Math. Z. 207, 379–390 (1991)

**Mathematische Zeitschrift**



# Free actions on products of spheres: The rational case

**James F. Davis** [1,*] **and R.J. Milgram** [2,**]

[1] Department of Mathematics, Indiana University, Bloomington IN 47405, USA
[2] Department of Mathematics, Stanford University, Stanford, CA 94305, USA

Received in final form April 18, 1990

Let $X = S^{n_1} \times \ldots \times S^{n_k}$ be a product of spheres of total dimension $n = n_1 + n_2 + \ldots + n_k$. A fundamental unanswered question is the determination of which finite groups can act freely on $X$ and what actions on the cohomology so arise. In particular it is conjectured that if an elementary abelian group acts freely, then its rank is less than or equal to $k$. Great progress has been made recently on this question by Carlsson [C], Adem-Browder [A-B], and Hoffman [H] in the case where all the spheres have the same dimension.

In this paper we completely solve the rational analogue of the above question. Given an action of a finite group $G$ on the rational cohomology ring $H^*(X; \mathbb{Q})$, we give necessary and sufficient conditions for $G$ to act freely on a closed manifold $Y$ having the rational homotopy type of $X$, so that the $G$-action induces the specified action on $H^*(Y; \mathbb{Q}) = H^*(X; \mathbb{Q})$. In particular the necessary conditions give new obstructions for $G$ to act freely on $X$ with a specified representation in $H^*(X; \mathbb{Q})$. Our method includes a general discussion as to when a space with finite fundamental group has the rational homotopy type of a closed manifold.

We now give our necessary and sufficient conditions:

($A$) For all $g \in G$, for all $n_i$ even, $g^*[S^{n_i}] = \alpha[S^{n_j}]$ for some nonzero rational number $\alpha$.

($B$) For all $g \in G - \{e\}$, $\sum(-1)^i(tr(g_*: H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q})) = 0$.

($C$) (i) For $n$ even, some $n_i$ odd, the equivariant intersection form on $H^{n/2}(X; \mathbb{Q})$ is hyperbolic,

(ii) for all $n_i$ even, no further condition,

(iii) for $n$ odd, $\chi_{\frac{1}{2}}(X; \mathbb{Q}) \in \sigma^*(\Omega_n(G, w)) \subset L_n(\mathbb{Q}G, w)$.

In condition ($A$), $[S^{n_i}] \in H^{n_i}(X; \mathbb{Q})$ is the image of a generator of $H^{n_i}(S^{n_i}; \mathbb{Q})$ under the projection $X \to S^{n_i}$. Condition ($A$) will be necessary and sufficient for an action on a $CW$ complex of the correct rational homotopy type. Condition ($B$) is the Lefschetz fixed point condition and can be rephrased in terms of a rational finiteness obstruction. Conditions ($A$) and ($B$) are necessary and suffi-

* Research supported in part by an NSF Postdoctoral Fellowship and an NSF grant
** Research supported in part by an NSF grant and the SFB 170

cient for constructing an action on a finite complex. Condition (*C*) will be necessary for surgery and we now explain it.

Given the action of $G$ on $H^*(X;\mathbb{Q})$ define the orientation character $w: G \to \{\pm 1\}$ by $w(g)=1$ if and only if $g$ is orientation preserving. If $V$ is a rational representation of $G$, give $V^* = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ a $\mathbb{Q}G$-module structure by setting $(g\cdot\phi)v = w(g)\phi(g^{-1}v)$. For $n$ odd define $L_n(\mathbb{Q}G, w)$ to be the abelian group with

*Generators.* $[V]$, isomorphism classes of finitely generated $\mathbb{Q}G$-modules $V$ so that $V \cong V^*$.

*Relations.* (i) $[V]+[W]=[V\oplus W]$,
(ii) $[\mathbb{Q}G]=0$.
(iii) $[V]=0$ if there is a $\mathbb{Q}G$-isomorphism $f$: $V \to V^*$ so that

$$f=(-1)^{(n+1)/2}f^*: V\{=V^{**}\} \to V^*.$$

(Note that $V\oplus V^*$ admits the hyperbolic form, so by relation (iii) these $L$-groups have exponent 2.)

If $M$ is a closed manifold with a free $G$-action having orientation character $w$, we define the *surgery semicharacteristic* (cf. [D1])

$$\chi_{\frac{1}{2}}(M;\mathbb{Q}) = \chi_{\frac{1}{2}}(M/G;\mathbb{Q}G) = \sum_{i=0}^{(n-1)/2} (-1)^i [H_i(M;\mathbb{Q})] \in L_n(\mathbb{Q}G, w).$$

*Remark.* To justify the notation we observe that $L_n(\mathbb{Q}G, w)$ can be indentified with $L_n^h(\mathbb{Q}G, w) = L_h^n(\mathbb{Q}G, w)$ and that for $n$ odd, $\chi_{1/2}$ can be identified with the Mischenko-Ranicki symmetric signature $\sigma^*$.

Let $\Omega_n(G, w)$ be the bordism group of free $G$-actions on oriented manifolds with orientation character $w$. Then $\chi_{1/2}$ is a bordism invariant and $\sigma^*(\Omega_n(G, w))$ is defined to be the set of all surgery semicharacteristics which arise from closed manifolds. This group has been completely computed in [D-M]. For $n \equiv 3 \bmod 4$ $\sigma^*(\Omega_n(G, w))$ is zero and for $n \equiv 1 \bmod 4$ it is given as

$$\{[\mathbb{Q}G \otimes_{\mathbb{Q}C} V] \mid C \text{ a cyclic subgroup of } G\}$$

where $V$ is a $\mathbb{Q}C$-module and $w|_C$ is trivial.

Sample computations are that for groups of odd order $L_n(\mathbb{Q}G, w)=0$, and

$$L_1(\mathbb{Q}(\mathbb{Z}/2)^m, 1) \cong (\mathbb{Z}/2)^{2^m-1},$$

while $\sigma^*(\Omega_n(G, w)) = (\mathbb{Z}/2)^m$.

**Theorem A** *Let $X = S^{n_1} \times \ldots \times S^{n_k}$ with $n = n_1 + \ldots + n_k$, $n > 3$, and $n_i > 1$ for all $i$. Fix an action of $G$ on $H^*(X;\mathbb{Q})$. Then $G$ acts freely on some closed manifold $Y$ with the same rational homotopy type as $X$ and with the specified action on $H^*(Y;\mathbb{Q})$ if and only if conditions (A), (B), and (C) hold.*

*Remark.* The proof shows somewhat more. For any $n$ and $n_i$, conditions (*A*), (*B*), and (*C*) are necessary for an action of $G$ on $Y$ having the same rational homology as $X$. If $X = S^3$, the fulfillment of the conditions gives an action on a manifold $Y$ having the rational homology of $S^3$.

*Remark.* When $X = S^n$ the theorem was known previously. When $n$ is even the group $G$ must be either $\{1\}$ or $\mathbb{Z}/2$ by the Lefschetz fixed point theorem. When $n \equiv 3 \bmod 4$ any $G$ can occur (Pardon [P]), and when $n \equiv 1 \bmod 4$, $G$ must be isomorphic to a product of a cyclic group with a group of odd order (Weinberger [W]-Davis [D1]).

The method of proof of Theorem A is the usual surgery theoretic one; first find a (rational) Poincaré complex and then do (rational) surgery. It might be worth emphasizing a result deduced from local surgery theory:

**Proposition B** *Let $Z$ be a CW complex with finite fundamental group $G$ which satisfies rational Poincaré duality in dimension $n$ with $n$ odd. Then $Z$ has the rational homotopy type of a closed manifold if and only if*

$$\chi_{\frac{1}{2}}(Z; \mathbb{Q}G) \in \sigma^*(\Omega_n(G, w)) \subset L_n(\mathbb{Q}G, w).$$

Condition $(C)$ is much more explicit when $n = \dim X$ is odd. Here are some examples which we discuss in § 3.

(1) (homological triviality): Any finite group $G$ acts on a closed manifold $Y \simeq_{\mathbb{Q}} X$, trivially on $H^*(Y; \mathbb{Q})$, provided $k > 1$.

(2) (maximal homological action): Suppose $n_1, \ldots, n_k$ are all distinct. If $(\mathbb{Z}/2)^m$ acts preserving orientation on $H^*(X; \mathbb{Q})$ so that $\operatorname{Im}((\mathbb{Z}/2)^m \to \operatorname{Aut} H^*(X; \mathbb{Q}))$ is isomorphic to $(\mathbb{Z}/2)^{k-1}$ where $k$ is the number of factors of $X$, then

(a) when $n \equiv 1 \bmod 4$ the action can be realized by a free action on a closed manifold $Y \simeq_{\mathbb{Q}} X$ if and only if $m \leqq k$,

(b) when $n \equiv 3 \bmod 4$ the action can always be realized by a free action on a closed manifold $Y \simeq_{\mathbb{Q}} X$.

A. Heller [He] showed that $(\mathbb{Z}/2)^3$ does not act freely on $S^p \times S^q$, but the rational result is new. For example this says that $(\mathbb{Z}/2)^3$ cannot act freely, orientation-preserving, non-trivially on rational homology on any rational $S^2 \times S^3$. For $k > 2$ the restriction $m \leqq k$ is new even in the integral case of actions on $X$ itself.

These results should be compared to the theorems of Carlsson [C] and A. Adem-W. Browder [A-B]. In [C], [A-B], the spheres all have the *same dimension*. However, our rational results indicate that the situation is likely to be much more involved when more than one dimension occurs.

## § 1 Preliminaries

In this section we show $(A)$, $(B)$ are necessary and sufficient to produce an action on a finite complex in the correct rational homotopy type. First we set some conventions. For a $CW$ complex $Z$ let $Z_{(0)}$ denote a complex equipped with a map $Z \to Z_{(0)}$ which induces an isomorphism on $\pi_1$ and a localization

$$\begin{array}{ccc} \pi_n(Z) & \longrightarrow & \pi_n(Z_{(0)}) \\ & \searrow \quad \nearrow{\scriptstyle \cong} & \\ & \pi_n(Z) \otimes \mathbb{Q} & \end{array}$$

for $n>1$. For $Z$ simply-connected the existence and uniqueness (up to homotopy) of $Z_{(0)}$ is due to Sullivan [S] and for general $Z$ follows from the fiberwise localization functor of Bousfield-Kan [B-K]. We say two spaces $Z$ and $Z'$ have the same rational homotopy type if there is a homotopy equivalence $Z_{(0)} \simeq Z'_{(0)}$. We write $Z \simeq_{\mathbb{Q}} Z'$.

**Definition.** Two actions $\phi, \phi'$: $G \to \mathrm{Aut}(H^*(X; \mathbb{Q}))$ are isomorphic if there are $\alpha \in \mathrm{Aut}(G)$, $\beta \in \mathrm{Aut}(H^*(X;\mathbb{Q}))$, so that $\beta\phi(\alpha(g))\beta^{-1} = \phi'(g)$ for all $g \in G$.

We now study the isomorphism classes of actions on the cohomology of spaces rationally equivalent to a product of spheres.

Write $X = X_1 \times X_2 \times X_3 \times \ldots$ where $X_i$ is a product of $m_i$ $i$-dimensional spheres. Consider $H^*(X_i; \mathbb{Q})$ and $H^*(S^{n_i}; \mathbb{Q})$ as subrings of $H^*(X;\mathbb{Q})$ via the projections $p_i$: $X \to X_i$ for $i$ odd, and $p_{n_i}$: $X \to S^{n_i}$ for $n_i$ even.

There is considerable freedom in the choice of projection $p_i$: $X \to X_i$ for $i$ odd since, rationally, $S^{2n+1}$ is an Eilenberg-MacLane space. However, $S^{2n}$ is, rationally, a two stage Postnikov system

$$K(\mathbb{Q}; 4n-1) \to S^{2n}_{(0)} \to K(\mathbb{Q}; 2n),$$

and the projection $p_{n_i}$ is quite rigid in this case. With this in mind we have

**Proposition 1.1** *Suppose a finite group $G$ acts on $H^*(X;\mathbb{Q})$ as above. Then the action is isomorphic to one so that the following two properties are satisfied.*

(a) *For all $g \in G$, $g^*(H^*(X_l; \mathbb{Q})) = H^*(X_l; \mathbb{Q})$.*

(b) *One can choose generators $[S^{n_1}], \ldots, [S^{n_k}]$ of $H^{n_1}(S^{n_1}; \mathbb{Q}), \ldots, H^{n_k}(S^{n_k}; \mathbb{Q})$ so that for all $g \in G$, and for all $n_i$ even, $g^*([S^{n_i}]) = \pm [S^{n_j}]$ for some $j$.*

*Proof.* Let $\mathcal{M}_l \subset H^l(X;\mathbb{Q})$ be the set of decomposables (the vector subspace spanned by cup products of lower dimensional classes). Clearly $\mathcal{M}_l$ is a $G$-submodule of $H^l(X;\mathbb{Q})$ for all $l$. If $l$ is odd, by Maschke's theorem there is a complimentary $G$-submodule $\mathcal{N}_l$ so that $H^l(X;\mathbb{Q}) = \mathcal{M}_l \oplus \mathcal{N}_l$. Then choose an isomorphic action so that $\mathcal{N}_l$ is $im(p_l^*)$.

Thus we may assume condition (a) is satisfied for $l$ odd. We complete the proof of (a) by induction. Write $a_{n_i} = p_{n_i}^*([S^{n_i}])$. Clearly $a_{n_i}^2 = 0$. For $l = n_i$ even, $g \in G$, let $g^*(a_{n_i}) = b + \sum \alpha_{ij} a_{n_j}$ where $b \in \mathcal{M}_l$ and the sum is over all $n_j = l$. By induction, $\sum \alpha_{ij} a_{n_j} \neq 0$. Then

$$0 = (g^*(a_{n_i}))^2 = b^2 + 2b \sum \alpha_{ij} a_{n_j} + 2 \sum_{j<k} \alpha_{ij}\alpha_{ik} a_{n_j} a_{n_k}.$$

Then $b=0$ so 1.1(a) follows, and at most one $\alpha_{ij}$ is non-zero so a weak form of 1.1(b) follows, namely, $g^*([S^{n_i}]) = \alpha [S^{n_j}]$ for some non-zero number $\alpha$.

For a group $H$, the wreath product $\Sigma_m \wr H$ is the semidirect product given by letting the permutation group $\Sigma_m$ act on $H^m$ by permuting coordinates. For $H \subset \mathbb{Q}^*$ we identify this with the set of $m \times m$ $H$-permutation matrices, that is, all matrices with exactly one nonzero element in each row or column, and all the nonzero elements are in $H$. By the above we have (for $i$ even) that the image of $G \to \mathrm{Aut}(H^i(X_i; \mathbb{Q})) = GL_{m_i}(\mathbb{Q})$ is contained in $\Sigma_{m_i} \wr \mathbb{Q}^*$. To complete the proof of 1.1 we show

**Lemma 1.2** *If $\rho$ is a finite subgroup of $\Sigma_m \wr \mathbb{Q}^* \subset GL_m(\mathbb{Q})$, then there is a diagonal matrix $D$ so that $D\rho D^{-1} \subset \Sigma_m \wr \{\pm 1\}$.*

*Proof.* We assume that $\rho$ acts transitively on the lines $\mathbb{Q}e_1, \ldots, \mathbb{Q}e_m$ where $\{e_1, \ldots, e_m\}$ is the standard basis; in the general case one considers each orbit separately. Clearly we can find a diagonal $D$ so that for every $j$, there exists a $g_j \in \rho$ such that $(Dg_j D^{-1})e_1 = e_j$. Thus for a general $g \in \rho$ and a general $j$, if $(DgD^{-1})e_j = ae_l$ then

$$(Dg_j D^{-1})(Dg_l D^{-1})^{-1}(DgD^{-1})e_j = ae_j.$$

Since $g_j g_l^{-1} g$ is of finite order so is $a \in \mathbb{Q}^*$. Thus $a = \pm 1$. □

*Remark.* 1.1 shows that condition $(A)$ is actually a consequence of having an action on the cohomology ring (or which respects the homology coproduct); its importance is due to:

**Proposition 1.3** *$(A)$ is necessary and sufficient to construct a free acton on a CW complex $Y(\simeq_{\mathbb{Q}} X)$ with specified action on its rational cohomology ring.*

*Proof.* Given an action on $H^*(X; \mathbb{Q})$ it suffices to construct a $G$-action on a space $Y' \simeq_{\mathbb{Q}} X$ and then to "free it up" by letting $Y$ be $EG \times Y'$. For $i$ odd, $X_{i(0)} \simeq K(\mathbb{Q}^{m_i}; i)$. Any element of $GL_m(\mathbb{Q})$ induces a homotopy equivalence of $X_{i(0)}$. A sufficiently functorial construction of Eilenberg-MacLane spaces (for example the construction given in [M]) gives an action of $GL_{m_i}(\mathbb{Q})$ on $K(\mathbb{Q}^{m_i}, i)$, and thus gives the desired action of $G$ on $K(\mathbb{Q}^{m_i}, i) \simeq X_{i(0)}$. For $i$ even $X_{i(0)}$ has a 2-stage Postnikov tower. We could either construct this space in a sufficiently canonical way or use 1.1(b) and construct an action on $X_i = S^i \times \ldots \times S^i$ by use of permuting coordinates and the antipodal map. One then takes the diagonal $G$-action on

$$X_{1(0)} \times X_2 \times X_{3(0)} \times X_4 \times \ldots . \quad \square$$

**Proposition 1.4** *$(A)$ and $(B)$ are necessary and sufficient for constructing a free action on a finite CW complex $Y(\simeq_{\mathbb{Q}} X)$ with the specified action on cohomology.*

This is a consequence of 1.3 and Lemma 1.5 below.

**Lemma 1.5** *Let $Z$ be a CW complex with finite fundamental group $G$ and universal cover $\tilde{Z}$. The following are equivalent.*

(a) *$H_*(\tilde{Z}; \mathbb{Q})$ is finitely generated, and for all $g \in G - \{e\}$*

$$\sum (-1)^i (tr(g_*: H_i(\tilde{Z}; \mathbb{Q}) \to H_i(\tilde{Z}; \mathbb{Q}))) = 0.$$

(b) *$H_*(\tilde{Z}; \mathbb{Q})$ is finitely generated and $\sum_{i=0}^{\infty} (-1)^i [H_i(\tilde{Z}; \mathbb{Q})] = 0 \in \tilde{K}_0(\mathbb{Q}G)$.*

(c) *$Z \simeq_{\mathbb{Q}}$ a finite complex.*

*Proof.* (a)$\Rightarrow$(b): $\tilde{K}_0(\mathbb{Q}G)$ is the Grothendieck group of virtual rational representations modulo regular representations. Since "characters determine representations"

$$\tilde{K}_0(\mathbb{Q}G) \to \bigoplus_{\substack{C \subset G, \\ C \text{ cyclic}}} \tilde{K}_0(\mathbb{Q}C)$$

is injective. Thus it suffices to prove it for cyclic groups in which case it is a direct calculation with characters.

(b)$\Rightarrow$(c): Since $\mathbb{Q}G$ is semisimple, $\{H_*(\tilde{Z};\mathbb{Q}), 0\}$ is a finite projective $\mathbb{Q}G$ chain complex which is chain homotopy equivalent to the cellular chain complex $\{C_*(\tilde{Z};\mathbb{Q}), \partial\}$. By (b) it follows that $\{C_*(\tilde{Z};\mathbb{Q}), \partial\}$ is chain homotopy equivalent to a finite free complex. The result then follows by mimicking the construction of the Wall finiteness obstruction [Wa].

(c)$\Rightarrow$(a): Follows from the Lefschetz fixed point theorem. □

## §2 Rational surgery theory

In this section we discuss the question of when a space $Y$ has the rational homotopy type of a closed manifold. The main tool is local surgery theory following the accounts of [Q], [T-W] and the computation of the rational symmetric signature in [D-M], [D2].

An *n-dimensional rational Poincaré complex* is a $CW$ complex $Y$ with zero rational finiteness obstruction (i.e. there is a rational homotopy equivalence $K \to Y$ with $K$ finite), orientation character $w: \pi_1(Y) \to \{\pm 1\}$ and a class $[Y] \in H_n(Y;\mathbb{Q}^w)$ inducing Poincaré duality

$$\cap[Y]: H^i(Y;\mathbb{Q}[\pi_1 Y]) \xrightarrow{\sim} H_{n-i}(Y;\mathbb{Q}[\pi_1 Y]).$$

($\mathbb{Q}^w$ is $\mathbb{Q}$ given a $\pi_1 Y$-action via $g\cdot a = w(g)\,a$.) Associated to $Y$ is the Mischenko-Ranicki symmetric signature [R2]

$$\sigma^* Y \in L^n_h(\mathbb{Q}[\pi_1 Y], w).$$

Since $\frac{1}{2} \in \mathbb{Q}$, $L^h_n(\mathbb{Q}[\pi_1 Y], w) = L^n_h(\mathbb{Q}[\pi_1 Y], w)$.

Recall the definition of a $\mathbb{Q}$-normal map from [T-W]; in the case where $w=1$ and CAT$=$DIFF this is a map $f: M^n \to Y$ together with a map $\bar{f}: Y \to \mathrm{BSO}_{(0)}$ so that $f_*[M] = [Y]$ and

$$\begin{array}{ccc} M & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \nu_M} & & \downarrow{\scriptstyle \bar{f}} \\ \mathrm{BSO} & \longrightarrow & \mathrm{BSO}_{(0)} \end{array} \quad \text{commutes.}$$

($M^n$ is a closed smooth manifold.) Let $N(Y;[Y])$ denote the set of bordism classes of $\mathbb{Q}$-normal maps to $Y$ with $f_*[M] = [Y]$. The main results of rational surgery theory can be summarized by:

**Theorem 2.1** *Let $Y$ be a rational Poincaré complex with fundamental group $\pi$ and orientation character $w$.*

(a) $N(Y;[Y]) \neq \varnothing$.

(b) *There is a surgery obstruction map $\sigma_*: N(Y;[Y]) \to L^h_n(\mathbb{Q}\pi, w)$ given by $\sigma_*(f,\bar{f}) = \sigma^*(M) - \sigma^*(Y)$. If $n>4$, then $\sigma_*(f,\bar{f})=0$ if and only if $(f,\bar{f})$ is normally bordant to a $\mathbb{Q}$-homotopy equivalence.*

(c) *There is an exact sequence of sets*

$$N(Y;[Y]) \xrightarrow{l\times\beta'} [Y,(G/O)]_{(0)} \oplus \Omega_n(\pi, w) \xrightarrow{\alpha} \Omega_n(\pi, w)\otimes \mathbb{Q}$$

*where* $\beta'(f,\bar{f})=[M\to B\pi]$. *Furthermore* $\Omega_{n+1}(\pi, w)\otimes\mathbb{Q}$ *acts on* $N(Y;[Y])$ *so that* $\alpha_1$ *and* $\alpha_2$ *are in the same orbit if and only if* $(l\times\beta')\alpha_1=(l\times\beta')\alpha_2$.

*Proof.* (b) is essentially ordinary surgery theory and (c) is discussed in [T-W], [Q] so we concentrate on (a). It suffices to show $N(K;[K])\neq\varnothing$ where $K$ is a finite complex and $K\to Y$ is a rational homotopy equivalence. The Spivak bundle $\nu_K\to K$ is a stable fibration with fiber $S^{l-1}_{(0)}$ equipped with a class $c_K\in\pi_{l+n}(T(\nu_K))$ which maps to [K] under the Hurewicz and Thom maps. As usual it suffices to find a lift of $\nu_k$ to $BO(l)$ and transversality gives us the required normal map. According to Sullivan [S], the classifying space for $S^l_{(0)}$ fibrations is

$$BSG_{(0)}\times B\mathbb{Q}^*\simeq B\mathbb{Q}^*.$$

However the finiteness of $K$ guarantees that $\nu_K$ factors as $\nu_K\colon K\to B\{\pm 1\}\to B\mathbb{Q}^*$, with the first map given by $w\colon \pi_1 K\to\{\pm 1\}$. A lift to $BO(l)$ is given by the pullback of the canonical line bundle over $B\{\pm 1\}$. □

**Addendum.** In fact this gives $f\colon M\to Y$ so that the rational Pontrajagin classes for $M$ are zero.

Now we restrict ourselves to the case where $\pi$ is finite. If $n$ is odd, $\sigma^*(Y)$ can be identified with the surgery semicharacteristic $\chi_{1/2}(\tilde{Y};\mathbb{Q})$ of its universal cover. For $n$ even, $\sigma^*(Y)$ can be identified with the Witt class of the direct sum of the equivariant intersection form on $H^{n/2}(\tilde{Y};\mathbb{Q})$ with the hyperbolic construction on the (virtual) $\mathbb{Q}\pi$-module

$$\sum_{i<n/2}(-1)^i[H_i(M;\mathbb{Q})]\in\tilde{K}_0(\mathbb{Q}\pi).$$

Clearly a necessary condition for $Y$ to have the rational homotopy type of a closed manifold is for

$$\sigma^*(Y)\in\mathrm{Im}(\sigma^*\colon \Omega_n(\pi_1(Y), w)\to L^h_n(\mathbb{Q}[\pi_1(Y)], w).$$

On the other hand we have

**Theorem 2.2** *Let* $\pi$ *be a finite group with orientation character* $w$.

(a) *For* $n$ *odd,* $\sigma^*(\Omega_n(\pi, w))=\sum i_*(L^h_n(\mathbb{Q}C, w))$ *where the sum is over all inclusions* $i\colon C\to\pi$ *of cyclic subgroups* $C$ *so that* $w(C)=1$.

(b) *For* $n\equiv 0 \bmod 4$, $\sigma^*(\Omega_n(\pi, w))=i_*(L^0(\mathbb{Z}))$ *where* $i\colon 1\to\pi$ *is the inclusion of the trivial group.* $L^0(\mathbb{Z})$ *is identified with* $\mathbb{Z}$ *by the usual signature map.*

(c) *For* $n\equiv 2 \bmod 4$, $\sigma^*(\Omega_n(\pi, w))\subset\ker(L^h_n(\mathbb{Q}\pi, w)\to L^p_n(\mathbb{Q}\pi, w))$.

The proof of (*a*) is in [D-M]; those of (*b*) and (*c*) are in [D2].

**Corollary 2.3** *Let* $Y$ *be a* $\mathbb{Q}$*-Poincaré complex of dimension* $n>4$ *with finite fundamental group* $\pi$ *and orientation character* $w$.

(a) *If* $n$ *is odd, the* $Y$ *has the rational homotopy type of a manifold if and only if* $\chi_{1/2}(Y;\mathbb{Q}\pi)\in\sigma^*(\Omega_n(\pi, w))$, *and*

(b) *under any one of the following hypotheses,*

(i) $n \equiv 0 \bmod 2$, (ii) $n \equiv 0 \bmod 4$, *w non-trivial, and* $\chi(Y)$ *even, or*

(iii) $n \equiv 0 \bmod 4$, *w trivial, and the signature of* $\tilde{Y}$ *is zero,*

*then Y has the rational homotopy type of a closed manifold if and only if the equivariant intersection form on* $H^{n/2}(\tilde{Y}; \mathbb{Q})$ *is hyperbolic.*

*Proof.* A transfer argument shows that

$$\text{res}: \Omega_n(\pi, 1) \otimes \mathbb{Q} \to \Omega_n \otimes \mathbb{Q}$$

is an isomorphism. By Thom's theorem, $\Omega_n \otimes \mathbb{Q}$ is detected by Pontrajagin numbers. An argument with the Gysin sequence shows that if $w \neq 1$, then $\Omega_n(\Pi, w)$ is torsion. It follows that $\Omega_n(\pi, w) \otimes \mathbb{Q}$ is zero for $n \not\equiv 0 \bmod 4$. Thus if $n \not\equiv 0 \bmod 4$ and $\sigma^*(Y) \in \sigma^*(\Omega_n(\pi, w))$ or $n \equiv 0 \bmod 4$ and $\sigma^*(Y) = 0$ then we can find $(M \to B_\pi) \in \text{torsion } \Omega_n(\pi, w)$ with $\sigma^*(M) = \sigma^*(Y)$. Then

$$* \times [M \to B_\pi] \in [Y, G/O]_{(0)} \oplus \Omega_n(\pi, w)$$

maps to zero so this is the image of a $\mathbb{Q}$-normal map with surgery obstruction $\sigma^*(M) - \sigma^*(Y) = 0$, so that $Y$ has the rational homotopy type of a closed manifold. Part (a) follow from 2.2 and the above observations.

Next suppose $n \equiv 0 \bmod 4$ and $w \neq 1$. Then $\sigma^*(\Omega_n(\pi, w))$ is generated by the form $\langle 1 \rangle$. Choosing $g \in \pi$ so that $w(g) = -1$ we see $-1 = g\bar{g}$ where $\bar{g} = w(g) g^{-1}$. It follows that $\langle 1 \rangle \perp \langle 1 \rangle \cong \langle 1 \rangle \perp \langle -1 \rangle$ which is zero in $L_n^h(\mathbb{Q}\pi, w)$. Thus if $\chi(Y)$ is even, then $\sigma^*(Y) \in \sigma^*(\Omega_n(\pi, w))$ if and only if $\sigma^*(Y)$ is zero. Next consider the two exact sequences

$$H^{n+1}(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Q}\pi)) \longrightarrow L_n^h(\mathbb{Q}\pi, w) \to L_n^p(\mathbb{Q}\pi, w),$$

$$H^{n+1}(\mathbb{Z}/2; \mathbb{Z}) \to H^{n+1}(\mathbb{Z}/2; K_0(\mathbb{Q}\pi)) \longrightarrow H^{n+1}(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Q}\pi))$$

$$\xrightarrow{\delta} H^{n+2}(\mathbb{Z}/2; \mathbb{Z}) \to \dots.$$

Since $\mathbb{Z}/2$ permutes the generators of the free abelian group $K_0(\mathbb{Q}\pi)$ and $n+1$ is odd, it follows that $H^{n+1}(\mathbb{Z}/2; K_0(\mathbb{Q}\pi)) = 0$. Thus $\ker(L_n^h(\mathbb{Q}\pi, w) \to L_n^p(\mathbb{Q}\pi, w))$ is at most of order 2 and any element in the kernel must be on an odd rank free module. Thus, if $n$ is even and $\chi(Y) = 0$, $\sigma^*(Y) = 0$ if and only if $im(\sigma^*(Y)) = 0 \in L_n^p(\mathbb{Q}\pi, w)$ which occurs if and only if the intersection form on $H^{n/2}(\tilde{Y}; \mathbb{Q})$ is hyperbolic. This takes care of case (b) (i).

Finally, if $n \equiv 0 \bmod 4$ and $w = 1$, then $\sigma^*(\Omega_n(\pi, w))$ is generated by $\langle 1 \rangle$. Applying the transfer to the trivial group we see that $tr(\langle 1 \rangle) = |\pi| \langle 1 \rangle$ which is detected by the signature. Thus, if the signature of $\tilde{Y}$ is zero, $\sigma^*(Y) \in \Omega_n(\pi, w)$ if and only if $\sigma^*(Y) = 0$, and reasoning as above this happens if and only if the intersection form on $H^{n/2}(\tilde{Y}; \mathbb{Q})$ is hyperbolic. □

## § 3 Proof of Theorem A and examples

Recalling earlier notation let $X = S^{n_1} \times \dots \times S^{n_k}$, and suppose $G$ acts on $H^*(X; \mathbb{Q})$ satisfying $(A)$, $(B)$, and $(C)$. There are three cases:

(i) $n$ is even but some $n_i$ is odd,

(ii) all $n_i$ are even,
(iii) $n = \dim X$ is odd.

In cases (i) and (iii) we use rational surgery theory while in case (ii) we rely on the work of Hoffman [H]. If conditions $(A)$ and $(B)$ are satisfied, by 1.4, $G$ acts freely on a finite complex $Y \simeq_{\mathbb{Q}} X$. By a standard transfer argument

$$\mathrm{tr}\colon H_*(Y/G; \mathbb{Q}^w) \xrightarrow{\sim} \{\alpha \in H_*(Y; \mathbb{Q}) \mid g\alpha = w(g)\alpha \text{ for all } g \in G\}.$$

Choose $[Y/G]$ to be the homology class mapping to $[X]$ under the transfer. Then $Y/G$ is a $\mathbb{Q}$-Poincaré complex. Case (iii) is complete by 2.3. Likewise in case (i) the signature of $Y$ and the Euler characteristic of $Y/G$ are zero so that 2.3 applies.

In case (ii) we show that $(A)$ and $(B)$ actually give a free linear action on $X$. By Lemma 1.2 we may assume the representation on cohomology is given by $\rho\colon G \to \Sigma_k \int \{\pm 1\}$. We let $\rho(g) = (a_{ij}) \in GL_k(\mathbb{Q})$ and $\sigma \in \Sigma_k$ be the underlying permutation of $\rho(g)$. Define an action on $\mathbb{R}^{n_1+1} \times \ldots \times \mathbb{R}^{n_k+1}$,

$$g \cdot (v_1, \ldots, v_k) = (a_{1\sigma(1)} v_{\sigma(1)}, \ldots, a_{k\sigma(k)} v_{\sigma(k)}).$$

By Hoffman's results, [H], this gives a free action on $X$. (It is in [H] where the hypothesis that the Lefschetz numbers all vanish is used. Also, we should note that Hoffman only deals with products of spheres of the same dimension, however his proof extends directly to our more general setting.) □

*Example* (*1*) (*homological triviality*). Let $n$ be odd and $X = S^{n_1} \times \ldots \times S^{n_k}$ with $k > 1$ and $n_i > 1$. Then any finite group $G$ acts on a closed manifold $Y \simeq_{\mathbb{Q}} X$, trivially on $H^*(Y; \mathbb{Q})$.

First just assume that $G$ acts on $H^*(X; \mathbb{Q})$ preserving orientation. Define

$$|\chi(X; \mathbb{Q})| = \sum [H_i(X; \mathbb{Q})] \in K_0(\mathbb{Q}(G)),$$
$$|\chi_{1/2}(X; \mathbb{Q})| = \sum_{i < n/2} [H_i(X; \mathbb{Q})] \in K_0(\mathbb{Q}(G)).$$

Since $w = 1$, $V^* \cong V$ for all representations, thus by Poincaré duality,

$$\tfrac{1}{2} |\chi(X; \mathbb{Q})| = |\chi_{1/2}(X; \mathbb{Q})|.$$

Since the $L$-group has exponent 2, $|\chi_{1/2}(X; \mathbb{Q})| = \chi_{1/2}(X; \mathbb{Q}) \in L_n(\mathbb{Q}(G), 1)$.

Now, if the action is trivial on homology, $|\chi(X; \mathbb{Q})| = 2^k [\mathbb{Q}]$ where $\mathbb{Q}$ is the trivial representation. Then $\chi_{1/2}(X; \mathbb{Q}) = 2^{k-1} [\mathbb{Q}] = 0 \in L_n(\mathbb{Q}(G), 1)$.

*Example* (*2*) (*Actions of elementary abelian groups when n is odd*). Here condition $(A)$ is always satisfied. If $w$ is non-trivial, then $L_n(\mathbb{Q}(G), w) = 0$ (see [D-M]), so one must only verify condition $(B)$ and it is easy to check whether or not $(B)$ holds. If, on the other hand, $w$ is trivial, then Poincaré duality implies that $(B)$ is satisfied. The group $L_n(\mathbb{Q}[(\mathbb{Z}/p)^m], 1)$ is zero unless $p = 2$ and $n \equiv 1 \bmod 4$. We will eventually concentrate on this case.

**Definition.** Given an action of $G$ on $H^*(X; \mathbb{Q})$ and $H$ on $H^*(Y; \mathbb{Q})$, the product action of $G \times H$ on $H^*(X \times Y; \mathbb{Q}) = H^*(X; \mathbb{Q}) \otimes H^*(Y; \mathbb{Q})$ is $(g, h)(x \otimes y) = gx \otimes hy$. Given actions of $G$ on $H^*(X; \mathbb{Q})$ and $H^*(Y; \mathbb{Q})$, the tensor product action of $G$ on $H^*(X \times Y; \mathbb{Q})$ is the restriction of the product action to the diagonal subgroup of $G \times G$.

If $X = S^{n_1} \times \ldots \times S^{n_k}$ we define the geometric rank $gr(X) = k$. If $(\mathbb{Z}/2)^m$ acts faithfully on $H^*(X; \mathbb{Q})$, then $m \leqq gr(X)$. Indeed, by 1.1(a), the action is determined by the (direct sum) action on the direct sum $\oplus H^l(X_l; \mathbb{Q})$ which has total dimension $k$. Since every rational representation is a direct sum of 1-dimensional representations the result follows.

**Proposition 3.1.** *If $(\mathbb{Z}/2)^k$ acts faithfully on $H^*(X; \mathbb{Q})$ with $gr(X) = k$, then the action is isomorphic to a product of the following types of actions.*

(a) $\mathbb{Z}/2 = \langle g \rangle$ *acts on* $H^*(S^j)$ *by* $g^*([S^j]) = -[S^j]$ *with $j$ odd,*
(b) $\mathbb{Z}/2 = \langle g \rangle$ *acts on* $H^*(S^j)$ *by* $g^*([S^j]) = -[S^j]$ *with $j$ even,*
(c) $\mathbb{Z}/2 \times \mathbb{Z}/2 = \langle g \rangle \times \langle h \rangle$ *acts on* $H^*(S^j \times S^j)$ *with $j$ even, by* $g^*[S^j \times 1] = [1 \times S^j]$, $h^*[1 \times S^j] = -[S^j \times 1]$.

*Proof.* If we have a decomposition $X = Y_1 \times Y_2$ so that $H^*(Y_i; \mathbb{Q})$ is invariant under $(\mathbb{Z}/2)^k$ then the action is a product action. Let

$$G_i = \ker((\mathbb{Z}/2)^k \to \operatorname{Aut}(H^*(Y_i; \mathbb{Q}))).$$

Then $k - rk(G_i) \leqq gr(Y_i)$, and so $k \leqq rk(G_1) + rk(G_2)$. Since the action is faithful $G_1 \cap G_2 = 0$, so $G = G_1 \times G_2$ and the action is a product.

By the above remarks, induction on $k$, and 1.1(a) we may assume $X = (S^j)^k$. If $j$ is odd then $H^*(X; \mathbb{Q}) = E(H^j(X; \mathbb{Q}))$. We may write $H^j(X; \mathbb{Q})$ as a direct sum of 1-dimensional representations $\oplus V_i$. Thus $H^*(X; \mathbb{Q}) = \otimes E(V_i)$. By an automorphism of $H^*(X; \mathbb{Q})$ we may assume the action is a product of faithful actions of $\mathbb{Z}/2$ on $H^*(S^j; \mathbb{Q})$ which must be of type (a) above.

If $j$ is even, then condition ($A$) and the fact that $(\mathbb{Z}/2)^k$ has exponent 2 imply that the action is a product of actions on $H^*(S^j; \mathbb{Q})$ or an indecomposable action on $H^*(S^j \times S^j; \mathbb{Q})$, which must be of type (b) or (c) respectively. □

Let $R[(\mathbb{Z}/2)^k] = \mathbb{Z}[x_1, \ldots, x_k]/\langle x_1^2 - 1, \ldots, x_k^2 - 1 \rangle$ be the rational representation ring. Here $x_i$ is the representation $\mathbb{Q} \to \mathbb{Q}$ given by $(a_1, \ldots, a_k)v = (-1)^{a_i} v$. Then for actions of

type (a), $\chi(X) = 1 - x_1$,
type (b), $\chi(X) = 1 + x_1$,
type (c), $\chi(X) = 2 + x_1 + x_2$.

Since the Euler characteristic is multiplicative we see that if $(\mathbb{Z}/2)^k$ acts faithfully on $H^*(X; \mathbb{Q})$ with $gr(X) = k$, then

(1) $\chi(X) \neq 0 \in K_0(\mathbb{Q}(G))/2K_0(\mathbb{Q}(G))$,
(2) $\chi(X) = [\mathbb{Q}((\mathbb{Z}/2)^k)] \in K_0(\mathbb{Q}(G))/2K_0(\mathbb{Q}(G))$, if and only if the action is a product of actions of type (a) and (b).
(3) $\chi(X) = [\mathbb{Q}(\mathbb{Z}/2)^k] \in K_0(\mathbb{Q}(G))$ if and only if the action is a product of actions of type (b).

In particular, if $n = \dim(X)$ is odd, then condition ($B$) will not be satisfied. We thus consider the case where $(\mathbb{Z}/2)^{k-1}$ acts faithfully on $H^*(X; \mathbb{Q})$. If $w \neq 1$ the $L$-group is zero so we concentrate on:

**Theorem 3.2 (Maximal homological triviality).** *Let $k=gr(X)$ and let $n=\dim(X)$ be congruent to* $1 \bmod 4$. *Suppose $G=(\mathbb{Z}/2)^m$ acts preserving orientation on $H^*(X;\mathbb{Q})$ so that $\operatorname{Im}((\mathbb{Z}/2)^m \to \operatorname{Aut}(H^*(X;\mathbb{Q})))$ is isomorphic to $(\mathbb{Z}/2)^{k-1}$. Then*

(a) *If $m>k$ then the action cannot be realized as a free action on a closed manifold $Y \simeq_{\mathbb{Q}} X$.*

(b) *If the dimensions of the spheres are all distinct and $m \leqq k$ then the action can be realized as a free action on a closed manifold $Y \simeq_{\mathbb{Q}} X$.*

*Proof.* Condition $(A)$ is automatic and condition $(B)$ holds by Poincaré duality. Here $\mathbb{Q}(G) \cong \mathbb{Q}^{2^m}$ with the trivial involution. No factor admits a skew-symmetric form, hence $L_n(\mathbb{Q}(G)) \cong \tilde{K}_0(\mathbb{Q}(G))/2\tilde{K}_0(\mathbb{Q}(G))$. Define a function $\varepsilon\colon L_n(\mathbb{Q}(G)) \to \{0, 1, 2, \ldots, 2^{m-1}\}$, where $\varepsilon(\alpha)=i$ if $\alpha$ can be represented as a sum of $i$ distinct 1-dimensional representations. Now $\sigma^*(\Omega_n(G))$ consists of representations induced from $\mathbb{Z}/2$-subgroups, so $\varepsilon(\operatorname{im}\sigma^*)=\{0, 2^{m-1}\}$.

Since there is an odd-dimensional sphere as a factor of $X$ and since the irreducible representations of $G$ are 1-dimensional, we can (by an automorphism of $H^*(X;\mathbb{Q})$) assume that the action is a tensor product of actions on $H^*(Z;\mathbb{Q}) := H^*(S^{n_1} \times \ldots \times S^{n_{k-1}};\mathbb{Q})$ and $H^*(S^{n_k};\mathbb{Q})$. We may further assume that the first $2^{k-1}$ coordinates, $(\mathbb{Z}/2)^{k-1} \subset (\mathbb{Z}/2)^m$, act faithfully on $H^*(Z;\mathbb{Q})$.

The action of $(\mathbb{Z}/2)^{k-1}$ on $H^*(Z;\mathbb{Q})$ is a product of actions of type (a), (b), and (c) above. To assure that the action is orientation preserving, the action on $H^{n_k}(S^{n_k};\mathbb{Q})$ must be a product

$$\phi = \prod_{i=1}^{k-1} x_i^{a_i}$$

where $a_i=1$ if $S^{n_i}$ is of type (a) or (b) and is zero otherwise. It follows that

$$(1 \pm \phi) \cdot \chi(Z;\mathbb{Q}) = \chi(X;\mathbb{Q}) = 0$$

and hence

$$|\chi_{1/2}(X;\mathbb{Q})| = \tfrac{1}{2}|\chi(X;\mathbb{Q})| = |\chi(Z;\mathbb{Q})| \in L_n(\mathbb{Q}(G)).$$

In particular $\varepsilon(\chi_{1/2}(X;\mathbb{Q}))$ is represented by an integer contained in $(0, 2^{k-1}]$. Part (a) of 3.2 follows immediately.

If all the spheres have distinct dimensions then the action on $H^*(Z;\mathbb{Q})$ has no type (c) factors. It follows that $\chi_{1/2}(X;\mathbb{Q}) \in L_n(\mathbb{Q}(G))$ is the representation induced from the trivial representation on $(\mathbb{Z}/2)^{m-k-1}$. Thus $\chi_{1/2} \in \operatorname{im}(\sigma^*)$ if and only if $m \leqq k$. □

*Example* (3) Assume $n$ is even and $H^*(X;\mathbb{Q})$ is the tensor product of actions on $H^*(Y_1;\mathbb{Q})$ and $H^*(Y_2;\mathbb{Q})$ with $\dim(Y_i)$ odd.

We will show that if conditions $(A)$ and $(B)$ are satisfied, then condition $(C)$ is automatic. By our construction in § 1

$$X \simeq_{\mathbb{Q}} V_1 \times V_2$$

where $G$ acts freely on $V_i$. The action on $V_1 \times V_2$ corresponds to the diagonal action and each $V_i/G$ satisfies odd dimensional Poincaré duality (but the rational finiteness condition is not necessarily satisfied for $V_i/G$).

If $M$ is a $\mathbb{Q}[G \times G] = \mathbb{Q}(G) \otimes_{\mathbb{Q}} \mathbb{Q}(G)$ module, let $\Delta^* M$ be the $\mathbb{Q}(G)$-module given by restricting the action to the diagonal subgroup. Since the projective

$L$-theory of a semisimple ring is zero [R1], $\sigma^*(V_i/G)$ vanishes in the projective $L$-groups, hence, applying the product formula, so does $\sigma^*((V_1/G)\times(V_2/G))$. Applying $\Delta^*$ we see that $\sigma^*((V_1\times V_2)/G)$ vanishes projectively. It follows that its intersection form is hyperbolic, so we are done.

We conclude with two questions:

(1) Are there examples of actions on $H^*(X;\mathbb{Q})$ with $\dim(X)$ even which satisfy $(A)$ and $(B)$ but not $(C)$?

(2) What are necessary and sufficient conditions for an even dimensional Poincaré complex with finite fundamental group to have the rational homotopy type of a closed manifold?

*Acknowledgement.* The authors would like to thank V. Puppe for suggesting that the results of [D-M] might be applied to study actions on a product of spheres.

## References

[A-B] Adem, A., Browder, W.: The free rank of symmetry of $(S^n)^k$. Invent. Math. **92**, 431–440 (1988)

[B-K] Bousfield, A.K., Kan, D.M.: Homotopy limits, completions, and localizations. Lecture Notes in Math, 304 (Springer, 1972)

[C] Carlsson, G.: On the rank of abelian groups acting freely on $(S^n)^k$. Invent. Math. **69**, 393–404 (1982)

[D1] Davis, J.F.: The surgery semicharacteristic. Proc. Lond. Math. Soc. (3) **47**, 411–428 (1983)

[D2] Davis, J.F.: The rational symmetric signature for manifolds with finite fundamental group. (In preparation)

[D-M] Davis, J.F., Milgram, R.J.: Semicharacteristics, bordism, and free group actions. Trans. A.M.S., **312**, 55–83 (1989)

[He] Heller, A.: A note on spaces with operators. Ill. J. Math. **3**, 98–100 (1959)

[H] Hoffman, M.: Free actions of abelian groups on a Cartesian power of an even sphere. Can. Math. Bull. **30**, 358–362 (1987)

[M] Milgram, R.J.: The bar construction and abelian $H$-spaces. Ill. J. Math. **11**, 242–250 (1967)

[M-S] Milnor, J.W., Stasheff, J.D.: Characteristic Classes. Ann. Math. Stud. **76**, Princeton: Princeton University Press 1974

[P] Pardon, W.: Mod 2 semicharacteristics and the converse to a theorem of Milnor. Math. Z. **171**, 247–268 (1980)

[Q] Quinn, F.: Semifree group actions and surgery on $PL$ homology manifolds, Geometric Topology. (Lect. Notes Math., Vol. 438, pp. 395–414) Berlin Heidelberg New York: Springer 1975

[R1] Ranicki, A.: On the algebraic $L$-theory of semisimple rings. J. Algebra **50**, 242–243 (1978)

[R2] Ranicki, A.: Exact sequences in the algebraic theory of surgery. Math. Notes **26**, Princeton: Princeton University Press 1981

[S] Sullivan, D.: Genetics of homotopy theory and the Adams conjecture. Ann. Math **100**, 1–80 (1974)

[T-W] Taylor, L., Williams, B.: Local surgery: foundations and applications, Proceedings, Aarhus 1978 (Lect. Notes Math., Vol. 763, pp. 673–695) Berlin Heidelberg New York: Springer 1979

[Wa] Wall, C.T.C.: Finiteness conditions for $CW$-complexes II. Proc. R. Soc. A**295**, 129–139 (1966)

[W] Weinberger, S.: Homologically trivial group actions II: Nonsimply connected manifolds. Am. J. Math. **108**, 1259–1276 (1986)

**Note added in proof**

This has been done by the authors in a preprint: The rational classification of manifolds with finite fundamental group.