c, d)$ has cardinality less than n . A proper morphism between groupoids induces a map of the associated C^* -categories. This gives our functor from $\text{GROUPOIDS}^{\text{proper}}$, the category of small groupoids and proper morphisms, to $C^*\text{-CATEGORIES}$.

Finally, for a group G , define the orbit category $\text{Or}(G)$ whose objects are the G -sets G/H where H is a subgroup of G and whose morphisms are the equivariant maps. The orbit category is a useful indexing set in group actions, for example, if X is a G -set there is an associated functor $\text{Or}(G) \rightarrow \text{SETS}$, $G/H \mapsto X/H$. For any G -set X , there is an associated groupoid \overline{X} whose objects are the elements of X and $\text{mor}_{\overline{X}}(x_1, x_2) = \{g \in G : gx_1 = x_2\}$, with the composition law given by group multiplication. This gives our last functor $\text{Or}(G) \rightarrow \text{GROUPOIDS}^{\text{proper}}$. Note the $\overline{G/H}$ is equivalent as a category to the group H .

The composite functor

$$\mathbf{KO}^{\text{top}} : \text{Or}(G) \rightarrow \text{SPECTRA}$$

satisfies $\pi_n \mathbf{KO}^{\text{top}}(G/H) = KO_n(C_r^*H)$. This functor is, according to [5], the basic building block of the assembly map (of which there are four equivalent descriptions in [5]). Given a family \mathcal{F} of subgroups of G , define the restricted orbit category $\text{Or}(G, \mathcal{F})$ to be the full subcategory of $\text{Or}(G)$ whose objects are G/H with $H \in \mathcal{F}$. Let 1 denote the family consisting of the trivial subgroup and $\mathcal{F}in$ the family of finite subgroups. Consider the maps

$$BG_+ \wedge \mathbf{KO} \simeq \text{hocolim}_{\text{Or}(G,1)} \mathbf{KO}^{\text{top}} \rightarrow \text{hocolim}_{\text{Or}(G,\mathcal{F}in)} \mathbf{KO}^{\text{top}} \rightarrow \text{hocolim}_{\text{Or}(G)} \mathbf{KO}^{\text{top}} \simeq \mathbf{KO}(C_r^*G).$$

The assembly map $KO_n(BG) \rightarrow KO_n(C_r^*(G))$ is, by definition, π_n applied to the composite. The induced map

$$\pi_n(\text{hocolim}_{\text{Or}(G,\mathcal{F}in)} \mathbf{KO}^{\text{top}}) \rightarrow KO_n(C_r^*(G))$$

has been identified with the Baum-Connes map in [11]. This identification is crucial for us, because the analytic map gives the obstructions to positive scalar curvature, and the topological map we need for computations.

2. PROOF OF THE MAIN THEOREM

Recall [29] that α is the composition $\alpha = A \circ p \circ D$,

$$\Omega_n^{\text{Spin}}(BG) \xrightarrow{D} ko_n(BG) \xrightarrow{p} KO_n(BG) \xrightarrow{A} KO_n(C_r^*G).$$

Here D is the ko -orientation of spin bordism, p is the covering of periodic K -theory by connective K -theory, and A is the assembly map. The groups $ko_n(*) = 0$ for $n < 0$ and $p : ko_n(*) \rightarrow KO_n(*)$ is an isomorphism for $n \geq 0$.

The utility of this factorization is the generalization of the Bordism Theorem by Jung and Stolz. Let $ko_n^+(BG)$ be the subgroup of $ko_n(BG)$ given by $D[N^n, v]$ where N is a positively curved spin manifold and $v : N \rightarrow BG$ is a continuous map.

Theorem 1 ([29]). *Let M^n be a closed spin manifold, $n \geq 5$, $G = \pi_1(M)$, and suppose $u : M \rightarrow BG$ induces the identity on the fundamental group. If $D[M, u] \in ko_n^+(BG)$, then M admits a metric of positive scalar curvature.*

Our direct sum in the theorem below and hereafter will be over the r conjugacy classes (C) of maximal finite subgroups of Γ . Thus $C \cong \mathbb{Z}/\nu_j$ for $j = 1, \dots, r$.

Proposition 2 ([1]). *Let Γ be a cocompact Fuchsian group with signature $(g; \nu_1, \nu_2, \dots, \nu_r)$, and let \mathbb{E} be any covariant functor from $\text{Or}(\Gamma, \mathcal{F}in)$ to SPECTRA. Then there is an exact sequence*

$$\begin{aligned} \cdots \rightarrow \oplus_{(C)} H_n(BC; \mathbb{E}(\Gamma/1)) &\rightarrow (\oplus_{(C)} \pi_n(\mathbb{E}(\Gamma/C))) \oplus H_n(B\Gamma; \mathbb{E}(\Gamma/1)) \\ &\rightarrow \pi_n(\text{hocolim}_{\text{Or}(\Gamma, \mathcal{F}in)}(\mathbb{E})) \rightarrow \oplus_{(C)} H_{n-1}(BC; \mathbb{E}(\Gamma/1)) \rightarrow \cdots \end{aligned}$$

This result is obtained by showing the spectral sequence from [6] collapses at E^2 . The proposition in [1] is stated for the algebraic K -theory functor \mathbb{K} , but holds for any \mathbb{E} . The reason why cocompact Fuchsian groups are so well behaved is due to the structure of the lattice of finite subgroups: if H is a non-trivial finite subgroup, then H is contained in a unique maximal finite subgroup M , the normalizer $N_\Gamma(H)$ equals M , and the set of finite subgroups containing H is totally ordered.

When \mathbb{E} is a spectrum and $\mathbb{E}_c : \text{Or}(\Gamma, \mathcal{F}in) \rightarrow \text{SPECTRA}$ is the constant functor $\Gamma/H \mapsto \mathbb{E}$, the following simplification to Proposition 2 can be made.

Corollary 3. *Let Γ be a cocompact Fuchsian group with signature $(g; \nu_1, \nu_2, \dots, \nu_r)$, let \mathbb{E} be a spectrum, and let X_g be the closed orientable surface of genus g . Then*

$$\cdots \rightarrow \oplus_{(C)} H_n(BC; \mathbb{E}) \rightarrow (\oplus_{(C)} \pi_n(\mathbb{E})) \oplus H_n(B\Gamma; \mathbb{E}) \rightarrow H_n(X_g; \mathbb{E}) \rightarrow \cdots$$

and

$$\cdots \rightarrow \oplus_{(C)} \tilde{H}_n(BC; \mathbb{E}) \rightarrow \tilde{H}_n(B\Gamma; \mathbb{E}) \rightarrow \tilde{H}_n(X_g; \mathbb{E}) \rightarrow \cdots$$

are exact.

Proof. By Proposition 2, it suffices to identify $\pi_n(\text{hocolim}_{\text{Or}(\Gamma, \mathcal{F}in)}(\mathbb{E}_c))$ with $H_n(X_g; \mathbb{E})$. Let $E_{\mathcal{F}in}\Gamma$ be the “universal Γ -space with finite isotropy”, i.e. a Γ -complex such that the fixed set $(E_{\mathcal{F}in}\Gamma)^H$ is contractible if H is a finite subgroup of Γ and empty if H is not finite (see [9], [8]). By definition of the homotopy colimit (see [5, sections 3 and 7]), $\pi_n(\text{hocolim}_{\text{Or}(\Gamma, \mathcal{F}in)}(\mathbb{E}_c)) \cong H_n(E_{\mathcal{F}in}\Gamma/\Gamma; \mathbb{E})$ for any constant $\text{Or}(\Gamma, \mathcal{F}in)$ -spectrum \mathbb{E}_c . A cocompact Fuchsian group acts with finite isotropy on the hyperbolic plane \mathbb{H}^2 with quotient X_g , and we can take \mathbb{H}^2 as a model for $E_{\mathcal{F}in}\Gamma$. □

Proposition 4. *Let Γ be a cocompact Fuchsian group with signature $(g; \nu_1, \nu_2, \dots, \nu_r)$, and let X_g be the closed orientable surface of genus g . Then there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccc} \widetilde{KO}_{n+1}(X_g) & \longrightarrow & \oplus_{(C)} \widetilde{KO}_n(BC) & \longrightarrow & \widetilde{KO}_n(B\Gamma) & \longrightarrow & \widetilde{KO}_n(X_g) \\ \downarrow \text{Id} & & \downarrow A & & \downarrow A & & \downarrow \text{Id} \\ \widetilde{KO}_{n+1}(X_g) & \longrightarrow & \oplus_{(C)} \widetilde{KO}_n(C_r^*C) & \longrightarrow & \widetilde{KO}_n(C_r^*\Gamma) & \longrightarrow & \widetilde{KO}_n(X_g) \end{array}$$

Proof. In [11], Hambleton-Pedersen identify the Baum-Connes map with the Davis-Lück assembly map [5]

$$\pi_n \text{hocolim}_{\text{Or}(G, \mathcal{F}in)} \mathbf{KO}^{\text{top}} \rightarrow KO_n(C_r^*G).$$

Kasparov [14], [4, p. 284] proved that the Baum-Connes map is an isomorphism for subgroups of $SO(n, 1)$, such as Γ . Thus we can (and will) replace $KO_n(C_r^*\Gamma)$ by $\pi_n \text{hocolim}_{\text{Or}(G, \mathcal{F}in)} \mathbf{KO}^{\text{top}}$ in the proof of this proposition.

Underlying Proposition 2 is an “excision” isomorphism,

$$\pi_*(\vee_{(C)} BC_+ \wedge \mathbb{E}(\Gamma/1) \rightarrow B\Gamma_+ \wedge \mathbb{E}(\Gamma/1)) \xrightarrow{\cong} \pi_*(\vee_{(C)} \mathbb{E}(\Gamma/C) \rightarrow \operatorname{hocolim}_{\operatorname{Or}(\Gamma, \mathcal{F}in)} \mathbb{E}),$$

defined for every functor $\mathbb{E} : \operatorname{Or}(\Gamma, \mathcal{F}in) \rightarrow \operatorname{SPECTRA}$. Now notice that for $H \in \mathcal{F}in$, there is a morphism in $\operatorname{GROUPOIDS}^{\operatorname{proper}}$

$$\overline{\Gamma/H} \rightarrow \overline{1/1},$$

where the target is the trivial groupoid. When $H = 1$, this is an equivalence. Hence there is a natural transformation of $\operatorname{Or}(\Gamma, \mathcal{F}in) - \operatorname{SPECTRA}$,

$$\mathbf{KO}^{\operatorname{top}} \rightarrow \mathbf{KO}_c,$$

inducing a homotopy equivalence $\mathbf{KO}^{\operatorname{top}}(\Gamma/1) \rightarrow \mathbf{KO}_c(\Gamma/1) = \mathbf{KO}$.

Proposition 4 now follows from a diagram chase. Indeed, consider the following diagram:

$$\begin{array}{ccc} \pi_*(\vee_{(C)} BC_+ \wedge \mathbf{KO} \rightarrow B\Gamma_+ \wedge \mathbf{KO}) & \xrightarrow{\cong} & \pi_*(\mathbf{KO} \rightarrow X_{g+} \wedge \mathbf{KO}) \\ \downarrow \cong & & \downarrow \operatorname{Id} \\ \pi_*(\vee_{(C)} \mathbf{KO}(C_r^* C) \rightarrow \mathbf{KO}(C_r^* \Gamma)) & \longrightarrow & \pi_*(\mathbf{KO} \rightarrow X_{g+} \wedge \mathbf{KO}) \end{array}$$

The top row is the excision isomorphism coming from \mathbf{KO}_c and the left column is the excision isomorphism coming from $\mathbf{KO}^{\operatorname{top}}$. Commutativity of the diagram comes from the natural transformation $\mathbf{KO}^{\operatorname{top}} \rightarrow \mathbf{KO}_c$. The horizontal isomorphisms give Mayer-Vietoris exact sequences and the commutativity of the square gives a map between the two Mayer-Vietoris sequences. \square

Proof of the Main Theorem. Let \mathbf{ko} be the spectrum for the connective cover of \mathbf{KO} . Since there is a natural transformation $p : \mathbf{ko}_c \rightarrow \mathbf{KO}_c$ of $\operatorname{Or}(\Gamma, \mathcal{F}in)$ - $\operatorname{SPECTRA}$, Corollary 3 and Proposition 4 give a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \widetilde{ko}_{n+1}(X_g) & \longrightarrow & \oplus_{(C)} \widetilde{ko}_n(BC) & \longrightarrow & \widetilde{ko}_n(B\Gamma) & \longrightarrow & \widetilde{ko}_n(X_g) \\ \downarrow p & & \downarrow A \circ p & & \downarrow A \circ p & & \downarrow p \\ \widetilde{KO}_{n+1}(X_g) & \longrightarrow & \oplus_{(C)} \widetilde{KO}_n(C_r^* C) & \longrightarrow & \widetilde{KO}_n(C_r^* \Gamma) & \longrightarrow & \widetilde{KO}_n(X_g). \end{array}$$

Now $\sum X_g \simeq (\vee_{2g} S^2) \vee S^3$, so

$$\begin{aligned} \widetilde{ko}_n(X_g) &\cong ko_{n-1}(*)^{2g} \oplus ko_{n-2}(*), \\ \widetilde{KO}_n(X_g) &\cong KO_{n-1}(*)^{2g} \oplus KO_{n-2}(*), \end{aligned}$$

and hence $p : \widetilde{ko}_n(X_g) \rightarrow \widetilde{KO}_n(X_g)$ is an isomorphism for $n \geq 2$; an alternative proof uses the Atiyah-Hirzebruch spectral sequence. Suppose now that $n \geq 5$ and that

$$\beta \in \ker(A \circ p : ko_n(B\Gamma) \rightarrow KO_n(C_r^* \Gamma)) \cong \ker(A \circ p : \widetilde{ko}_n(B\Gamma) \rightarrow \widetilde{KO}_n(C_r^* \Gamma)).$$

Then using the commutative diagram and the fact that the p 's are isomorphisms, one finds an element

$$\gamma \in \ker(A \circ p : \oplus_{(C)} ko_n(BC) \rightarrow \oplus_{(C)} KO_n(C_r^* C))$$

which maps to β . In [3], Botvinnik-Gilkey-Stolz show a strong version of the GLR conjecture for finite cyclic groups, namely that

$$ko_n^+(BC) = \ker(A \circ p : ko_n(BC) \rightarrow KO_n(C_r^*C)).$$

Thus $\gamma \in \oplus_{(C)} ko_n^+(BC)$, and hence its image β lies in $ko_n^+(B\Gamma)$. Thus the strong version of the GLR conjecture for Γ is true, namely that

$$ko_n^+(B\Gamma) = \ker(A \circ p : ko_n(B\Gamma) \rightarrow KO_n(C_r^*\Gamma)).$$

Hence, by Theorem 1, our Main Theorem is proved. □

Remark 5. In fact, [3] shows that $ko_n^+(BC)$ is generated by lens spaces and simply-connected spaces, and our proof shows that the same is true for $ko_n^+(B\Gamma)$.

Remark 6. Fuchsian groups have almost periodic cohomology, i.e. $H^m(\Gamma) \cong H^{m+2}(\Gamma)$ provided $m > 2$. Thus our main result is not too surprising in view of [3]; the K -theory terms arising from X_g in a sense measure the failure of $H^*(\Gamma)$ to be periodic.

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