

# Manifolds Homotopy Equivalent to Certain Torus Bundles over Lens Spaces

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## Abstract

We compute the topological simple structure set of closed manifolds that occur as total spaces of flat bundles over lens spaces  $S^l/(\mathbb{Z}/p)$  with fiber  $T^n$  for an odd prime  $p$  and  $l \geq 3$  provided that the induced  $\mathbb{Z}/p$ -action on  $\pi_1(T^n) = \mathbb{Z}^n$  is free outside the origin. To the best of our knowledge this is the first computation of the structure set of a topological manifold whose fundamental group is not obtained from torsionfree and finite groups using amalgamated and HNN-extensions. We give a collection of classical surgery invariants such as splitting obstructions and  $\rho$ -invariants that decide whether a simple homotopy equivalence from a closed topological manifold to  $M$  is homotopic to a homeomorphism. © 2020 Wiley Periodicals LLC

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## 0 Introduction

### 0.1 Flat Torus Bundles over Lens Spaces

Throughout this paper we will consider the following setup and notation:

- Let  $p$  be an odd prime.
- Let  $\rho: \mathbb{Z}/p \rightarrow \text{Aut}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Z})$  be a group homomorphism so that the induced action of  $\mathbb{Z}/p$  on  $\mathbb{Z}^n - \{0\}$  is free.
- The homomorphism  $\rho$  defines an action of  $\mathbb{Z}/p$  on the torus  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ . If we want to emphasize the  $\mathbb{Z}/p$ -action, we write  $\mathbb{T}_\rho^n = B\mathbb{Z}_\rho^n$ .
- Fix a free action of  $\mathbb{Z}/p$  on a sphere  $\mathbb{S}^l$  for an odd integer  $l \geq 3$ . We refer to the orbit space  $L^l := \mathbb{S}^l/(\mathbb{Z}/p)$  as a *lens space*.
- Define a closed  $(n + l)$ -manifold

$$M := \mathbb{T}_\rho^n \times_{\mathbb{Z}/p} \mathbb{S}^l.$$

- The fundamental group of  $M^{n+l}$  is the semidirect product denoted by

$$\Gamma := \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/p.$$

We computed the  $K$ -theory of the  $C^*$ -algebra of  $\Gamma$  in [13].

An example of such an action  $\rho$  is given by the regular representation  $\mathbb{Z}[\mathbb{Z}/p]$  modulo the ideal generated by the norm element, in which case we have  $\rho: \mathbb{Z}/p \rightarrow \text{Aut}(\mathbb{Z}^{p-1})$ .

The action of  $\mathbb{Z}/p$  on  $\mathbb{Z}^n - \{0\}$  is free if and only if the fixed point set  $(\mathbb{T}^n)^{\mathbb{Z}/p}$  is finite. Equip the torus and the sphere with the standard orientations; this determines an orientation on  $M$ . Since the  $\mathbb{Z}/p$ -action on  $\mathbb{S}^l$  is free, there is a fiber bundle  $\mathbb{T}_\rho^n \rightarrow M^{n+l} \rightarrow L^l$ . It is worth noting that  $\mathbb{T}^n$ ,  $L^\infty$ , and  $\mathbb{T}_\rho^n \times_{\mathbb{Z}/p} \mathbb{S}^\infty$  are models for  $B\mathbb{Z}^n$ ,  $B\mathbb{Z}/p$ , and  $B\Gamma$ , respectively. Notice that our assumptions imply that  $\dim(M) = n + l \geq 5$ .

Next we summarize our main results.

### 0.2 Geometric Topological Simple Structure Set of $M$

We will prove the following theorem in Theorem 9.2 (5).

**THEOREM 0.1** (Geometric topological simple structure set of  $M$ ). *As an abelian group we have*

$$\mathcal{S}^{\text{geo},s}(M) \cong \mathbb{Z}^{p^k (p-1)/2} \oplus \bigoplus_{i=0}^{n-1} L_{n-i}(\mathbb{Z})^{r_i},$$

where the natural number  $k$  is determined by the equality  $n = k(p - 1)$  and the numbers  $r_i$  are defined in (4.6).

In particular, the structure set is infinite.

To our knowledge this is the first computation of the structure set of a topological manifold whose fundamental group is not obtained from torsionfree and finite groups using amalgamated and HNN-extensions. The computation is rather

involved and based on the Farrell-Jones conjecture. We also compute the periodic structure sets of  $B\Gamma$  and of  $M$  and prove detection results for these structure sets. The notion of a structure set is recalled in Section 5. Its study is motivated by the question of determining the homeomorphism classes of closed manifolds homotopy equivalent to  $M$ .

### 0.3 Homotopy Equivalence Versus Homeomorphism

The following result gives a criterion when a simple homotopy equivalence of closed manifolds with  $M$  as target is homotopic to a homeomorphism. It is proved in Section 10.

**THEOREM 0.2** (Simple homotopy equivalence versus homeomorphism). *Let  $h: N \rightarrow M$  be a simple homotopy equivalence with a closed topological manifold  $N$  as source and the manifold  $M$  of Section 0.1 as target. Then  $h$  is homotopic to a homeomorphism if and only if the following conditions are satisfied:*

- *Vanishing of splitting obstructions:*

*Let  $\bar{h}: \bar{N} \rightarrow \mathbb{T}^n \times \mathbb{S}^l$  be obtained from  $h$  by pulling back the  $\mathbb{Z}/p$ -covering  $\mathbb{T}^n \times \mathbb{S}^l \rightarrow M$ . Consider any nonempty subset  $J \subset \{1, 2, \dots, n\}$ . Let  $T^J \times \bullet \subset \mathbb{T}^n \times \mathbb{S}^l$  be the obvious  $|J|$ -dimensional submanifold. By making  $\bar{h}$  transversal to  $T^J \times \bullet$ , we obtain a normal map  $(\bar{h})^{-1}(T^J \times \bullet) \rightarrow T^J \times \bullet$  that defines a surgery obstruction in  $L_{|J|}(\mathbb{Z})$ .*

*This obstruction has to be zero.*

- *Equality of  $\rho$ -invariants:*

*Consider any subgroup  $P \subset \Gamma$  of order  $p$ . Let  $P'$  be the image of  $P$  under the abelianization map  $\text{pr}: \Gamma \rightarrow \Gamma_{\text{ab}}$ . Let  $M_P \rightarrow M$  be the cover corresponding to the subgroup  $\text{pr}^{-1}(P')$ . Let  $h_P: N_P \rightarrow M_P$  be the corresponding covering simple homotopy equivalence. Then we must have the equality of  $\rho$ -invariants in  $\tilde{R}(P')^{(-1)^{(n+l+1)/2}}[1/p]$ :*

$$\rho(N_P \rightarrow BP') = \rho(M_P \rightarrow BP'),$$

Here is a detailed outline of the paper. Using the Farrell-Jones conjecture, it is straightforward to compute  $K_1$  and  $K_0$  of the group ring  $\mathbb{Z}\Gamma$  (see Section 3). Computing the algebraic  $L$ -theory (Section 4) is more difficult; we use homological computations from our previous paper [13] and a result of Land and Nikolaus [21]. This result generalizes a result of Sullivan [33, 34] comparing  $L$ -theory spectra with topological  $K$ -theory spectra after inverting 2. As with much of the rest of the paper, we compute the algebraic  $L$ -theory at  $p$  and away from  $p$ .

Our goal is to compute and detect the geometric structure set of  $M$ . However, it is much easier to compute the periodic structure set (also known as the algebraic structure set) of the classifying space  $B\Gamma$ . Indeed, as a simple application of the Farrell-Jones conjecture, we show in Section 7

$$\bigoplus_{P \in (\mathcal{P})} \mathcal{S}_m^{\text{per},s}(BP) \xrightarrow{\cong} \mathcal{S}_m^{\text{per},s}(B\Gamma).$$

In Section 6 we use equivariant  $KO$ -homology to show for any odd order  $p$ -group  $G$ , that  $\mathcal{S}_m^{\text{per},s}(BG)$  is a finitely generated free  $\mathbb{Z}[1/p]$ -module. Section 8 is the heart of the paper, where we compute the periodic structure set of  $M$ , working at  $p$  and away from  $p$ . In Section 9 we give the computation of the geometric structure set of  $M$ , as well as detection by algebraic topological invariants. In Section 10, we detect the structure set by the geometric invariants given in Theorem 0.2: splitting obstructions and the  $\rho$ -invariant. Finally, in Section 11 we mention some basic questions that we did not answer, in the hope that these questions are accessible and will stimulate future work.

### 1 Preliminaries about the Group $\mathbb{Z}^n \rtimes \mathbb{Z}/p$

In this section we collect various facts about  $\Gamma$  from [13, lemma 1.9].

LEMMA 1.1.

- (1) Let  $\zeta = e^{2\pi i/p}$ . There are nonzero ideals  $I_1, \dots, I_k$  of  $\mathbb{Z}[\zeta]$  and isomorphisms of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules

$$\mathbb{Z}^n \cong I_1 \oplus \dots \oplus I_k, \quad \mathbb{Z}^n \otimes \mathbb{Q} \cong \mathbb{Q}(\zeta)^k.$$

Hence  $n = k(p - 1)$ .

- (2) Each nontrivial finite subgroup  $P$  of  $\Gamma$  is isomorphic to  $\mathbb{Z}/p$  and its Weyl group  $W_\Gamma P := N_\Gamma P/P$  is trivial.  
 (3) There are isomorphisms

$$H^1(\mathbb{Z}/p; \mathbb{Z}^n) \xrightarrow{\cong} \text{cok}(\rho - \text{id}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n) \cong (\mathbb{Z}/p)^k$$

and a bijection

$$\text{cok}(\rho - \text{id}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n) \xrightarrow{\cong} \mathcal{P} := \{(P) \mid P \subset \Gamma, 1 < |P| < \infty\}.$$

Here  $\mathcal{P}$  is the set of conjugacy classes  $(P)$  of nontrivial subgroups of finite order. If we fix an element  $s \in \Gamma$  of order  $p$ , the bijection sends the element  $\bar{u} \in \mathbb{Z}^n / (1 - \rho)\mathbb{Z}^n$  to the conjugacy class of the subgroup of order  $p$  generated by  $us$ .

- (4) We have  $|\mathcal{P}| = p^k$ .  
 (5) There is a bijection from the  $\mathbb{Z}/p$ -fixed set of the  $\mathbb{Z}/p$ -space  $\mathbb{T}_\rho^n := \mathbb{R}_\rho^n / \mathbb{Z}_\rho^n$  to  $H^1(\mathbb{Z}/p; \mathbb{Z}_\rho^n)$ . In particular,  $(\mathbb{T}_\rho^n)^{\mathbb{Z}/p}$  consists of  $p^k$  points.  
 (6)  $[\Gamma, \Gamma] = \text{im}(\rho - \text{id}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n)$ .  
 (7)  $\Gamma/[\Gamma, \Gamma] \cong \text{cok}(\rho - \text{id}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n) \oplus \mathbb{Z}/p = (\mathbb{Z}/p)^{k+1}$ .

### 2 Preliminaries about the Farrell-Jones Conjecture

To classify high-dimensional manifolds one uses the surgery exact sequence. One of the terms in the surgery exact sequence is the 4-periodic  $L$ -group  $L_*(\mathbb{Z}G)$ , where  $G$  is the fundamental group of the manifold under consideration. Although the  $L$ -groups are algebraically defined, when  $G$  is infinite the computation of the  $L$ -groups is done by a mix of algebraic, topological, and geometric methods. This

is encoded in the Farrell-Jones conjecture, which can be stated in terms of an equivariant homology theory in the sense of [23, sec. 1] as follows.

Let  $\mathbf{E}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$  be a covariant functor from the category of small groupoids to the category of spectra, which is *homotopy invariant*; i.e., it sends an equivalence of groupoids to a weak homotopy equivalence of spectra. Given a cellular map  $X \rightarrow Y$  of  $G$ -CW-complexes for a (discrete) group  $G$ , Davis-Lück [11] define

$$H_m^G(X \rightarrow Y; \mathbf{E}) := \pi_m(\text{map}_G(G/-, \text{cone}(X \rightarrow Y)) \wedge_{\text{Or}(G)} \mathbf{E}(\overline{G/-})),$$

where cone refers to the mapping cone,  $\text{Or}(G)$  to the orbit category of  $G$ , and  $\overline{G/H}$  to the groupoid associated to the  $G$ -set  $G/H$ . This defines a  $G$ -homology theory  $H_*^G$  on the category of  $G$ -CW-complexes. Its coefficients are given by  $H_m^G(G/H; \mathbf{E}) = \pi_m(\mathbf{E}(H))$ .

An equivariant homology theory  $\mathcal{H}_*^?$  in the sense of [23, sec. 1] assigns to every discrete group  $G$  a  $G$ -homology theory  $\mathcal{H}_*^G$  on the category of  $G$ -CW-complexes. Given a group homomorphism  $\alpha: G \rightarrow H$ , there is a corresponding map of abelian groups  $\text{ind}_\alpha: \mathcal{H}_m^G(X, A) \rightarrow \mathcal{H}_m^H(H \times_\alpha (X, A))$ . The axioms for an equivariant homology theory are satisfied when  $\mathbf{E}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$  is a homotopy invariant functor and  $\mathcal{H}_m^G(X)$  is defined as above, see [25, prop. 157, p. 796].

Examples of such GROUPOIDS-spectra are  $\mathbf{K}$  and  $\mathbf{L}^{(-\infty)}$  defined in [11, sec. 2]. Here  $\pi_m(\mathbf{K}(\overline{G/H})) = K_m(\mathbb{Z}H)$  and  $\pi_m(\mathbf{L}^{(-\infty)}(\overline{G/H})) = L_m^{(-\infty)}(\mathbb{Z}H)$ .

A family  $\mathcal{F}$  of subgroups of  $G$  is a collection of subgroups that is nonempty and closed under conjugation and under taking subgroups. The *classifying space*  $E_{\mathcal{F}}G$  for group actions with isotropy in  $\mathcal{F}$  is characterized up to  $G$ -homotopy equivalence as a  $G$ -CW-complex where  $(E_{\mathcal{F}}G)^H$  is empty if  $H \notin \mathcal{F}$  and contractible if  $H \in \mathcal{F}$ . We write  $\underline{E}G$  for the classifying space when  $\mathcal{F}$  is the family of finite subgroups and  $\underline{G}$  for the classifying space when the  $\mathcal{F}$  is the family of virtually cyclic subgroups. For more information about these spaces we refer to [24].

The Farrell-Jones conjecture for the group  $G$ , which was originally stated in [14, 1.6 on p. 257], predicts that for all  $m \in \mathbb{Z}$  the projection  $\underline{G} \rightarrow \bullet$  induces isomorphisms

$$\begin{aligned} H_m^G(\underline{G}; \mathbf{K}) &\rightarrow H_m^G(\bullet; \mathbf{K}) = K_m(\mathbb{Z}G), \\ H_m^G(\underline{G}; \mathbf{L}^{(-\infty)}) &\rightarrow H_m^G(\bullet; \mathbf{L}^{(-\infty)}) = L_m^{(-\infty)}(\mathbb{Z}G). \end{aligned}$$

We now specialize to the group  $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$ . The first point is that the Farrell-Jones conjecture in  $K$ - and  $L$ -theory holds for  $\Gamma$  by [7]. Since the only subgroups

that are virtually cyclic and not finite are infinite cyclic, and since the Farrell-Jones conjecture holds for infinite cyclic groups, the transitivity principle [14, theorem A.10], or [25, theorem 65, p. 742] shows that

$$H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(-\infty)}) \xrightarrow{\cong} H_m^\Gamma(\underline{\Gamma}; \mathbf{L}^{(-\infty)}),$$

$$H_m^\Gamma(\underline{E}\Gamma; \mathbf{K}) \xrightarrow{\cong} H_m^\Gamma(\underline{\Gamma}; \mathbf{K}),$$

are bijective for all  $m \in \mathbb{Z}$ . Hence we get:

**THEOREM 2.1** (Farrell-Jones conjecture for  $\Gamma$ ). *The projection  $\underline{E}\Gamma \rightarrow \bullet$  induces for all  $m \in \mathbb{Z}$  bijections*

$$H_m^\Gamma(\underline{E}\Gamma; \mathbf{K}) \rightarrow H_m^\Gamma(\bullet; \mathbf{K}) = K_m(\mathbb{Z}\Gamma),$$

$$H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(-\infty)}) \rightarrow H_m^\Gamma(\bullet; \mathbf{L}^{(-\infty)}) = L_m^{(-\infty)}(\mathbb{Z}\Gamma).$$

*Remark 2.2.* Note that  $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$  acts affinely on  $\mathbb{Z}^n$  where  $\mathbb{Z}^n$  acts by translation and  $\mathbb{Z}/p$  acts by the map  $\rho$ . By extending scalars,  $\Gamma$  acts properly and cocompactly on  $\mathbb{R}^n$ . Hence  $\Gamma$  is a crystallographic group and  $\mathbb{R}^n$  can be taken as a model for  $\underline{E}\Gamma$ .

### 3 Algebraic $K$ -theory of the Group Ring

For a group  $G$  and an integer  $m$ , define  $\text{Wh}_m(G)$  to be the homotopy groups of the homotopy cofiber of the assembly map in algebraic  $K$ -theory, i.e.,

$$\text{Wh}_m(G) = H_m^G(EG \rightarrow \bullet; \mathbf{K})$$

where  $\mathbf{K} = \mathbf{K}_{\mathbb{Z}}$  is the algebraic  $K$ -theory spectrum over the orbit category of [11, sec. 2] with  $\pi_n(\mathbf{K}(\overline{G/H})) = K_n(\mathbb{Z}H)$ . Hence  $\text{Wh}_1(G)$  is the classical Whitehead group  $\text{Wh}(G) = \text{cok}(\{\pm 1\} \times G^{\text{ab}} \rightarrow K_1(\mathbb{Z}G))$ ,  $\text{Wh}_0(G)$  is  $\tilde{K}_0(\mathbb{Z}G) = \text{cok}(K_0(\mathbb{Z}) \rightarrow K_0(\mathbb{Z}G))$ , and  $\text{Wh}_n(G)$  is  $K_n(\mathbb{Z}G)$  for  $n \leq -1$ .

The space  $\underline{E}\Gamma$  can be profitably analyzed using the following cellular  $\Gamma$ -pushout; see [28, cor. 2.11],

$$(3.1) \quad \begin{array}{ccc} \bigsqcup_{(P) \in \mathcal{P}} \Gamma \times_P EP & \longrightarrow & E\Gamma \\ \downarrow & & \downarrow \\ \bigsqcup_{(P) \in \mathcal{P}} \Gamma/P & \longrightarrow & \underline{E}\Gamma. \end{array}$$

This leads to the following result taken from [13, lemma 7.2(ii)].

LEMMA 3.1. Let  $\mathcal{H}_*^?$  be an equivariant homology theory in the sense of [23, sec. 1]. Then there is a long exact sequence

$$\begin{aligned} \cdots \rightarrow \mathcal{H}_{m+1}^\Gamma(\underline{E}\Gamma) \xrightarrow{\text{ind}_{\Gamma \rightarrow 1}} \mathcal{H}_{m+1}(\underline{B}\Gamma) \rightarrow \bigoplus_{P \in (\mathcal{P})} \widetilde{\mathcal{H}}_m^P(\bullet) \\ \xrightarrow{\varphi_m} \mathcal{H}_m^\Gamma(\underline{E}\Gamma) \xrightarrow{\text{ind}_{\Gamma \rightarrow 1}} \mathcal{H}_m(\underline{B}\Gamma) \rightarrow \cdots, \end{aligned}$$

where  $\widetilde{\mathcal{H}}_m^P(\bullet)$  is the kernel of the induction map

$$\text{ind}_{P \rightarrow 1}: \mathcal{H}_m^P(\bullet) \rightarrow \mathcal{H}_m(\bullet),$$

the map  $\varphi_m$  is induced by the various inclusions  $P \rightarrow \Gamma$ , and  $\underline{B}\Gamma := \Gamma \backslash \underline{E}\Gamma$ .

The map

$$\text{ind}_{\Gamma \rightarrow 1}[1/p]: \mathcal{H}_m^\Gamma(\underline{E}\Gamma)[1/p] \rightarrow \mathcal{H}_m(\underline{B}\Gamma)[1/p]$$

is split surjective.

THEOREM 3.2 (Computation of  $\text{Wh}_m(\Gamma)$ ). For every  $m \in \mathbb{Z}$ ,

$$\bigoplus_{P \in (\mathcal{P})} \text{Wh}_m(P) \xrightarrow{\cong} \text{Wh}_m(\Gamma).$$

Furthermore, for  $p$  an odd prime,  $\text{Wh}(\mathbb{Z}/p) \cong \mathbb{Z}^{(p-3)/2}$ ,  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$  is the ideal class group  $C(\mathbb{Z}[\exp(2\pi i/p)])$  and hence is finite, and  $K_m(\mathbb{Z}[\mathbb{Z}/p]) = 0$  for  $m \leq -1$ .

PROOF. The isomorphism  $\bigoplus \text{Wh}_m(P) \cong \text{Wh}_m(\Gamma)$  is a direct consequence of Theorem 2.1 and the  $G$ -pushout (3.1). See also [26, theorem 1.8], [12, theorem 5.1(d)], and [27, theorem 0.2].

The computation of  $\text{Wh}(\mathbb{Z}/p)$  for an odd prime  $p$  (and much more information about the Whitehead group for finite groups) can be found in [30], a discussion of  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$  in [29], and the vanishing of  $K_m(\mathbb{Z}[\mathbb{Z}/p])$  for  $m \leq -1$  in [8].  $\square$

Theorem 3.2 is consistent with [26, theorem 1.8(i)].

## 4 Algebraic $L$ -theory of the Group Ring

In this section we compute the  $L$ -groups of  $\mathbb{Z}\Gamma$  for all decorations.

### 4.1 Decorated $L$ -groups

We first discuss the so-called decorated versions of  $L$ -theory  $L_*^{(-i)}(\mathbb{Z}G)$  for any group  $G$  for  $i = 2, 1, 0, -1, -2, \dots$  and  $i = -\infty$ . We recall briefly a few facts; more information can be found in [31]. For  $m \in \mathbb{Z}$ , these are functors  $L_m^{(i)}: \text{GROUPS} \rightarrow \text{ABELIAN GROUPS}$  that are 4-periodic in the sense that  $L_m^{(i)}(\mathbb{Z}G) = L_{m+4}^{(i)}(\mathbb{Z}G)$ . There are natural maps

$$L_*^{(i+1)}(\mathbb{Z}G) \rightarrow L_*^{(i)}(\mathbb{Z}G),$$

and one defines

$$L_*^{(-\infty)}(\mathbb{Z}G) = \operatorname{colim}_{i \rightarrow -\infty} L_*^{(i)}(\mathbb{Z}G).$$

One sometimes writes

$$L_*^s(\mathbb{Z}G) = L_*^{(2)}(\mathbb{Z}G), \quad L_*^h(\mathbb{Z}G) = L_*^{(1)}(\mathbb{Z}G); \quad L_*^p(\mathbb{Z}G) = L_*^{(0)}(\mathbb{Z}G).$$

The  $L^s$ -groups are bordism groups of algebraic Poincaré complexes with based modules and simple Poincaré duality; they are useful in classifying manifolds. The  $L^h$ -groups are bordism groups of algebraic Poincaré complexes with free modules; they are useful for studying the existence question of when a space has the homotopy type of a manifold. The  $L^p$ -groups are bordism groups of algebraic Poincaré complexes with projective modules. The  $L^{(-\infty)}$ -groups are useful for the Farrell-Jones conjecture.

We will often use the Ranicki-Rothenberg exact sequence for a group  $G$ ; see [31, theorem 7.12, p. 146]:

$$(4.1) \quad \cdots \rightarrow L_m^{(i+1)}(\mathbb{Z}G) \rightarrow L_m^{(i)}(\mathbb{Z}G) \rightarrow \widehat{H}^m(\mathbb{Z}/2; \operatorname{Wh}_i(G)) \\ \rightarrow L_{m-1}^{(i+1)}(\mathbb{Z}G) \rightarrow L_{m-1}^{(i)}(\mathbb{Z}G) \rightarrow \cdots$$

For any decoration  $i$ , there is a homotopy invariant functor

$$\mathbf{L}^{(i)}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$$

satisfying  $\pi_m(\mathbf{L}^{(i)}(\overline{G/H})) = L_m^{(i)}(\mathbb{Z}H)$ . Farrell and Jones [14] conjecture that

$$H_*^G(\underline{G}; \mathbf{L}^{(-\infty)}) \xrightarrow{\cong} H_*^G(\bullet; \mathbf{L}^{(-\infty)})$$

for all groups  $G$ . However, the decorated version of the assembly map

$$H_*^G(\underline{G}; \mathbf{L}^{(i)}) \xrightarrow{\cong} H_*^G(\bullet; \mathbf{L}^{(i)})$$

need not be bijective in general; for example, it is not bijective for the group  $G = \mathbb{Z}^2 \times \mathbb{Z}/29$  for the decorations  $p$ ,  $h$ , and  $s$ ; see [15, example 14]. However, the Farrell-Jones conjecture for the group  $\Gamma$  holds for all  $i$  as we show below. This will be important since the  $L^s$ -version is the geometrically significant one.

### 4.2 The $\langle -\infty \rangle$ -decoration

In this subsection we compute  $L_m^{(-\infty)}(\mathbb{Z}\Gamma)$  using the Farrell-Jones conjecture; see Theorem 2.1. The  $L$ -theory of  $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$  is, in some sense, built from the  $L$ -theory of  $\mathbb{Z}^n$  and  $\mathbb{Z}/p$ , so we first review these.

The Farrell-Jones conjecture in  $K$ -theory holds for the torsion-free group  $\mathbb{Z}^n$ . It follows that  $\operatorname{Wh}_m(\mathbb{Z}^n) = 0$  for all  $m \in \mathbb{Z}$  and for all  $n \in \mathbb{Z}_{\geq 0}$ . Thus the maps

$$(4.2) \quad L_m^{(i)}(\mathbb{Z}[\mathbb{Z}^n]) \xrightarrow{\cong} L_m^{(-\infty)}(\mathbb{Z}[\mathbb{Z}^n])$$



are bijections for  $i, m$ , and  $n$ , and the map of spectra

$$(4.3) \quad \mathbf{L}^{(i)}(\mathbb{Z}[\mathbb{Z}^n]) \xrightarrow{\simeq} \mathbf{L}^{(-\infty)}(\mathbb{Z}[\mathbb{Z}^n])$$

is a weak homotopy equivalence for all  $i$  and  $n$ . Hence we will omit the decoration and refer to  $L_m(\mathbb{Z}[\mathbb{Z}^n])$  and  $\mathbf{L}(\mathbb{Z}[\mathbb{Z}^n])$ .

When  $n = 0$ , the  $L$ -groups are well-known, essentially due to Kervaire-Milnor,

$$L_m(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & m \equiv 0 \pmod{4}, \\ 0, & m \equiv 1 \pmod{4}, \\ \mathbb{Z}/2, & m \equiv 2 \pmod{4}, \\ 0, & m \equiv 3 \pmod{4}, \end{cases}$$

where the map to  $\mathbb{Z}$  is given by the signature divided by 8 and the map to  $\mathbb{Z}/2$  is given by the Arf invariant. Since the Farrell-Jones conjecture in  $L$ -theory holds for the torsion-free group  $\mathbb{Z}^n$ ,

$$(4.4) \quad L_m(\mathbb{Z}[\mathbb{Z}^n]) \xleftarrow{\cong} H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z})) = \bigoplus_{i=0}^n L_{m-i}(\mathbb{Z}) \binom{n}{i}.$$

For  $p$  an odd prime, we get from [4, theorem 1], [5, theorems 1, 2, and 3], and [16, theorem 10.1]

$$L_m^{(0)}(\mathbb{Z}[\mathbb{Z}/p]) = \begin{cases} \mathbb{Z}^{(p-1)/2} \oplus L_m(\mathbb{Z}), & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

Since  $\text{Wh}_i(\mathbb{Z}/p) = 0$  for  $i < 0$  by [8], we have

$$\begin{aligned} L_m^{(0)}(\mathbb{Z}[\mathbb{Z}/p]) &\xrightarrow{\cong} L_m^{(-1)}(\mathbb{Z}[\mathbb{Z}/p]) \\ &\xrightarrow{\cong} L_m^{(-2)}(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{\cong} \dots \xrightarrow{\cong} L_m^{(-\infty)}(\mathbb{Z}[\mathbb{Z}/p]). \end{aligned}$$

Thus

$$(4.5) \quad L_m^{(-\infty)}(\mathbb{Z}[\mathbb{Z}/p]) = \begin{cases} \mathbb{Z}^{(p-1)/2} \oplus L_m(\mathbb{Z}), & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

**THEOREM 4.1** ( $L^{(-\infty)}$ -theory). *There is a long exact sequence*

$$\begin{aligned} \dots \rightarrow H_{m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) &\rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(-\infty)}(\mathbb{Z}P) \rightarrow L_m^{(-\infty)}(\mathbb{Z}\Gamma) \\ &\xrightarrow{\beta_m} H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{m-1}^{(-\infty)}(\mathbb{Z}P) \rightarrow \dots, \end{aligned}$$

where  $\tilde{L}_m^{(-\infty)}(\mathbb{Z}P)$  is the kernel of the map  $L_m^{(-\infty)}(\mathbb{Z}P) \rightarrow L_m(\mathbb{Z})$  induced by induction with  $P \rightarrow 1$ .

The map  $\beta_m[1/p]$  is a split surjection, and thus there is an isomorphism of  $\mathbb{Z}[1/p]$ -modules

$$L_m^{(-\infty)}(\mathbb{Z}\Gamma)[1/p] \cong \left( \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(-\infty)}(\mathbb{Z}P)[1/p] \right) \oplus H_m(B\Gamma; \mathbf{L}(\mathbb{Z}))[1/p].$$

PROOF. This follows directly from Theorem 2.1 and Lemma 3.1. □

Next we want to improve Theorem 4.1 by comparing it with the computation of  $KO_*(C_r^*(G))$  of [13, theorem 10.1].

The next result is motivated by Sullivan’s thesis [33], but its proof requires much more, in particular recent results of Land and Nikolaus.

**THEOREM 4.2** (Comparing  $L$ -theory and topological  $K$ -theory). *There is a natural transformation of equivariant homology theories*

$$T_*^?(-): H_*^?(-; \mathbf{L}^{(-\infty)})[1/2] \rightarrow KO_*^?(-)[1/2]$$

such that for every group  $G$ , every proper  $G$ - $CW$ -complex  $X$ , and every  $m \in \mathbb{Z}$  the map

$$T_m^G(X): H_m^G(X; \mathbf{L}^{(-\infty)})[1/2] \rightarrow KO_m^G(X)[1/2]$$

is an isomorphism.

PROOF. Notice that we invert 2 so that the decorations do not matter and we therefore ignore them. The key ingredient is a rigorous construction of a weak homotopy equivalence of spectra  $\mathbf{KO}(A)[1/2] \rightarrow \mathbf{L}(A)[1/2]$  for a real  $C^*$ -algebra  $A$  from its topological  $K$ -theory spectrum to its algebraic  $L$ -theory spectrum after inverting  $1/2$ , which is natural in  $A$ , [21, theorem C]. In particular, this applies to the real group  $C^*$ -algebra and can be extended from groups to groupoids; see [21, sec. 5.2]. For any group there is a natural map from the integral group ring to the real group  $C^*$ -algebra, which yields a natural map between the corresponding  $L$ -theory spectra. This construction also extends to groupoids. Combining these two transformations and using the fact that the one in [21, theorem C] is a weak homotopy equivalence yields the desired transformation  $T_*^?(-)$ . Recall that a  $G$ - $CW$ -complex is proper if all isotropy groups are finite. In order to show that  $T_n^G(X)$  is an isomorphism for every proper  $G$ - $CW$ -complex  $X$ , it suffices to do this in the case  $X = G/H$  for a finite subgroup  $H \subseteq G$ ; see [6, lemma 1.2]. Hence one needs to show that the change-of-coefficients map  $L_m(\mathbb{Z}H)[1/2] \rightarrow L_m(\mathbb{R}H)[1/2]$  is bijective for all  $m$ , which is done in [32, prop. 22.34, p. 253]. □

If  $G$  is a group and  $M$  is a left  $\mathbb{Z}G$ -module, the *invariants* of  $M$  are  $M^G := \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) = H^0(BG; M)$ , and the *coinvariants* of  $M$  are  $M_G := \mathbb{Z} \otimes_{\mathbb{Z}G} M = H_0(BG; M)$ . If  $G$  is finite of order  $q$ , the norm map  $M_G \rightarrow M^G$  sending  $1 \otimes x$  to  $\sum_{g \in G} gx$  is an isomorphism after tensoring with  $\mathbb{Z}[1/q]$ .

If  $\mathcal{H}_*$  is a (nonequivariant) generalized homology theory, and  $G$  acts on  $X$ , then the homology quotient map factors as  $\mathcal{H}_*(X) \rightarrow \mathcal{H}_*(X)_G \rightarrow \mathcal{H}_*(G \backslash X)$ . If the quotient map  $X \rightarrow G \backslash X$  is a regular  $G$ -cover and  $G$  is finite of order  $q$ , then

there is a transfer map  $\mathcal{H}_*(G \backslash X) \rightarrow \mathcal{H}_*(X)^G$  so that the composite  $\mathcal{H}_*(X)_G \rightarrow \mathcal{H}_*(G \backslash X) \rightarrow \mathcal{H}_*(X)^G$  is an isomorphism after tensoring with  $\mathbb{Z}[1/q]$ .

LEMMA 4.3. *Let  $\mathcal{H}_*$  be any generalized homology theory taking values in  $\mathbb{Z}[1/p]$ -modules. For all  $m \in \mathbb{Z}$ , the following maps are isomorphisms:*

$$\begin{aligned} \mathcal{H}_m(B\mathbb{Z}_\rho^n)_{\mathbb{Z}/p} &\xrightarrow{\cong} \mathcal{H}_m(B\Gamma) \xrightarrow{\cong} \mathcal{H}_m(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}, \\ \mathcal{H}_m(B\Gamma) &\xrightarrow{\cong} \mathcal{H}_m(\underline{B}\Gamma). \end{aligned}$$

PROOF. Consider the diagram

$$\begin{array}{ccc} \mathcal{H}_*(B\mathbb{Z}_\rho^n)_{\mathbb{Z}/p} & \xrightarrow{\alpha} & \mathcal{H}_*(B\Gamma) & \xrightarrow{\beta} & \mathcal{H}_*(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}, \\ & & \downarrow \gamma & & \\ & & \mathcal{H}_*(\underline{B}\Gamma) & & \end{array}$$

where  $\alpha$  and  $\gamma$  are the functorial  $\mathcal{H}_*$ -maps and  $\beta$  is the transfer. We will show that  $\alpha$ ,  $\beta$ , and  $\gamma$  are isomorphisms.

Given a  $\mathbb{Z}/p$ - $CW$ -complex  $X$ , the map

$$j_m: \mathcal{H}_m(X)_{\mathbb{Z}/p} \rightarrow \mathcal{H}_m((\mathbb{Z}/p) \backslash X)$$

is natural in  $X$ . Since the functor sending a  $\mathbb{Z}[1/p][\mathbb{Z}/p]$ -module  $M$  to  $M_{\mathbb{Z}/p}$  is an exact functor, the assignments sending a  $\mathbb{Z}/p$ - $CW$ -complex to  $\mathcal{H}_m(X)_{\mathbb{Z}/p}$  and to  $\mathcal{H}_m((\mathbb{Z}/p) \backslash X)$  are  $\mathbb{Z}/p$ -homology theories and  $j_*$  is a transformation of  $\mathbb{Z}/p$ -homology theories. One easily checks that  $j_m$  is a bijection if  $X$  is  $(\mathbb{Z}/p)/H$  for any subgroup  $H \subseteq \mathbb{Z}/p$ . It follows that  $j_m$  is a bijection for any  $\mathbb{Z}/p$ - $CW$ -complex. Taking  $X$  to be  $\mathbb{Z}^n \backslash E\Gamma$ , we see  $\alpha$  is an isomorphism and taking  $X$  to be  $\mathbb{Z}^n \backslash \underline{E}\Gamma$  we see that  $\gamma \circ \alpha$  is an isomorphism. We commented above that  $\beta \circ \alpha$  is an isomorphism. It follows that all the maps are isomorphisms.  $\square$

We also need some numbers defined in our previous paper [13]. For  $j, k \in \mathbb{Z}_{\geq 0}$  and for  $p$  an odd prime, define

$$(4.6) \quad r_j := \text{rk}(\Lambda^j(\mathbb{Z}[\zeta_p]^k)^{\mathbb{Z}/p}),$$

where  $\Lambda^j$  means the  $j^{\text{th}}$  exterior power of a  $\mathbb{Z}$ -module and  $\zeta_p$  is a primitive  $p^{\text{th}}$  root of unity. Thus  $r_j = \text{rk} H^j(\mathbb{T}_\rho^n)^{\mathbb{Z}/p}$ . When  $k = 1$  we worked out these numbers in [13, lemma 1.22]:  $r_j = 0$  for  $j \geq p$ , and for  $0 \leq j \leq (p - 1)$ ,

$$r_j = \frac{1}{p} \left( \binom{p-1}{j} + (-1)^j (p-1) \right).$$

THEOREM 4.4 (Computation of  $L^{(-\infty)}(\mathbb{Z}\Gamma)$ ).

(1) *There is an isomorphism*

$$L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) \cong H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})).$$

Transferring to the finite index subgroup  $\mathbb{Z}^n$  of  $\Gamma$  induces an isomorphism

$$L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}.$$

(2) There is an exact sequence

$$0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{2m}^{(-\infty)}(\mathbb{Z}P) \rightarrow L_{2m}^{(-\infty)}(\mathbb{Z}\Gamma) \rightarrow H_{2m}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow 0,$$

which splits after inverting  $p$ .

(3) We have

$$L_m^{(-\infty)}(\mathbb{Z}\Gamma) \cong \begin{cases} \mathbb{Z}p^{k(p-1)/2} \oplus \left(\bigoplus_{i=0}^n L_{m-i}(\mathbb{Z})^{r_i}\right), & m \text{ even,} \\ \bigoplus_{i=0}^n L_{m-i}(\mathbb{Z})^{r_i}, & m \text{ odd.} \end{cases}$$

PROOF. (1) and (2). The computation of  $L_*^{(-\infty)}(\mathbb{Z}[\mathbb{Z}/p])$  in (4.5) and Theorem 4.1 implies that for every  $m \in \mathbb{Z}$  we obtain an isomorphism

$$L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) \xrightarrow{\cong} H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z}))$$

and a short exact sequence

$$0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{2m}^{(-\infty)}(\mathbb{Z}P) \rightarrow L_{2m}^{(-\infty)}(\mathbb{Z}\Gamma) \rightarrow H_{2m}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow 0,$$

which splits after inverting  $p$ .

It remains to show that transferring to subgroup  $\mathbb{Z}^n$  of  $\Gamma$  induces an isomorphism

$$L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}.$$

It suffices to do this after localizing at  $p$  and after inverting  $p$ .

We first invert 2. Because of Theorem 4.2 and the fact that the assembly maps

$$\begin{aligned} H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(-\infty)})[1/2] &\xrightarrow{\cong} H_m^\Gamma(\bullet; \mathbf{L}^{(-\infty)})[1/2] = L_m^{(-\infty)}(\mathbb{Z}\Gamma)[1/2], \\ KO_m^\Gamma(\underline{E}\Gamma)[1/2] &\xrightarrow{\cong} KO_m(C_r^*(\Gamma; \mathbb{R}))[1/2], \end{aligned}$$

are bijections (see Theorem 2.1 and [17]), we obtain a commutative diagram of  $\mathbb{Z}[1/2]$ -modules

$$\begin{array}{ccc} L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma)[1/2] & \xrightarrow{\iota^*[1/2]} & L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}[1/2] \\ \downarrow \cong & & \downarrow \cong \\ KO_{2m+1}(C_r^*(\Gamma; \mathbb{R}))[1/2] & \xrightarrow{\iota^*[1/2]} & KO_{2m+1}(C_r^*(\mathbb{Z}^n; \mathbb{R}))^{\mathbb{Z}/p}[1/2] \end{array}$$

with bijective vertical arrows. The lower horizontal arrow is bijective by theorem [13, sec. 11.2]. Hence

$$\iota_{(p)}^*: L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma)_{(p)} \xrightarrow{\cong} L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])_{(p)}^{\mathbb{Z}/p}$$

is bijective for every  $m \in \mathbb{Z}$ .

Next consider the commutative diagram

$$(4.7) \quad \begin{CD} H_{2m+1}(B\Gamma; \mathbf{L}(\mathbb{Z})) @>\iota^*>> H_{2m+1}(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p} \\ @V A_\Gamma VV @V A_{\mathbb{Z}^n/\mathbb{Z}}^{\mathbb{Z}/p} VV \\ L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) @>\iota^*>> L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}. \end{CD}$$

We will show that all maps are isomorphisms after inverting  $p$ . Lemma 4.3 shows this is true for the top horizontal map. The Farrell-Jones conjecture in  $L$ -theory for  $\mathbb{Z}^n$  (see (4.4)) shows this holds for the right vertical map. To show this holds for  $A_\Gamma$ , first rewrite it as  $A_\Gamma : H_{2m+1}(B\Gamma; \mathbf{L}(\mathbb{Z})) = H_{2m+1}^\Gamma(E\Gamma; \mathbf{L}^{(-\infty)}) \rightarrow H_{2m+1}^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(-\infty)})$ , using the Farrell-Jones conjecture in  $L$ -theory for  $\Gamma$ . The map  $\text{ind}_{\Gamma \rightarrow 1} : H_{2m+1}^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(-\infty)}) \rightarrow H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z}))$  is an isomorphism after inverting  $p$  by Lemma 3.1 and the vanishing of  $L_{2m+1}^{(-\infty)}(\mathbb{Z}[\mathbb{Z}/p])$ . The composite  $\text{ind}_{\Gamma \rightarrow 1} \circ A_\Gamma$  is an isomorphism after inverting  $p$  by Lemma 4.3. Thus we can conclude that  $A_\Gamma$  is an isomorphism after inverting  $p$ .

Thus the bottom row  $\iota^* : L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) \rightarrow L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}$  of (4.7) is also an isomorphism after inverting  $p$ . We have shown that  $\iota_{(p)}^*$  and  $\iota^*[1/p]$  are isomorphisms, so we conclude that  $\iota^*$  is an isomorphism.

(3) It suffices to show that the abelian groups in question are isomorphic after inverting 2 and after inverting  $p$ . We conclude from [13, theorem 10.1] and Theorem 4.2 that this is the case after inverting 2.

It remains to treat the case where we invert  $p$ . Because of assertion (2), (4.5), and Lemma 4.3, it remains to prove

$$(4.8) \quad L_m(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}[1/p] \cong \bigoplus_{i=0}^n (L_{m-i}(\mathbb{Z}))^{r_i}[1/p].$$

Since  $L_m(\mathbb{Z}[\mathbb{Z}^n]) \cong H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))$ , it remains to show

$$(4.9) \quad H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p}[1/p] \cong \bigoplus_{i=0}^n (L_{m-i}(\mathbb{Z}))^{r_i}[1/p].$$

The Atiyah-Hirzebruch spectral sequence converging to  $H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))$  collapses, since  $B\mathbb{Z}^n$  is  $\mathbb{T}^n$  and therefore one can compute  $H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))$  directly. Hence we obtain a filtration of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules

$$0 = F_{n+1, m-n-1} \subseteq F_{n, m-n} \subseteq \dots \subseteq F_{1, m-1} \subseteq F_{0, m} = H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))$$

together with exact sequences of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules

$$0 \rightarrow F_{i+1, m-i-1} \rightarrow F_{i, m-i} \xrightarrow{q} H_i(B\mathbb{Z}_p^n) \otimes_{\mathbb{Z}} L_{m-i}^{(-\infty)}(\mathbb{Z}) \rightarrow 0,$$

which splits as short exact sequence of  $\mathbb{Z}$ -modules. Let  $s: H_i(B\mathbb{Z}_\rho^n) \otimes_{\mathbb{Z}} L_{m-i}(\mathbb{Z}) \rightarrow F_{i,m-i}$  be a  $\mathbb{Z}$ -map with  $q \circ s = \text{id}$ . Then

$$\tilde{s}: H_i(B\mathbb{Z}_\rho^n) \otimes_{\mathbb{Z}} L_{m-i}(\mathbb{Z}) \rightarrow F_{i,m-i}, \quad x \mapsto \sum_{g \in \mathbb{Z}/p} g \cdot s(g^{-1} \cdot x),$$

is a  $\mathbb{Z}[\mathbb{Z}/p]$ -map with  $q \circ \tilde{s} = p \cdot \text{id}$ .

We obtain an exact sequence of  $\mathbb{Z}[1/p]$ -modules,

$$0 \rightarrow (F_{i+1,n-i+1})^{\mathbb{Z}/p}[1/p] \rightarrow (F_{i,n-i})^{\mathbb{Z}/p}[1/p] \xrightarrow{q^{\mathbb{Z}/p}[1/p]} H_i(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}[1/p] \otimes_{\mathbb{Z}} L_{m-i}(\mathbb{Z}) \rightarrow 0.$$

Since  $q^{\mathbb{Z}/p}[1/p] \circ \tilde{s}^{\mathbb{Z}/p}[1/p]$  is the automorphism  $p \cdot \text{id}$  of  $H_i(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}[1/p] \otimes_{\mathbb{Z}} L_{m-i}(\mathbb{Z})$ , this short exact sequence of  $\mathbb{Z}[1/p]$ -modules splits and we obtain an  $\mathbb{Z}[1/p]$ -isomorphism

$$(F_{i,m-i})^{\mathbb{Z}/p}[1/p] \cong (F_{i+1,m-i-1})^{\mathbb{Z}/p}[1/p] \oplus (H_i(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}[1/p] \otimes_{\mathbb{Z}} L_{m-i}(\mathbb{Z})).$$

This implies by induction over  $i$  that there is an isomorphism of  $\mathbb{Z}[1/p]$ -modules

$$H_m(B\mathbb{Z}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p}[1/p] \cong \bigoplus_{i=0}^n H_i(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}[1/p] \otimes_{\mathbb{Z}} L_{m-i}(\mathbb{Z}).$$

Since  $H_i(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p} \cong \mathbb{Z}^{r_i}$ , the claim follows. This finishes the proof of Theorem 4.4.  $\square$

### 4.3 Arbitrary Decorations

Finally, we extend Theorem 4.4 to all decorations. Recall we abbreviate  $\mathbf{L}^{(i)}(\mathbb{Z})$  by  $\mathbf{L}(\mathbb{Z})$ , which is justified by (4.3).

**THEOREM 4.5** (Computation of  $L^{(i)}(\mathbb{Z}\Gamma)$ ). *Let  $i \in \{2, 1, 0, -1, -2, \dots\} \sqcup \{-\infty\}$ . Then:*

- (1) *The assembly map*

$$A_m^{(i)}: H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}) \xrightarrow{\cong} H_m^\Gamma(\bullet; \mathbf{L}^{(i)}) = L_m^{(i)}(\mathbb{Z}\Gamma)$$

*is an isomorphism for  $m \in \mathbb{Z}$ .*

- (2) *For  $m \in \mathbb{Z}$  there is an exact sequence*

$$0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) \rightarrow L_m^{(i)}(\mathbb{Z}\Gamma) \xrightarrow{\beta_m^{(i)}} H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow 0$$

*that splits after inverting  $p$ . The first map is given by the various inclusions  $P \rightarrow \Gamma$ , and  $\beta_m^{(i)}$  is defined by the composite*

$$\beta_m^{(i)}: L_m^{(i)}(\mathbb{Z}\Gamma) \xrightarrow{(A_m^{(i)})^{-1}} H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}) \xrightarrow{\text{ind}_{\Gamma \rightarrow \{1\}}} H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})).$$

(3) For  $m \in \mathbb{Z}$  the maps

$$L_{2m+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma),$$

$$\beta_{2m+1}^{(i)}: L_{2m+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\cong} H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})),$$

are bijective and transferring to the finite index subgroup  $\mathbb{Z}^n$  of  $\Gamma$  induces an isomorphism

$$L_{2m+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_{2m+1}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}.$$

(4) We have

$$L_m^s(\mathbb{Z}\Gamma) \cong \begin{cases} \mathbb{Z}p^{k(p-1)/2} \oplus (\bigoplus_{i=0}^n L_{m-i}(\mathbb{Z})^{r_i}), & m \text{ even,} \\ \bigoplus_{i=0}^n L_{m-i}(\mathbb{Z})^{r_i}, & m \text{ odd.} \end{cases}$$

PROOF. (1) and (2). Since  $\text{Wh}_i(P) = 0$  because of [8] and  $\text{Wh}_i(\Gamma) = 0$  by Theorem 3.2 for  $i \leq -2$ , we obtain from (4.1) for  $i \leq -1, m \in \mathbb{Z}$ , isomorphisms

$$L_m^{(-\infty)}(\mathbb{Z}P) \xrightarrow{\cong} L_m^{(i-1)}(\mathbb{Z}P) \xrightarrow{\cong} L_m^{(i)}(\mathbb{Z}P),$$

$$L_m^{(-\infty)}(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_m^{(i-1)}(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_m^{(i)}(\mathbb{Z}\Gamma),$$

$$H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(-\infty)}) \xrightarrow{\cong} H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i-1)}) \xrightarrow{\cong} H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}).$$

This together with Theorem 4.4 implies that assertions (1) and (2) are true for  $i \in \{-1, -2, \dots\} \sqcup \{-\infty\}$ .

It remains to prove assertions (1) and (2) for  $i = 0, 1, 2$ , which we will do by induction on  $i$ . So we want to show for  $i \leq 2$  that assertions (1) and (2) are true for  $i$  if they are true for  $i - 1$ . Consider the commutative diagram in Figure 4.1. The left vertical column is the direct sum over  $\mathcal{P}$  of the Rothenberg sequences (4.1) associated to  $P$ . The middle column is the Rothenberg sequences (4.1) associated to  $\Gamma$ . The isomorphism in the right column come from (4.3). Notice that all columns are exact. In each row the left arrow comes from the various inclusions  $P \rightarrow \Gamma$ . The rows involving the Tate cohomology are exact sequences

$$0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \widehat{H}^m(\mathbb{Z}/2; \text{Wh}_{i-1}(P)) \rightarrow \widehat{H}^m(\mathbb{Z}/2; \text{Wh}_{i-1}(\Gamma)) \rightarrow 0 \rightarrow 0.$$

The sequences for the decorations  $\langle i - 1 \rangle$  are short exact sequences by induction hypothesis

$$0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \widetilde{L}_m^{(i-1)}(\mathbb{Z}P) \rightarrow L_m^{(i-1)}(\mathbb{Z}\Gamma) \xrightarrow{\beta_m^{(i-1)}} H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow 0.$$

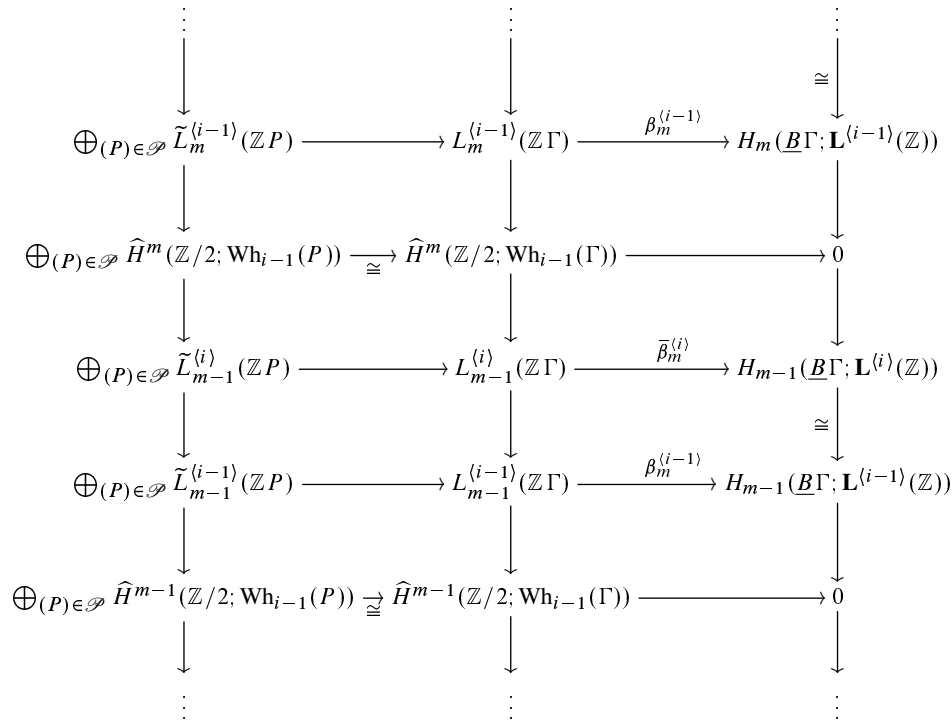


FIGURE 4.1. Rotherberg exact sequences.

The map  $\bar{\beta}_m^{(i)}: L_m^{(i)}(\mathbb{Z}\Gamma) \rightarrow H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z}))$  is defined so that the diagram commutes. The remaining columns yield short exact sequences

$$(4.10) \quad 0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) \rightarrow L_m^{(i)}(\mathbb{Z}\Gamma) \xrightarrow{\bar{\beta}_m^{(i)}} H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow 0.$$

Lemma 3.1 shows that there is a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_{m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) &\rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) \rightarrow H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}) \\ &\rightarrow H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{m-1}^{(i)}(\mathbb{Z}P) \rightarrow \cdots \end{aligned}$$

and that the  $\mathbb{Z}[1/p]$ -map

$$H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)})[1/p] \rightarrow H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z}))[1/p]$$

is split surjective. We have already shown (see (4.10)) that the composite

$$\bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) \rightarrow H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}) \rightarrow L_m^{(i)}(\mathbb{Z}\Gamma)$$



is injective. Hence the long exact sequence reduces to short exact sequences

$$0 \rightarrow \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) \rightarrow H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}) \rightarrow H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \rightarrow 0$$

which split after inverting  $p$ . We obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) & \longrightarrow & H_m^\Gamma(\underline{E}\Gamma; \mathbf{L}^{(i)}) & \longrightarrow & H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \longrightarrow 0 \\ & & \text{id} \downarrow & & \downarrow A_m^{(i)} & & \text{id} \downarrow \\ 0 & \longrightarrow & \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^{(i)}(\mathbb{Z}P) & \longrightarrow & L_m^{(i)}(\mathbb{Z}\Gamma) & \longrightarrow & H_m(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \longrightarrow 0. \end{array}$$

Since the rows are exact and the first and third vertical arrows are bijective, the middle arrow is bijective by the Five Lemma. This finishes the proof of assertions (1).

A direct computation shows that the map  $\bar{\beta}_m^{(i)}$  agrees with the map  $\beta_m^{(i)}$ . Now assertion (2) follows from (4.10).

(3) The following isomorphism is due independently to Bak [3] and Wall [35, cor. 2.4.3]:

$$(4.11) \quad \tilde{L}_m^s(\mathbb{Z}P) \cong \begin{cases} \mathbb{Z}^{(p-1)/2}, & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

Hence assertion (2) implies that we obtain an isomorphism

$$L_{2m+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\cong} H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})).$$

The following diagram commutes:

$$\begin{array}{ccc} L_{2m+1}^s(\mathbb{Z}\Gamma) & \longrightarrow & H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \\ \downarrow & & \downarrow \text{id} \\ L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) & \longrightarrow & H_{2m+1}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})). \end{array}$$

The lower horizontal arrow is an isomorphism by Theorem 4.4 (1), and the right vertical arrow is an isomorphism by (4.3). Hence the map

$$L_{2m+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\cong} L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma)$$

is an isomorphism. The following diagram commutes

$$\begin{array}{ccc} L_{2m+1}^s(\mathbb{Z}\Gamma) & \longrightarrow & L_{2m+1}^s(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p} \\ \downarrow & & \downarrow \\ L_{2m+1}^{(-\infty)}(\mathbb{Z}\Gamma) & \longrightarrow & L_{2m+1}^{(-\infty)}(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p}. \end{array}$$

The lower horizontal arrow is an isomorphism by Theorem 4.4 (1). Since for  $i \leq 1$  we have  $\text{Wh}_i(\mathbb{Z}^n) = 0$ , the right vertical arrow is an isomorphism by the Rothenberg sequence (4.1). Hence the upper horizontal arrow is an isomorphism. This finishes the proof of assertion (3).

(4) It suffices to prove the claim after inverting 2 and after inverting  $p$ . If we invert 2, the natural comparison maps between the various decorated  $L$ -groups are isomorphisms by the Rothenberg sequence (4.1), and the claim follows from Theorem 4.4 (3). If we invert  $p$ , the claim follows from assertion (2), Lemma 4.3, and isomorphisms (4.2), (4.9), and (4.11).  $\square$

### 5 Structure Sets

Given a closed oriented  $m$ -dimensional manifold  $N$ , we denote its *geometric simple structure set* by  $\mathcal{S}^{\text{geo},s}(N)$ . An element is represented by a simple homotopy equivalence  $N' \rightarrow N$  with a closed manifold  $N'$  as source and  $N$  as target. Two such maps  $g': N' \rightarrow N$  and  $g'': N'' \rightarrow N$  define the same element in  $\mathcal{S}^{\text{geo},s}(N)$  if and only if there is a homeomorphism  $u: N' \rightarrow N''$  such that  $g'' \circ u$  and  $g'$  are homotopic. This structure set appears in the geometric surgery exact sequence due to Browder, Novikov, Sullivan, and Wall; see [36, theorem 10.3] and [22, chap. 5], valid when  $m = \dim N \geq 5$ :

$$(5.1) \quad \cdots \rightarrow \mathcal{N}(N \times (D^1, S^0)) \rightarrow L_{m+1}^s(\mathbb{Z}[\pi_1(N)]) \rightarrow \mathcal{S}^{\text{geo},s}(N) \\ \rightarrow \mathcal{N}(N) \rightarrow L_m^s(\mathbb{Z}[\pi_1(N)]).$$

In the following we abbreviate

$$\mathbf{L} := \mathbf{L}^s = \mathbf{L}^{(2)}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$$

so that we have  $\pi_m(\mathbf{L}(\overline{G/H})) = L_m^s(\mathbb{Z}H)$  for a group  $G$  and subgroup  $H \subseteq G$ .

Given any  $CW$ -complex  $X$ , there is an exact algebraic surgery sequence of abelian groups

$$(5.2) \quad \cdots \xrightarrow{\eta_{m+2}(X)} H_{m+1}(X; \mathbf{L}(\mathbb{Z})) \xrightarrow{A_{m+1}(X)} L_{m+1}^s(\mathbb{Z}[\pi_1(X)]) \\ \xrightarrow{\xi_{m+1}(X)} \mathcal{S}_{m+1}^{\text{per},s}(X) \xrightarrow{\eta_{m+1}(X)} H_m(X; \mathbf{L}(\mathbb{Z})) \\ \xrightarrow{A_m(X)} L_m^s(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\xi_m(X)} \cdots,$$

natural in  $X$ . There is also a 1-connective version of the sequence (5.2)

$$(5.3) \quad \cdots \xrightarrow{\eta_{m+2}^{(1)}(X)} H_{m+1}(X; \mathbf{L}\langle 1 \rangle) \xrightarrow{A_{m+1}^{(1)}(X)} L_{m+1}^s(\mathbb{Z}[\pi_1(X)]) \\ \xrightarrow{\xi_{m+1}^{(1)}(X)} \mathcal{S}_{m+1}^{(1),s}(X) \xrightarrow{\eta_{m+1}^{(1)}(X)} H_m(X; \mathbf{L}\langle 1 \rangle) \\ \xrightarrow{A_m^{(1)}(X)} L_m^s(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\xi_m^{(1)}(X)} \cdots,$$

where  $\mathbf{p}: \mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$  is the 1-connective cover of the  $L$ -theory spectrum  $\mathbf{L} := \mathbf{L}(\mathbb{Z})$ . Here  $\pi_i(\mathbf{p}): \pi_i(\mathbf{L}\langle 1 \rangle) \rightarrow \pi_i(\mathbf{L}(\mathbb{Z}))$  is an isomorphism for  $i \geq 1$  and  $\pi_i(\mathbf{L}\langle 1 \rangle) = 0$  for  $i \leq 0$ .

The algebraic surgery exact sequences can be constructed in two ways. It can be constructed by defining the structure groups to be the homotopy groups of the cofiber of a spectrum-level assembly map (defined, for example, in [11, example 5.5]) or at the level of representatives (see the quadratic structure group  $\mathbb{S}_m(\mathbb{Z}, X)$  appearing in [32, def. 14.6, p. 148]). Using the second definition, Ranicki identified the geometric surgery sequence (5.1) with the 1-connective algebraic surgery sequence (5.3) truncated at  $L_5^s(\mathbb{Z}[\pi_1(X)])$ , see [32, theorem 18.5, p. 198] and [20]. In particular, we get an identification

$$(5.4) \quad \mathcal{S}^{\text{geo},s}(N) \cong \mathcal{S}_{n+l+1}^{(1),s}(N)$$

and a map

$$(5.5) \quad j(N): \mathcal{S}^{\text{geo},s}(N) \cong \mathcal{S}_{n+l+1}^{(1),s}(N) \rightarrow \mathcal{S}_{n+l+1}^{\text{per},s}(N)$$

from the canonical map  $\mathbf{p}: \mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$ .

### 6 Periodic Simple Structure Set of $BP$ for a Finite $p$ -group

In this section we compute the periodic simple structure groups  $\mathcal{S}_*^{\text{per},s}(BP)$  for a finite  $p$ -group  $P$  and  $p$  an odd prime.

Let  $k$  be a nonzero integer. A map of abelian groups  $f: A \rightarrow B$  is a  $1/k$ -equivalence if  $f \otimes \text{id}: A \otimes \mathbb{Z}[1/k] \rightarrow B \otimes \mathbb{Z}[1/k]$  is an isomorphism. An abelian group  $A$  is  $1/k$ -local if the map  $A \otimes \mathbb{Z} \rightarrow A \otimes \mathbb{Z}[1/k]$  is an isomorphism. A map  $A \rightarrow B$  is a  $1/k$ -localization if it is a  $1/k$ -equivalence and  $B$  is  $1/k$ -local. If  $k$  and  $l$  are relatively prime integers, then  $A$  is  $1/k$ -local if and only if  $A \otimes \mathbb{Z}[1/l]$  is  $1/k$ -local.

**THEOREM 6.1** (Periodic structure set of  $BP$  for a finite  $p$ -group  $P$ ). *Let  $p$  be an odd prime and  $P$  be a finite  $p$ -group. Let  $\alpha(P; \mathbb{C})$  be the number of irreducible real representations of  $P$  of complex type.*

$$(1) \quad \tilde{L}_m^s(\mathbb{Z}P) \cong \begin{cases} \mathbb{Z}^{\alpha(P; \mathbb{C})}, & m \text{ even,} \\ 0, & m \text{ odd,} \end{cases}$$

where  $\tilde{L}_m^s(\mathbb{Z}P)$  is the cokernel of the map  $L_m^s(\mathbb{Z}) \rightarrow L_m^s(\mathbb{Z}P)$  induced by the inclusion  $1 \rightarrow P$ .

$$(2) \quad \text{The homomorphism } \xi_*(BP): L_*^s(\mathbb{Z}P) \rightarrow \mathcal{S}_*^{\text{per},s}(BP) \text{ induces a } 1/p\text{-localization } \tilde{\xi}_*(BP): \tilde{L}_*^s(\mathbb{Z}P) \rightarrow \mathcal{S}_*^{\text{per},s}(BP).$$

**PROOF.**

(1) This is a result due independently to Bak and Wall; see [3] and [35, cor. 2.4.3].

(2) The assembly map  $A_*(\bullet): H_*(\bullet; \mathbf{L}(\mathbb{Z})) \rightarrow L_*^s(\mathbb{Z})$  is an isomorphism, essentially by definition. Thus there is a reduced algebraic surgery exact sequence

$$(6.1) \quad \dots \xrightarrow{\tilde{\eta}_{m+1}(X)} \tilde{H}_m(X; \mathbf{L}(\mathbb{Z})) \xrightarrow{\tilde{A}_m(X)} \tilde{L}_m^s(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\tilde{\xi}_m(X)} \mathcal{S}_m^{\text{per},s}(X) \\ \xrightarrow{\tilde{\eta}_m(X)} \tilde{H}_{m-1}(X; \mathbf{L}(\mathbb{Z})) \xrightarrow{\tilde{A}_{m-1}(X)} \tilde{L}_{m-1}^s(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\tilde{\xi}_{m-1}(X)} \dots$$

and

$$(6.2) \quad \dots \xrightarrow{\tilde{\eta}_{m+1}^{(1)}(X)} \tilde{H}_m(X; \mathbf{L}\langle 1 \rangle) \xrightarrow{\tilde{A}_m^{(1)}(X)} \tilde{L}_m^s(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\tilde{\xi}_m^{(1)}(X)} \mathcal{S}_{\text{geo},s}^s(X) \\ \xrightarrow{\tilde{\eta}_m^{(1)}(X)} \tilde{H}_{m-1}(X; \mathbf{L}\langle 1 \rangle) \xrightarrow{\tilde{A}_{m-1}^{(1)}(X)} \tilde{L}_{m-1}^s(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\tilde{\xi}_{m-1}^{(1)}(X)} \dots$$

Using the Atiyah-Hirzebruch spectral sequence and the  $\mathbb{Z}$ -flatness of  $\mathbb{Z}[1/p]$ , one sees that  $\tilde{H}_*(BP; \mathbf{L}(\mathbb{Z})) \otimes \mathbb{Z}[1/p] = 0$ . Using the flatness of  $\mathbb{Z}[1/p]$  again it follows that  $\tilde{\xi}_*(BP)$  is a  $1/p$ -equivalence. Thus it suffices to show that  $\mathcal{S}_*^{\text{per},s}(BP)$  is  $1/p$ -local.

We will express the structure set in terms of an equivariant homology theory. We get from [9, theorem B.1] an isomorphism

$$\mathcal{S}_m^{\text{per},s}(X) \cong H_m^G(\tilde{X} \rightarrow \bullet; \mathbf{L}(\mathbb{Z}))$$

for a connected CW-complex  $X$  with fundamental group  $G$ . Thus it suffices to show that  $H_*^P(EP \rightarrow \bullet; \mathbf{L}(\mathbb{Z}))$  is  $1/p$ -local. Since  $p$  is odd, it remains to show that  $H_*^P(EP \rightarrow \bullet; \mathbf{L}(\mathbb{Z}))[1/2]$  is  $1/p$ -local since  $p$  is odd. Because of Theorem 4.2 it is enough to show that  $KO_*(EP \rightarrow \bullet; \mathbf{L}(\mathbb{Z}))[1/2]$  is  $1/p$ -local.

The composite of complexification  $c$  with the forgetful map  $r$

$$\mathbf{KO} \xrightarrow{c} \mathbf{K} \xrightarrow{r} \mathbf{KO}$$

is multiplication by 2, so

$$KO_m^P(EP \rightarrow \bullet)[1/2] \text{ is a summand of } K_m^P(EP \rightarrow \bullet)[1/2].$$

Since a summand of a  $1/p$ -local abelian group is  $1/p$ -local, it suffices to show that  $K_*^P(EP \rightarrow \bullet)$  is  $1/p$ -local. Hence the proof of Theorem 6.1 is finished after we have proved the next result.  $\square$

LEMMA 6.2. *Let  $p$  be a prime and  $P$  be a  $p$ -group. Let  $\alpha(P; \mathbb{C})$  be the number of irreducible complex representations of  $P$ .*

- (1)  $\tilde{K}_m^P(\bullet) \cong \begin{cases} \mathbb{Z}^{\alpha(P; \mathbb{C})-1}, & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$
- (2) *The map  $\tilde{K}_*^P(\bullet) \rightarrow K_*^P(EP \rightarrow \bullet)$  is a  $1/p$ -localization.*

PROOF.

(1) Let  $R(P)$  be the complex representation ring. Note that equivariant  $K$ -cohomology and equivariant  $K$ -homology are 2-periodic,

$$K_P^0(\bullet) = R(P), \quad K_0^P(\bullet) = \text{Hom}_{\mathbb{Z}}(R(P), \mathbb{Z}), \quad K_P^1(\bullet) = 0 = K_1^P(\bullet).$$

(2) Our method for computing the relative equivariant  $K$ -homology will be to apply a universal coefficient theorem. This universal coefficient theorem applies only to finite complexes. Hence we take models of  $BP$  and  $EP$  whose  $n$ -skeletons  $BP^n$  and  $EP^n$  have a finite number of cells. For each  $n$  we have the exact sequence

$$(6.3) \quad 0 \rightarrow K_P^1(EP^n) \rightarrow K_P^0(EP^n \rightarrow \bullet) \rightarrow K_P^0(\bullet) \\ \rightarrow K_P^0(EP^n) \rightarrow K_P^1(EP^n \rightarrow \bullet) \rightarrow 0.$$

It induces an exact sequence of pro- $\mathbb{Z}$ -modules indexed by  $n = 0, 1, 2, 3, \dots$

$$(6.4) \quad 0 \rightarrow \{K_P^1(EP^n)\} \rightarrow \{K_P^0(EP^n \rightarrow \bullet)\} \rightarrow \{K_P^0(\bullet)\} \\ \rightarrow \{K_P^0(EP^n)\} \rightarrow \{K_P^1(EP^n \rightarrow \bullet)\} \rightarrow 0,$$

where  $\{K_P^0(\bullet)\}$  is a constant pro- $\mathbb{Z}$ -module. For an introduction to the abelian category of pro- $\mathbb{Z}$ -modules, see [1]. By the Atiyah-Segal completion theorem [1], there is an isomorphism of pro- $\mathbb{Z}$ -modules  $\{K_P^1(EP^n)\} \cong \{0\}$  and a commutative diagram of pro- $\mathbb{Z}$ -modules,

$$\begin{array}{ccc} \{R(P)\} & \longrightarrow & \{R(P)/I^n\} \\ \downarrow \cong & & \downarrow \cong \\ \{K_P^0(\bullet)\} & \longrightarrow & \{K_P^0(EP^n)\}, \end{array}$$

where  $I = \ker(R(P) \rightarrow \mathbb{Z})$  is the augmentation ideal of the representation ring. It follows that the sequence (6.4) can be identified with

$$0 \rightarrow \{0\} \rightarrow \{I^n\} \rightarrow \{R(P)\} \rightarrow \{R(P)/I^n\} \rightarrow \{0\} \rightarrow 0,$$

and in particular,

$$\{K_P^1(EP^n \rightarrow \bullet)\} = \{0\}, \quad \{K_P^0(EP^n \rightarrow \bullet)\} = \{I^n\}.$$

A universal coefficient theorem for  $K_*^P$  proven in [18] states that for an equivariant map  $X \rightarrow Y$  of finite  $P$ -CW-complexes, there is a short exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}(K_P^{*+1}(X \rightarrow Y), \mathbb{Z}) \rightarrow K_*^P(X \rightarrow Y) \\ \rightarrow \text{Hom}_{\mathbb{Z}}(K_P^*(X \rightarrow Y), \mathbb{Z}) \rightarrow 0.$$

It is an algebraic fact that  $\{I^n\} = \{p^n I\}$  and, setting  $I^* = \text{Hom}_{\mathbb{Z}}(I, \mathbb{Z}) \subset I^* \otimes \mathbb{Q}$ , that  $(p^n I)^* = p^{-n} I^*$ . Thus

$$\begin{aligned} K_0^P(EP \rightarrow \bullet) &\cong \text{colim}_{n \rightarrow \infty} K_0^P(EP^n \rightarrow \bullet) \\ &\cong \text{colim}_{n \rightarrow \infty} \text{Hom}_{\mathbb{Z}}(K_P^0(EP^n \rightarrow \bullet), \mathbb{Z}) \\ &\cong \text{colim}_{n \rightarrow \infty} (p^n I)^* \\ &\cong I^* \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]. \end{aligned}$$

The map  $\tilde{K}_0^P(\bullet) \rightarrow K_0^P(EP \rightarrow \bullet)$  can be identified with the inclusion  $I^* = I^* \otimes \mathbb{Z} \rightarrow I^* \otimes \mathbb{Z}[1/p]$ . Similar reasoning shows that  $K_1^P(EP \rightarrow \bullet) = 0$ . Hence Lemma 6.2 follows.  $\square$

### 7 Periodic Simple Structure Set of $B\Gamma$

In this section we compute the periodic simple structure set of  $B\Gamma$ . Recall that  $\mathcal{P}$  is the set of conjugacy classes of subgroups of  $\Gamma$  of order  $p$  and that the order of  $\mathcal{P}$  is  $p^k$ .

**THEOREM 7.1** (Computation of  $\mathcal{S}^{\text{per},s}(B\Gamma)$ ).

(1) *The map*

$$\bigoplus_{(P) \in \mathcal{P}} \mathcal{S}_m^{\text{per},s}(BP) \xrightarrow{\cong} \mathcal{S}_m^{\text{per},s}(B\Gamma)$$

*induced by the various inclusions  $P \rightarrow \Gamma$  is for all  $m \in \mathbb{Z}$  an isomorphism.*

*We get*

$$\mathcal{S}_m^{\text{per},s}(B\Gamma) \cong \begin{cases} (\mathbb{Z}[1/p])^{p^k \cdot (p-1)/2} & m \text{ odd,} \\ 0 & m \text{ even,} \end{cases}$$

(2) *Let  $\text{pr}: \Gamma \rightarrow \Gamma_{\text{ab}} = \Gamma/[\Gamma, \Gamma]$  be the projection onto the abelianization of  $\Gamma$ . Let  $\mathcal{S}_m^{\text{per},s}(B\text{pr}): \mathcal{S}_m^{\text{per},s}(B\Gamma) \rightarrow \mathcal{S}_m^{\text{per},s}(\Gamma_{\text{ab}})$  be the induced map. Recall  $\Gamma_{\text{ab}} \cong (\mathbb{Z}/p)^{k+1}$ . For every  $(P) \in \mathcal{P}$ , let  $P' \subset \Gamma_{\text{ab}}$  be the image of  $P$  under  $\text{pr}$ . Let*

$$\text{res}_{\Gamma_{\text{ab}}}^{P'}: \mathcal{S}_m^{\text{per},s}(B\Gamma_{\text{ab}}) \rightarrow \mathcal{S}_m^{\text{per},s}(BP')$$

*be the map induced by transfer to the subgroup  $P' \subset \Gamma_{\text{ab}}$ .*

*Then the map*

$$\prod_{(P) \in \mathcal{P}} \text{res}_{\Gamma_{\text{ab}}}^{P'} \circ \mathcal{S}_m^{\text{per},s}(B\text{pr}): \mathcal{S}_m^{\text{per},s}(B\Gamma) \xrightarrow{\cong} \prod_{(P) \in \mathcal{P}} \mathcal{S}_m^{\text{per},s}(BP')$$

*is an isomorphism.*

PROOF.

(1) By [9, theorem B.1], we have the identifications

$$\begin{aligned} \mathcal{S}_m^{\text{per},s}(B\Gamma) &= H_m^\Gamma(E\Gamma \rightarrow \bullet; \mathbf{L}^s), \\ \mathcal{S}_m^{\text{per},s}(BP) &= H_m^P(EP \rightarrow \bullet; \mathbf{L}^s). \end{aligned}$$

Theorem 4.5(1) implies

$$H_*^\Gamma(E\Gamma \rightarrow \underline{E}\Gamma; \mathbf{L}^s) = H_*^\Gamma(E\Gamma \rightarrow \bullet; \mathbf{L}^s).$$

We get for  $m \in \mathbb{Z}$  from the induction structure (see [23, sec. 1]) isomorphisms

$$(7.1) \quad H_m^{\{1\}}(BP; \mathbf{L}^s) \xleftarrow{\cong} H_m^P(EP; \mathbf{L}^s) \xrightarrow{\cong} H_m^\Gamma(\Gamma \times_P EP; \mathbf{L}^s).$$

We conclude from the  $\Gamma$ -pushout (3.1) and the identification (7.1),

$$\bigoplus_{(P) \in \mathcal{P}} H_*^P(EP \rightarrow \bullet; \mathbf{L}^s) = H_*^\Gamma(E\Gamma \rightarrow \underline{E}\Gamma; \mathbf{L}^s).$$

Hence

$$\bigoplus_{(P) \in \mathcal{P}} \mathcal{S}_m^{\text{per},s}(BP) \xrightarrow{\cong} \mathcal{S}_m^{\text{per},s}(B\Gamma)$$

is an isomorphism. Assertion (1) follows from Theorem 6.1 and Lemma 1.1(4).

(2) Because of assertion (1) it suffices to show that the composite

$$\bigoplus_{(P) \in \mathcal{P}} \mathcal{S}_m^{\text{per},s}(BP) \xrightarrow{\cong} \mathcal{S}_m^{\text{per},s}(B\Gamma) \xrightarrow{\prod_{(P) \in \mathcal{P}} \text{res}_{\Gamma_{\text{ab}}}^{P'} \circ \mathcal{S}_m^{\text{per},s}(B\text{pr})} \prod_{(P) \in \mathcal{P}} \mathcal{S}_m^{\text{per},s}(BP')$$

is an isomorphism. We conclude from Theorem 6.1(2) that it suffices to show that the composite

$$\bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^s(\mathbb{Z}P) \rightarrow L_m^s(\mathbb{Z}\Gamma) \xrightarrow{\text{pr}_*} \tilde{L}_m^s(\mathbb{Z}[\Gamma_{\text{ab}}]) \xrightarrow{\prod_{(P) \in \mathcal{P}} \text{res}_{\Gamma_{\text{ab}}}^{P'}} \prod_{(P) \in \mathcal{P}} \tilde{L}_m^s(\mathbb{Z}P')$$

is an isomorphism after inverting  $p$ , where the first map is given by induction with the various inclusions  $P \rightarrow \Gamma$ , the second by induction with  $\text{pr}: \Gamma \rightarrow \Gamma_{\text{ab}}$ , and the third is the product over the various transfer homomorphisms  $\text{res}_{\Gamma_{\text{ab}}}^{P'}$ . This composite agrees with the composite

$$\bigoplus_{(P) \in \mathcal{P}} \tilde{L}_m^s(\mathbb{Z}P') \xrightarrow{\left(\prod_{(Q) \in \mathcal{P}} \text{res}_{\Gamma_{\text{ab}}}^{Q'}\right) \circ \left(\bigoplus_{(P) \in \mathcal{P}} \text{ind}_{P'}^{\Gamma_{\text{ab}}}\right)} \prod_{(Q) \in \mathcal{P}} \tilde{L}_m^s(\mathbb{Z}Q').$$

Hence it suffices to show that for  $(P), (Q) \in \mathcal{P}$

$$\text{res}_{\Gamma_{\text{ab}}}^{Q'} \circ \text{ind}_{P'}^{\Gamma_{\text{ab}}}: \tilde{L}_m^s(\mathbb{Z}P') \rightarrow \tilde{L}_m^s(\mathbb{Z}Q')$$

is trivial for  $(P) \neq (Q)$  and  $p^m \cdot \text{id}$  for  $(P) = (Q)$ . Notice that  $P' = Q' \Leftrightarrow (P) = (Q)$  holds for  $(P), (Q) \in \mathcal{P}$  by Lemma 1.1, items (3) and (7). By the double coset formula the composite

$$\text{res}_{Q'}^{\Gamma_{\text{ab}}} \circ \text{ind}_{P'}^{\Gamma_{\text{ab}}}: L_m^s(\mathbb{Z}P') \rightarrow L_m^s(\mathbb{Z}Q')$$

factorizes through  $L_m^s(\mathbb{Z})$  if  $P' \neq Q'$ , and is  $\sum_{\Gamma_{\text{ab}}/P} \text{id}$  if  $P' = Q'$ . Since  $\Gamma_{\text{ab}}/P$  contains  $p^k$  elements by Lemma 1.1(3) and (7), Theorem 7.1 follows.  $\square$

The splitting of the periodic structure set appearing in Theorem 7.1(1) has already been established in [26, theorem 2.12].

### 8 Periodic Simple Structure Set of $M$

In this section we compute the periodic simple structure set  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  of  $M$ . Recall that  $M = \mathbb{T}_\rho^n \times_{\mathbb{Z}/p} \mathbb{S}^l$  and  $\pi_1(M) = \Gamma = \mathbb{Z}^n \times_\rho \mathbb{Z}/p$ . Let  $f: M \rightarrow B\Gamma$  be a classifying map for the universal covering of  $M$ .

**THEOREM 8.1** (Periodic simple structure set of  $M$ ). *There is a homomorphism*

$$\sigma: \mathcal{S}_{n+l+1}^{\text{per},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p}$$

such that the following holds:

(1) *The map*

$$\sigma \times \mathcal{S}_{n+l+1}^{\text{per},s}(f): \mathcal{S}_{n+l+1}^{\text{per},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p} \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma)$$

*is injective.*

(2) *The cokernel of  $\sigma$  is a finite abelian  $p$ -group.*

(3) *Consider the composite*

$$v: \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) \rightarrow \tilde{L}_{n+l+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\tilde{\xi}_{n+l+1}(M)} \mathcal{S}_{n+l+1}^{\text{per},s}(M)$$

where the first map is given by induction with the various inclusions  $P \rightarrow \Gamma$  and  $\tilde{\xi}_{n+l+1}(M)$  comes from (6.1).

Then  $v$  is injective, the image of  $v$  is contained in the kernel of  $\sigma$ , and  $\ker(\sigma)/\text{im}(v)$  is a finite abelian  $p$ -group.

(4) *After inverting  $p$  we obtain an isomorphism*

$$\begin{aligned} (\sigma \times \mathcal{S}_{n+l+1}^{\text{per},s}(f))[1/p]: \mathcal{S}_{n+l+1}^{\text{per},s}(M)[1/p] \\ \rightarrow H_n(\mathbb{T}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p}[1/p] \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma)[1/p]. \end{aligned}$$

(5) *As an abelian group we have*

$$\mathcal{S}_{n+l+1}^{\text{per},s}(M) \cong \mathbb{Z}p^{k(p-1)/2} \oplus \bigoplus_{i=0}^n L_{n-i}(\mathbb{Z})^{r_i},$$

where the numbers  $r_i$  are defined in (4.6).



*Remark 8.2.* There are several different points of views on the codomain of  $\sigma$ . Indeed, there are isomorphisms

$$\begin{aligned} L_n(\mathbb{Z}[\mathbb{Z}^n])^{\mathbb{Z}/p} &\cong H_n(\mathbb{T}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p} \cong \bigoplus_{i=0}^n (H_i(\mathbb{T}^n)^{\mathbb{Z}/p} \otimes L_{n-i}(\mathbb{Z})) \\ &\cong \bigoplus_{i=0}^n L_{n-i}(\mathbb{Z})^{r_i}. \end{aligned}$$

The first isomorphism is due to the Shaneson splitting/Farrell-Jones conjecture, the second isomorphism is due to the collapse of the Atiyah-Hirzebruch spectral sequence, and the last isomorphism comes from (4.6).

In the proof of Theorem 8.1 it will be convenient to define  $\mu: \mathcal{S}_{n+l+1}^{\text{per},s}(M) \rightarrow L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}$  and then define the composite

$$\sigma: \mathcal{S}_{n+l+1}^{\text{per},s}(M) \xrightarrow{\mu} L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p} \cong H_n(\mathbb{T}^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p},$$

noting that  $\ker \mu = \ker \sigma$  and  $\text{cok } \mu \cong \text{cok } \sigma$ .

**PROOF OF ASSERTION (1) OF THEOREM 8.1.** We will have proved assertion (1) once we accomplish the following two goals:

Goal (1) Construct the commutative diagram in Figure 8.1.

Goal (2) Show that the induced maps

$$\begin{aligned} \ker(\mathcal{S}_{n+l+1}^{\text{per},s}(f)) &\rightarrow \ker(H_{n+l}(f; \mathbf{L}(\mathbb{Z}))) \rightarrow \ker(E_{l,n}^\infty(f)) \\ &\text{are isomorphisms.} \end{aligned}$$

Once we accomplish these two goals, a diagram chase gives the proof of Theorem 8.1(1).

We now construct the commutative diagram of abelian groups (Figure 8.1). Some explanations are in order. Here and elsewhere in the paper if  $A$  is a nontrivial  $\mathbb{Z}G$ -module and  $X$  is a free  $G$ -space, we write  $H_i^G(X; A)$  for  $H_i(C_*(X) \otimes_{\mathbb{Z}G} A)$ , and we also write  $H_i(G; A)$  for  $H_i^G(EG; A)$ .

The symbol  $\text{inc}$  always stands for an obvious inclusion.

Given a free  $\mathbb{Z}/p$ -CW-complex  $X$  and any homology theory  $\mathcal{H}_*$ , there is a Leray-Serre spectral sequence converging to  $\mathcal{H}_{i+j}(X \times_{\mathbb{Z}/p} \mathbb{T}_\rho^n)$  whose  $E^2$ -term is  $H_i^{\mathbb{Z}/p}(X; \mathcal{H}_j(\mathbb{T}_\rho^n))$ . In particular, we have a spectral sequence

$$E_{i,j}^2 = H_i^{\mathbb{Z}/p}(X; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) \implies H_{i+j}(X \times_{\mathbb{Z}/p} \mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z})).$$

The symbols  $F_{l,n}(M)$ ,  $F_{l,n}(B\Gamma)$ ,  $E_{l,n}^r(M)$ , and  $E_{l,n}^r(B\Gamma)$  denote the corresponding filtration terms and  $E^r$ -terms of the spectral sequences applied to the free  $\mathbb{Z}/p$ -CW-complex  $X = \mathbb{S}^l$  and  $X = E\mathbb{Z}/p$ . This explains the third, fourth, and fifth row in the diagram of Figure (8.1) except for the map  $g_{n+l}$ , which we describe below. In order to show the equality of the fourth, fifth, and the sixth row, we need to show the following:

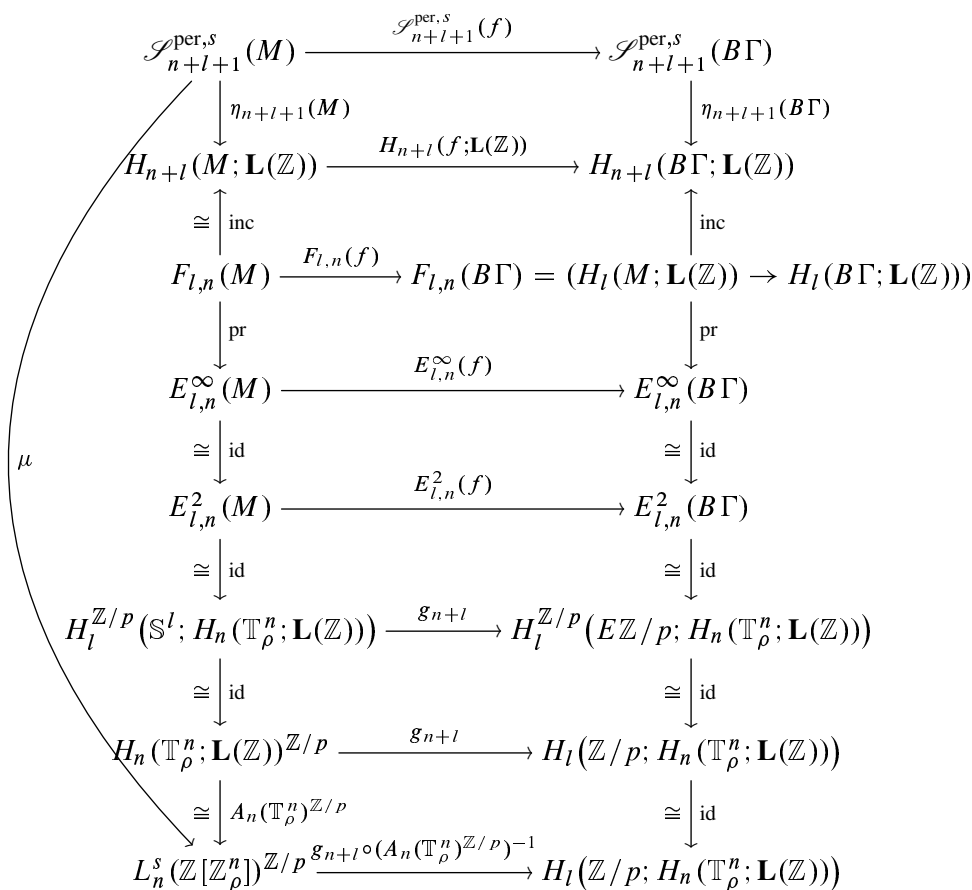


FIGURE 8.1. Commutative surgery diagram.

LEMMA 8.3. All differentials in the following two spectral sequences vanish:

- (1)  $E_{i,j}^2 = H_i(\mathbb{Z}/p; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) \implies H_{i+j}(B\Gamma; \mathbf{L}(\mathbb{Z})).$
- (2)  $E_{i,j}^2 = H_i^{\mathbb{Z}/p}(\mathbb{S}^l; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) \implies H_{i+j}(M; \mathbf{L}(\mathbb{Z})).$

PROOF.

(1) It suffices to show that all differentials vanish after inverting  $p$  and after localizing at  $p$ . Since for  $i \neq 0$

$$E_{i,j}^2[1/p] = H_i(\mathbb{Z}/p; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))[1/p]) = 0,$$

this is obvious after inverting  $p$ .

If we localize at  $p$ , we get a natural isomorphism of homology theories

$$KO_*(-)_{(p)} \xrightarrow{\cong} H_*(-; \mathbf{L}(\mathbb{Z}))_{(p)},$$

since  $p$  is odd, by Sullivan’s  $KO[1/2]$ -orientation; see Theorem 4.2. Hence it suffices to show that all the differentials of the Leray-Serre spectral sequence converging to  $KO_{i+j}(B\Gamma)$  with  $E^2$ -term  $H_i(\mathbb{Z}/p, KO_j(\mathbb{T}_\rho^n))$  are trivial. The edge homomorphism

$$H_0(\mathbb{Z}/p; KO_m(\mathbb{T}_\rho^n)) = KO_m(\mathbb{T}_\rho^n) \otimes_{\mathbb{Z}[\mathbb{Z}/p]} \mathbb{Z} \xrightarrow{\cong} KO_m(B\Gamma)$$

is bijective for even  $m$  by [13, theorem 6.1(ii)]. Thus all differentials involving  $E_{0,m}^r$  are trivial for  $m$  even. Hence it suffices to show

$$H_i(\mathbb{Z}/p; KO_j(\mathbb{T}_\rho^n))_{(p)} = 0 \quad \text{if } i > 0, i + j \equiv 0 \pmod{2}.$$

Since there are natural transformation of homology theories  $i: KO_* \rightarrow K_*$  and  $r: K_* \rightarrow KO_*$  with  $r \circ i = 2 \cdot \text{id}$ , it suffices to show

$$H_i(\mathbb{Z}/p; K_j(\mathbb{T}_\rho^n))_{(p)} = 0 \quad \text{if } i > 0, i + j \equiv 0 \pmod{2}.$$

This vanishing is explicitly given in the proof of [13, theorem 4.1 (ii)].

(2) Let  $g: \mathbb{S}^l \rightarrow E\mathbb{Z}/p$  be the classifying map of the free  $\mathbb{Z}/p$ -CW-complex  $\mathbb{S}^l$ . It induces a map of the spectral sequence of assertion (2) to the one of assertion (1). We know already that all the differentials of the latter one vanish. The induced maps on the  $E^2$ -terms

$$(8.1) \quad \begin{aligned} E_{i,j}^2(M) &= H_i^{\mathbb{Z}/p}(\mathbb{S}^l; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) \rightarrow E_{i,j}^2(B\Gamma) \\ &= H_i^{\mathbb{Z}/p}(E\mathbb{Z}/p; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) \end{aligned}$$

are bijective for  $i \leq l - 1$  and surjective for  $i = l$  since  $\mathbb{S}^l \rightarrow E\mathbb{Z}/p$  is  $l$ -connected. Since  $\mathbb{S}^l$  is  $l$ -dimensional, we have

$$E_{i,j}^2(M) = 0 \quad \text{if } i \geq l + 1.$$

This finishes the proof of Lemma 8.3. □

Let

$$g_{n+l}: H_l^{\mathbb{Z}/p}(\mathbb{S}^l; H_n(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) \rightarrow H_l^{\mathbb{Z}/p}(E\mathbb{Z}/p; H_n(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z})))$$

be the map induced by the classifying map  $g: \mathbb{S}^l \rightarrow E\mathbb{Z}/p$ .

For an appropriate choice of generator  $t \in \mathbb{Z}/p$  the chain  $\mathbb{Z}[\mathbb{Z}/p]$ -chain complex of  $\mathbb{S}^l$  is  $\mathbb{Z}[\mathbb{Z}/p]$ -chain homotopy equivalent to the  $l$ -dimensional  $\mathbb{Z}[\mathbb{Z}/p]$ -chain complex

$$\dots \rightarrow 0 \rightarrow \mathbb{Z}[\mathbb{Z}/p] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/p] \xrightarrow{N} \dots \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/p] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/p] \rightarrow 0 \rightarrow \dots$$

Thus we obtain an identification

$$H_l^{\mathbb{Z}/p}(\mathbb{S}^l; H_n(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))) = H_n(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z}))^{\mathbb{Z}/p}.$$

The assembly map

$$A_n(\mathbb{T}_\rho^n): H_n(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z})) \xrightarrow{\cong} L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])$$

is an isomorphism because of the Shaneson splitting (or the Farrell-Jones conjecture in  $L$ -theory for the group  $\mathbb{Z}^n$ ) and the Rothenberg sequences since  $\text{Wh}(\mathbb{Z}^n) = 0$  and  $\tilde{K}_m(\mathbb{Z}[\mathbb{Z}^n]) = 0$  for  $m \leq 0$ , and is a map of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules because of naturality of the assembly map.

Define the map

$$(8.2) \quad \mu: \mathcal{S}_{n+l+1}^{\text{per},s}(M) \rightarrow L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}$$

to be the composite of the vertical arrows in the diagram (8.1) from  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  to  $L_n^s(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}$ , namely, to be the composite  $A_n(T_\rho^n)^{\mathbb{Z}/p} \circ \text{pr} \circ \eta_{n+l+1}(M)$ .

We have explained all modules and maps in the diagram (8.1). One easily checks that it commutes. Hence we have accomplished Goal (1).

LEMMA 8.4. *Consider the following commutative square:*

$$\begin{array}{ccc} \mathcal{S}_{n+l+1}^{\text{per},s}(M) & \xrightarrow{\mathcal{S}_{n+l+1}^{\text{per},s}(f)} & \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma) \\ \downarrow \eta_{n+l+1}(M) & & \downarrow \eta_{n+l+1}(B\Gamma) \\ H_{n+l}(M; \mathbf{L}(\mathbb{Z})) & \xrightarrow{H_{n+l}(f; \mathbf{L}(\mathbb{Z}))} & H_{n+l}(B\Gamma; \mathbf{L}(\mathbb{Z})) \end{array}$$

where the vertical maps come from (5.2).

Then the vertical maps induce an isomorphism

$$\ker(\mathcal{S}_{n+l+1}^{\text{per},s}(f)) \xrightarrow{\cong} \ker(H_{n+l}(f; \mathbf{L}(\mathbb{Z}))).$$

PROOF. Consider the following commutative diagram of abelian groups, which comes from (5.2):

$$\begin{array}{ccc} H_{n+l+1}(M; \mathbf{L}(\mathbb{Z})) & \xrightarrow{H_{n+l+1}(f; \mathbf{L}(\mathbb{Z}))} & H_{n+l+1}(B\Gamma; \mathbf{L}(\mathbb{Z})) \\ \downarrow A_{n+l+1}(M) & & \downarrow A_{n+l+1}(B\Gamma) \\ L_{n+l+1}^s(\mathbb{Z}\Gamma) & \xrightarrow{\text{id}} & L_{n+l+1}^s(\mathbb{Z}\Gamma) \\ \downarrow \xi_{n+l+1}(M) & & \downarrow \xi_{n+l+1}(B\Gamma) \\ \mathcal{S}_{n+l+1}^{\text{per},s}(M) & \xrightarrow{\mathcal{S}_{n+l+1}^{\text{per},s}(f)} & \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma) \\ \downarrow \eta_{n+l+1}(M) & & \downarrow \eta_{n+l+1}(B\Gamma) \\ H_{n+l}(M; \mathbf{L}(\mathbb{Z})) & \xrightarrow{H_{n+l}(f; \mathbf{L}(\mathbb{Z}))} & H_{n+l}(B\Gamma; \mathbf{L}(\mathbb{Z})) \\ \downarrow A_{n+l}(M) & & \downarrow A_{n+l}(B\Gamma) \\ L_{n+l}^s(\mathbb{Z}\Gamma) & \xrightarrow{\text{id}} & L_{n+l}^s(\mathbb{Z}\Gamma). \end{array}$$

An easy diagram chase shows that it suffices to show that the map

$$H_{n+l+1}(f; \mathbf{L}(\mathbb{Z})): H_{n+l+1}(M; \mathbf{L}(\mathbb{Z})) \rightarrow H_{n+l+1}(B\Gamma; \mathbf{L}(\mathbb{Z}))$$

is surjective. We prove surjectivity after localizing at  $p$  and after inverting  $p$ .

We begin with localizing at  $p$ . Since  $p$  is odd, for every  $CW$ -complex  $X$  and  $m \in \mathbb{Z}$  there is a natural isomorphism (see Theorem 4.2),

$$H_m(X; \mathbf{L}(\mathbb{Z}))_{(p)} \xrightarrow{\cong} KO_m(X)_{(p)}.$$

Hence it suffices to show that

$$KO_{n+l+1}(f)_{(p)}: KO_{n+l+1}(M)_{(p)} \rightarrow KO_{n+l+1}(B\Gamma)_{(p)}$$

is surjective. This follows from the following commutative diagram:

$$\begin{CD} KO_{n+l+1}(\mathbb{S}^l \times B\mathbb{Z}^n) @>>> KO_{n+l+1}(M) \\ @V KO_{n+l+1}(\text{pr}) VV @VV KO_{n+l+1}(f) V \\ KO_{n+l+1}(B\mathbb{Z}^n) @>>> KO_{n+l+1}(B\Gamma) \end{CD}$$

since the left vertical arrow is induced by the projection and is hence surjective and the lower horizontal arrow is surjective, as  $KO_k(B\mathbb{Z}^n) \rightarrow KO_k(B\Gamma)$  is surjective for  $k$  even by [13, theorem 6.1(ii)].

Next we invert  $p$ . Then a standard transfer argument (see, for example, proposition A.4 of [13]) shows that

$$H_m(B\mathbb{Z}_\rho^n; \mathbf{L}(\mathbb{Z})) \otimes_{\mathbb{Z}[\mathbb{Z}/p]} \mathbb{Z}[1/p] \xrightarrow{\cong} H_m(B\Gamma; \mathbf{L}(\mathbb{Z}))[1/p]$$

is an isomorphism for all  $m \in \mathbb{Z}$ . It follows, as above, that  $H_{n+l+1}(M; \mathbf{L}(\mathbb{Z})) \rightarrow H_{n+l+1}(B\Gamma; \mathbf{L}(\mathbb{Z}))$  is surjective, as desired.  $\square$

LEMMA 8.5. *The composite*

$$H_{n+l}(M; \mathbf{L}(\mathbb{Z})) \xrightarrow{\text{id}} F_{l,n}(M) \xrightarrow{\text{pr}} E_{l,n}^\infty(M)$$

*induces a bijection*

$$\begin{aligned} \ker(H_{n+l}(f; \mathbf{L}(\mathbb{Z})): H_{n+l}(M; \mathbf{L}(\mathbb{Z})) \rightarrow H_{n+l}(B\Gamma; \mathbf{L}(\mathbb{Z}))) \\ \xrightarrow{\cong} \ker(E_{l,n}^\infty(f): E_{l,n}^\infty(M) \rightarrow E_{l,n}^\infty(B\Gamma)). \end{aligned}$$

PROOF. We have shown in Lemma 8.3, items (1) and 2, that all the differentials of the spectral sequence converging to  $H_{i+j}(B\Gamma; \mathbf{L}(\mathbb{Z}))$  with

$$E_{i,j}^2 = H_i(\mathbb{Z}/p; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z})))$$

and all the differentials of the spectral sequence converging to  $H_{i+j}(M; \mathbf{L}(\mathbb{Z}))$  with  $E_{i,j}^2 = H_i^{\mathbb{Z}/p}(\mathbb{S}^l; H_j(\mathbb{T}_\rho^n; \mathbf{L}(\mathbb{Z})))$  vanish. The map

$$E_{i,j}^2(f): E_{i,j}^2(M) \rightarrow E_{i,j}^2(B\Gamma)$$

is bijective for  $i \leq l - 1$  and all  $j$  since the map  $\mathbb{S}^l \rightarrow E\mathbb{Z}/p$  is  $l$ -connected. Hence the map

$$F_{i,j}(f): F_{i,j}(M) \rightarrow F_{i,j}(B\Gamma)$$

is bijective for  $i \leq l - 1$  and all  $j$ . This implies

$$F_{l-1,n+1}(M) \cap \ker(H_{n+l}(f; \mathbf{L}(\mathbb{Z}))) : H_{n+l}(M; \mathbf{L}(\mathbb{Z})) \rightarrow H_{n+l}(B\Gamma; \mathbf{L}(\mathbb{Z})) = 0.$$

Since  $S^l$  is  $l$ -dimensional and hence  $H_{n+l}(M; \mathbf{L}(\mathbb{Z})) = F_{l,n}(M)$ , Lemma 8.5 follows.  $\square$

Lemma 8.4 and Lemma 8.5 give Goal (2) and hence complete the proof of assertion (1) of Theorem 8.1  $\square$

PROOF OF ASSERTION (2) OF THEOREM 8.1. Consider the commutative diagram

$$(8.3) \quad \begin{array}{ccc} F_{l-1,n+l}(M) & \xrightarrow{\cong} & F_{l-1,n+l}(B\Gamma) \\ \text{inc} \downarrow & & \downarrow \cong_{1/p} \\ H_{n+l}(M; \mathbf{L}(\mathbb{Z})) & \xrightarrow{H_{n+l}(f; \mathbf{L}(\mathbb{Z}))} & H_{n+l}(B\Gamma; \mathbf{L}(\mathbb{Z})) \\ A_{n+l}(M) \downarrow & \swarrow A_{n+l}(B\Gamma) & \downarrow \cong_{1/p} \\ L_{n+l}^s(\mathbb{Z}\Gamma) & \xrightarrow{\cong} & H_{n+l}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z})) \end{array}$$

where the two maps ending at  $L_{n+l}^s(\mathbb{Z}\Gamma)$  are the assembly maps  $A_{n+l}(M)$  and  $A_{n+l}(B\Gamma)$  and the bottom horizontal isomorphism is discussed in Theorem 4.5(3). We have already explained that the top horizontal arrow is an isomorphism; see (8.1). The spectral sequence appearing in Lemma 8.3(1) implies that the vertical inclusion at the top right induces an isomorphism after tensoring with  $\mathbb{Z}[1/p]$ . We conclude from Lemma 4.3 that  $H_{n+l}(B\Gamma; \mathbf{L}(\mathbb{Z}))[1/p] \rightarrow H_{n+l}(\underline{B}\Gamma; \mathbf{L}(\mathbb{Z}))[1/p]$  is an isomorphism.

Now the diagram (8.3) shows that

$$(A_{n+l}(M) \circ \text{inc})[1/p] : F_{l-1,n+1}(M)[1/p] \xrightarrow{\cong} L_{n+l}^s(\mathbb{Z}\Gamma)[1/p]$$

is bijective.

We have the following commutative diagram:

$$(8.4) \quad \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & F_{l-1,n+1}(M) & \xrightarrow{A_{n+l}(M) \circ \text{inc}} & \\ & \eta_{n+l+1}(M) \rightarrow & \downarrow \text{inc} & \xrightarrow{A_{n+l}(M)} & L_{n+l}^s(\mathbb{Z}\Gamma) \\ \mathcal{S}_{n+l+1}^{\text{per},s}(M) & & H_{n+l}(M; \mathbf{L}(\mathbb{Z})) & & \\ & \searrow \mu & \downarrow A_{n+l}(\mathbb{T}_\rho^n)^{\mathbb{Z}/p} \circ \text{pr} & & \\ & & L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p} & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

with exact row and exact column. An easy diagram chase proves that

$$\mu[1/p]: \mathcal{S}_{n+l+1}^{\text{per},s}(M)[1/p] \rightarrow L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}[1/p]$$

is surjective, and we get

$$(8.5) \quad \ker(\mu[1/p]) = \ker(\eta_{n+l+1}(M)[1/p]).$$

Since  $L_n^s(\mathbb{Z}[\mathbb{Z}_\rho^n])$  is a finitely generated abelian group (see Theorem 6.1(1), the cokernel of  $\mu$  is a finite abelian  $p$ -group. Recalling that the cokernel of  $\sigma$  is isomorphic to the cokernel of  $\mu$ , this finishes the proof of assertion (2) of Theorem 8.1  $\square$

PROOF OF ASSERTION 3 OF THEOREM 8.1. The composite of

$$\nu: \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) \rightarrow \mathcal{S}_{n+l+1}^{\text{per},s}(M)$$

with

$$\mathcal{S}_{n+l+1}^{\text{per},s}(f): \mathcal{S}_{n+l+1}^{\text{per},s}(M) \rightarrow \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma)$$

becomes an isomorphism after inverting  $p$  since the composite is also the composite of

$$\bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) \rightarrow \bigoplus_{(P) \in \mathcal{P}} \mathcal{S}_{n+l+1}^{\text{per},s}(BP) \rightarrow \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma),$$

where the first map is an isomorphism after inverting  $p$  by Theorem 6.1(2) and the second map is an isomorphism by Theorem 7.1(1). Hence  $\nu$  is injective after inverting  $p$ . Since  $\tilde{L}_{n+l+1}^s(\mathbb{Z}P)$  is a finitely generated free abelian group,  $\nu$  is injective.

Note

$$\text{im}(\nu) \subset \text{im}(\tilde{\xi}_{n+l+1}(M)) = \text{im}(\xi_{n+l+1}(M)) = \ker(\eta_{n+l+1}(M)) \subset \ker \mu,$$

where the first equality holds since the simply connected surgery exact sequence is short exact and the second equality holds because of the periodic surgery exact sequence (5.2). We have shown in (8.5) that  $\ker(\mu[1/p]) = \ker(\eta_{n+l+1}(M)[1/p])$

holds. Therefore  $\ker(\mu)/\ker \eta_{n+l+1}(M) = \ker(\mu)/\text{im}(\xi_{n+l+1}(M))$  is a  $p$ -torsion abelian group. Since  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  is a finitely generated abelian group because of the surgery exact sequence (5.2), we conclude that  $\ker(\mu)/\text{im}(\xi_{n+l+1}(M))$  is a finite abelian  $p$ -group.

Next we consider the following commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) & & \\
 & & \downarrow & \searrow v & \\
 H_{n+l+1}(B\Gamma; \mathbf{L}(\mathbb{Z})) & \xrightarrow{A_{n+l+1}(B\Gamma)} & L_{n+l+1}^s(\mathbb{Z}\Gamma) & \xrightarrow{\xi_{n+l+1}(B\Gamma)} & \mathcal{S}_{n+l+1}^{\text{per},s}(M) \\
 & \searrow H_{n+l+1}(i; \mathbf{L}(\mathbb{Z})) & \downarrow & & \\
 & & H_{n+l+1}(B\underline{\Gamma}; \mathbf{L}(\mathbb{Z})) & & \\
 & & \downarrow & & \\
 & & 0 & & 
 \end{array}$$

Here the exact column is taken from Theorem 4.5(2), the row is exact because of the surgery exact sequence (5.2) for  $M$  and the surjectivity of

$$H_{n+l+1}(M; \mathbf{L}(\mathbb{Z})) \rightarrow H_{n+l+1}(B\underline{\Gamma}; \mathbf{L}(\mathbb{Z})),$$

which we have shown in the proof of Lemma 8.4. The map  $H_{n+l+1}(i; \mathbf{L}(\mathbb{Z}))$  induced by the obvious map  $i: B\Gamma \rightarrow B\underline{\Gamma}$  is bijective after inverting  $p$ ; see the proof of part 8.1(2). Since there is a finite  $CW$ -model for  $B\underline{\Gamma}$  and hence  $H_{n+l+1}(B\underline{\Gamma}; \mathbf{L}(\mathbb{Z}))$  is finitely generated, the cokernel of  $H_{n+l+1}(i; \mathbf{L}(\mathbb{Z}))$  is a finite abelian  $p$ -group. Since this cokernel is isomorphic to  $\text{im}(\xi_{n+l+1}(B\Gamma))/\text{im}(v)$  by the commutative diagram above, we have shown that both

$$\ker(\mu)/\text{im}(\xi_{n+l+1}(M)) \quad \text{and} \quad \text{im}(\xi_{n+l+1}(B\Gamma))/\text{im}(v)$$

are finite abelian  $p$ -groups. Hence  $\ker(\mu)/\text{im}(v)$  is a finite abelian  $p$ -group. Recalling that  $\ker(\mu) = \ker(\sigma)$ , this finishes the proof of assertion (3) of Theorem 8.1.  $\square$

PROOF OF ASSERTION (4) OF THEOREM 8.1. Because of assertion (1) it suffices to show that

$$\begin{aligned}
 (\mu \times \mathcal{S}_{n+l+1}^{\text{per},s}(f))[1/p]: \mathcal{S}_{n+l+1}^{\text{per},s}(M)[1/p] \\
 \rightarrow L_n(\mathbb{Z}[\mathbb{Z}_p^n])^{\mathbb{Z}/p}[1/p] \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma)[1/p]
 \end{aligned}$$



is surjective. We have already shown that  $\mu[1/p]$  is surjective and that its kernel agrees with the image of  $\nu[1/p]$ . We have already explained that  $\mathcal{S}_{n+l+1}^{\text{per},s}(f)[1/p] \cong \nu[1/p]$  is an isomorphism. Assertion (4) follows.  $\square$

PROOF OF ASSERTION (5). If we invert  $p$ , we conclude from assertion (4) using Theorem 7.1(1), Theorem 6.1, and Lemma 1.1(4),

$$\mathcal{S}_{n+l+1}^{\text{per},s}(M)[1/p] \cong \mathbb{Z}p^{k(p-1)/2}[1/p] \oplus L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}[1/p].$$

The abelian groups  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  and  $\mathbb{Z}p^{k(p-1)/2} \oplus L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}$  are finitely generated. The abelian group  $\mathbb{Z}p^{k(p-1)/2} \oplus L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}$  contains no  $p$ -torsion. We conclude from assertion (1) that the abelian group  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  contains no  $p$ -torsion. Hence

$$\mathcal{S}_{n+l+1}^{\text{per},s}(M) \cong \mathbb{Z}p^{k(p-1)/2} \oplus L_n(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p}.$$

It remains to prove

$$L_{n+l}(\mathbb{Z}[\mathbb{Z}_\rho^n])^{\mathbb{Z}/p} \cong \bigoplus_{i=0}^n L_{n-i}(\mathbb{Z})^{r_i}.$$

This follows from Remark 8.2. This finishes the proof of Theorem 8.1.  $\square$

### 9 Geometric Simple Structure Set of $M$

In this section we compute the geometric simple structure set  $\mathcal{S}^{\text{geo},s}(M)$  of  $M$ . For the rest of this paper, we simplify our notation and write  $\mathbf{L}$  for the spectrum  $\mathbf{L}(\mathbb{Z})$  and  $\mathbf{L}\langle 1 \rangle$  for its 1-connective cover.

In general, we have the following relationship between the periodic and the geometric simple structure set.

LEMMA 9.1. *Let  $N$  be a connected, oriented, closed  $m$ -dimensional manifold. Then we obtain an exact sequence*

$$0 \rightarrow \mathcal{S}^{\text{geo},s}(N) \xrightarrow{j(N)} \mathcal{S}_{m+1}^{\text{per},s}(N) \xrightarrow{\partial(N)} H_m(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle)$$

where the map  $j(N)$  is taken from (5.5), the map  $\partial(N)$  factors as

$$\mathcal{S}_{m+1}^{\text{per},s}(N) \xrightarrow{\eta_{m+1}(N)} H_m(N; \mathbf{L}) \rightarrow H_m(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle),$$

and we have  $H_m(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle) \cong H_m(N; L_0(\mathbb{Z})) \cong \mathbb{Z}$ .

PROOF. We have the following commutative diagram with exact columns:

$$\begin{array}{ccc}
 H_{m+1}(N; \mathbf{L}\langle 1 \rangle) & \longrightarrow & H_{m+1}(N; \mathbf{L}) \\
 \downarrow & & \downarrow \\
 L_{m+1}^s(\mathbb{Z}\Gamma) & \xrightarrow{\text{id}} & L_{m+1}^s(\mathbb{Z}\Gamma) \\
 \downarrow & & \downarrow \\
 \mathcal{S}_{m+1}^{\text{geo},s}(N) & \xrightarrow{j(N)} & \mathcal{S}_{m+1}^{\text{per},s}(N) \\
 \downarrow & & \downarrow \\
 H_m(N; \mathbf{L}\langle 1 \rangle) & \longrightarrow & H_m(N; \mathbf{L}) \\
 \downarrow & & \downarrow \\
 L_m^s(\mathbb{Z}\Gamma) & \xrightarrow{\text{id}} & L_m^s(\mathbb{Z}\Gamma),
 \end{array}$$

where the columns are the exact sequences (5.2) and (5.3) using the identification (5.4) and the horizontal maps are given by the passage  $\mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$ . Let  $\mathbf{L}/\mathbf{L}\langle 1 \rangle$  be the homotopy cofiber of the canonical map  $\mathbf{i}: \mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$  and denote by  $\mathbf{pr}: \mathbf{L} \rightarrow \mathbf{L}/\mathbf{L}\langle 1 \rangle$  the canonical map of spectra. We get an exact sequence

$$\begin{aligned}
 H_{m+1}(N; \mathbf{L}\langle 1 \rangle) &\rightarrow H_{m+1}(N; \mathbf{L}) \rightarrow H_{m+1}(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle) \\
 &\rightarrow H_m(N; \mathbf{L}\langle 1 \rangle) \rightarrow H_m(N; \mathbf{L}) \rightarrow H_m(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle).
 \end{aligned}$$

Since  $\pi_q(\mathbf{L}/\mathbf{L}\langle 1 \rangle)$  vanishes for  $q \geq 1$  and is  $L_0(\mathbb{Z})$  for  $q = 0$ , an easy spectral sequence argument shows that  $H_{m+1}(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle) = 0$ . Thus the top horizontal map is surjective. The fourth horizontal map is injective and its cokernel maps injectively to  $H_m(N; \mathbf{L}/\mathbf{L}\langle 1 \rangle) \cong H_m(N; L_0(\mathbb{Z})) \cong \mathbb{Z}$ . A diagram chase yields the desired exact sequence.  $\square$

Recall that  $M = \mathbb{T}_\rho^n \times_{\mathbb{Z}/p} \mathbb{S}^l$  and  $\Gamma = \mathbb{Z}^n \times_\rho \mathbb{Z}/p$  is  $\pi_1(M)$ . Let  $f: M \rightarrow B\Gamma$  be a classifying map for the universal covering of  $M$ .

**THEOREM 9.2** (Geometric simple structure set of  $M$ ). *There is a homomorphism*

$$\sigma^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p},$$

such that the following holds:

- (1) The map

$$\begin{aligned}
 \sigma^{\text{geo}} \times (\mathcal{S}_{n+l+1}^{\text{per},s}(f) \circ j(M)): \\
 \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma)
 \end{aligned}$$

is injective.

- (2) The cokernel of  $\sigma^{\text{geo}}$  is a finite abelian  $p$ -group.

(3) Consider the composite

$$\nu^{\text{geo}}: \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) \rightarrow \tilde{L}_{n+l+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\tilde{\xi}_{n+l+1}^{(1)}(M)} \mathcal{S}^{\text{geo},s}(M)$$

where the first map is given by induction with the various inclusions  $P \rightarrow \Gamma$  and  $\tilde{\xi}_{n+l+1}^{(1)}(M)$  comes from (6.2).

Then  $\nu^{\text{geo}}$  is injective, the image of  $\nu^{\text{geo}}$  is contained in the kernel of  $\sigma^{\text{geo}}$  and  $\ker(\sigma^{\text{geo}}) / \text{im}(\nu^{\text{geo}})$  is a finite abelian  $p$ -group.

(4) After inverting  $p$  we obtain an isomorphism

$$\begin{aligned} (\sigma^{\text{geo}} \times (\mathcal{S}_{n+l+1}^{\text{per},s}(f) \circ j(M))) [1/p]: \mathcal{S}^{\text{geo},s}(M)[1/p] \\ \rightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} [1/p] \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma)[1/p]. \end{aligned}$$

(5) As an abelian group we have

$$\mathcal{S}^{\text{geo},s}(M) \cong \mathbb{Z} p^{k(p-1)/2} \oplus \bigoplus_{i=0}^{n-1} L_{n-i}(\mathbb{Z})^{r_i},$$

where the numbers  $r_i$  are defined in (4.6).

(6) The cokernel of the map  $\partial(M): \mathcal{S}_{d+l+1}^{\text{per},s}(M) \xrightarrow{\eta_{d+l+1}(M)} H_{d+l}(M; \mathbf{L}/\mathbf{L}\langle 1 \rangle)$  appearing in Lemma 9.1 is a finite cyclic  $p$ -group.

**Remark 9.3.** There are several different points of views on the codomain of  $\sigma^{\text{geo}}$  (see Remark 8.2). Indeed there are isomorphisms

$$H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \cong \bigoplus_{i=0}^{n-1} (H_i(\mathbb{T}^n)^{\mathbb{Z}/p} \otimes L_{n-i}(\mathbb{Z})) \cong \bigoplus_{i=0}^{n-1} L_{n-i}(\mathbb{Z})^{r_i}.$$

**PROOF.** We first prove (4). We construct a commutative diagram whose columns are exact after inverting  $p$ .

$$(9.1) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{S}^{\text{geo},s}(M) & \xrightarrow{\sigma^{\text{geo}} \times (\mathcal{S}_{n+l+1}^{\text{per},s}(f) \circ j(M))} & H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma) \\ \downarrow & & \downarrow \\ \mathcal{S}_{n+l+1}^{\text{per},s}(M) & \xrightarrow[\cong_{1/p}]{\sigma \times \mathcal{S}_{n+l+1}^{\text{per},s}(f)} & H_n(\mathbb{T}_\rho^n; \mathbf{L})^{\mathbb{Z}/p} \times \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma) \\ \downarrow & & \downarrow \\ H_{n+l}(M; \mathbf{L}/\mathbf{L}\langle 1 \rangle) & \xrightarrow[\cong_{1/p}]{\alpha} & H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p}. \end{array}$$

The exact left column is due to Lemma 9.1. In the right column, the first nontrivial map is induced by the product of the change of coefficients map  $\mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$  with

the identity on the structure group and the second nontrivial map is induced by the composite of projection on the torus factor and the change of coefficients map  $\mathbf{L} \rightarrow \mathbf{L}/\mathbf{L}\langle 1 \rangle$ . The right column is exact after inverting  $p$ , since  $H_{n+1}(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle) = 0$ ; thus we have the exact sequence of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules

$$0 \rightarrow H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle) \rightarrow H_n(\mathbb{T}_\rho^n; \mathbf{L}) \rightarrow H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle).$$

Recall that all differentials of the spectral sequence

$$E_{i,j}^2 = H_i^{\mathbb{Z}/p}(\mathbb{S}^l; H_j(\mathbb{T}_\rho^n; \mathbf{L})) \implies H_{i+j}(M; \mathbf{L})$$

vanish by Lemma 8.3(2). This implies that for the spectral sequence

$$E_{i,j}^2 = H_i^{\mathbb{Z}/p}(\mathbb{S}^l; H_j(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)) \implies H_{i+j}(M; \mathbf{L}\langle 1 \rangle)$$

all differentials which end or start at place  $(s, t)$  vanish, provided that  $s + t = n + l$ . Hence we can define analogously to  $\sigma$  a map

$$(9.2) \quad \sigma^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}^{\text{geo},s}(M) & \xrightarrow{\sigma^{\text{geo}}} & H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \\ j(M) \downarrow & & \downarrow H_n(\mathbb{T}_\rho^n; \mathbf{i})^{\mathbb{Z}/p} \\ \mathcal{S}_{n+l+1}^{\text{per},s}(M) & \xrightarrow{\sigma} & H_n(\mathbb{T}_\rho^n; \mathbf{L})^{\mathbb{Z}/p}. \end{array}$$

The homomorphism  $\alpha$  in diagram (9.1) is given by the edge isomorphism

$$H_l^{\mathbb{Z}/p}(\mathbb{S}^l; H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)) \xrightarrow{\cong} H_{n+l}(M; \mathbf{L}/\mathbf{L}\langle 1 \rangle)$$

at  $i = l$  and  $j = n$  of the spectral sequence

$$E_{i,j}^2 = H_i^{\mathbb{Z}/p}(\mathbb{S}^l; H_j(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)) \implies H_{i+j}(M; \mathbf{L}/\mathbf{L}\langle 1 \rangle),$$

the canonical map

$$H_l^{\mathbb{Z}/p}(\mathbb{S}^l; H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)) \rightarrow H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p},$$

which is an isomorphism after inverting  $p$  since  $l$  is odd, the isomorphism of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules

$$H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle) \xrightarrow{\cong} H_n(\mathbb{T}_\rho^n; L_0(\mathbb{Z})),$$

which is the obvious edge homomorphism in the spectral sequence

$$E_{i,j}^2 = H_i(\mathbb{T}^n; \pi_j(\mathbf{L}/\mathbf{L}\langle 1 \rangle)) \implies H_{i+j}(\mathbb{T}^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle),$$

and the bijectivity of the inclusion

$$H_n(\mathbb{T}_\rho^n; L_0(\mathbb{Z}))^{\mathbb{Z}/p} \xrightarrow{\cong} H_n(\mathbb{T}_\rho^n; L_0(\mathbb{Z})).$$

The second horizontal arrow  $\sigma \times \mathcal{S}_{n+l+1}^{\text{per},s}(f)$  in diagram (9.1) is an isomorphism after inverting  $p$  by Theorem 8.1(4). We leave it to the reader to check that the diagram (9.1) commutes. By the Five Lemma the upper horizontal arrow

$$\sigma^{\text{geo}} \times (\mathcal{S}_{n+l+1}^{\text{per},s}(f) \circ j(M)): \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \times \mathcal{S}_{n+l+1}^{\text{per},s}(M)$$

is an isomorphism after inverting  $p$ . This finishes the proof of assertion (4).

(1) This follows from assertion (4) since  $\mathcal{S}^{\text{geo},s}(M)$  is a subgroup of  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  and  $\mathcal{S}_{n+l+1}^{\text{per},s}(M)$  contains no  $p$ -torsion by Theorem 8.1(5).

(2) This follows from assertion (4) since  $H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p}$  is a finitely generated abelian group.

(3) We get from the diagram (9.1) the following commutative diagram with exact columns:

$$\begin{array}{ccccc}
 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) & \xrightarrow{\nu^{\text{geo}}} & \mathcal{S}^{\text{geo},s}(M) & \xrightarrow{\sigma^{\text{geo}}} & H_n(\mathbb{T}_\rho^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \\
 \downarrow \text{id} & & \downarrow j(M) & & \downarrow H_n(\mathbb{T}_\rho^n; \mathbf{i})^{\mathbb{Z}/p} \\
 \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) & \xrightarrow{\nu} & \mathcal{S}_{n+l+1}^{\text{per},s}(M) & \xrightarrow{\sigma} & H_n(\mathbb{T}_\rho^n; \mathbf{L})^{\mathbb{Z}/p} \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H_{n+l}(M; \mathbf{L}/\mathbf{L}\langle 1 \rangle) & \xrightarrow{\alpha} & H_n(\mathbb{T}_\rho^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p}.
 \end{array}$$

We have already shown that the map  $\alpha$  is an isomorphism after inverting  $p$ , and its source is an infinite cyclic group. Hence  $\alpha$  is injective. Theorem 8.1(3) shows that  $\nu$  is injective, the kernel of  $\sigma$  contains  $\text{im}(\nu)$ , and  $\ker(\sigma)/\text{im}(\nu)$  is a finite abelian  $p$ -group. An easy diagram chase shows that  $\sigma^{\text{geo}}$  is injective, the image of  $\sigma^{\text{geo}}$  is contained in the kernel of  $\sigma^{\text{geo}}$ , and there is obvious isomorphism

$$\ker(\sigma^{\text{geo}})/\text{im}(\nu^{\text{geo}}) \xrightarrow{\cong} \ker(\sigma)/\text{im}(\nu).$$

(5) This follows from Theorem 8.1(5) and the diagram (9.1) by inspecting the definition of the various maps and the identification  $H_{n+l}(M; L_0(\mathbb{Z})) \cong L_0(\mathbb{Z})$ .

(6) This follows from the diagram (9.1) after we have shown that the the map  $H_d(\mathbb{T}^n; \mathbf{L}) \rightarrow H_d(\mathbb{T}^n; \mathbf{L}/\mathbf{L}\langle 1 \rangle)$  is surjective. The latter claim follows from the fact  $\mathbb{T}^n$  is stably homotopy equivalent to a wedge of spheres. This finishes the proof of Theorem 9.2.  $\square$

### 10 Invariants for Detecting the Structure Set of $M$

Let  $G$  be a finite group; let  $R(G)$  be its complex representation ring. Let  $\tilde{R}(G) = R(G)/\langle \text{reg} \rangle$  be the reduced regular representation ring. (Additively  $R(G)$  and  $\tilde{R}(G)$  are  $K_0(\mathbb{C}G)$  and  $\tilde{K}_0(\mathbb{C}G)$ , respectively.) For  $\varepsilon = \pm 1$ , let

$R(G)^\varepsilon$  and  $\tilde{R}(G)^\varepsilon$  be the subgroups invariant under  $V \mapsto \varepsilon\bar{V}$ . For example,  $\tilde{R}(\mathbb{Z}/p) \cong \mathbb{Z}[e^{2\pi i/p}]$  and both  $\tilde{R}(\mathbb{Z}/p)^{+1}$  and  $\tilde{R}(\mathbb{Z}/p)^{-1}$  are free abelian of rank  $(p - 1)/2$  for  $p$  an odd prime.

Let  $V$  be a finitely generated  $\mathbb{R}G$ -module and let

$$B: V \times V \rightarrow \mathbb{R}$$

be a  $G$ -equivariant  $\varepsilon$ -symmetric form. Atiyah-Singer [2, sec. 6] define the  $G$ -signature  $\text{sign}_G(V, B) \in R(G)^\varepsilon$ . If  $W$  is a compact  $2d$ -dimensional oriented manifold with a  $G$ -action, define its  $G$ -signature  $\text{sign}_G(W)$  to be the  $G$ -signature of its intersection form.

This defines the *multisignature* maps

$$(10.1) \quad \begin{aligned} \text{sign}_G: L_{2d}^s(\mathbb{Z}G) &\rightarrow R(G)^{(-1)^d}, \\ \widetilde{\text{sign}}_G: \tilde{L}_{2d}^s(\mathbb{Z}G) &\rightarrow \tilde{R}(G)^{(-1)^d}. \end{aligned}$$

LEMMA 10.1. *The multisignature homomorphisms  $\text{sign}_G$  and  $\widetilde{\text{sign}}_G$  are  $\mathbb{Z}[1/2]$ -isomorphisms.*

PROOF. See Wall [36, theorems 13A.3 and 13A.4] or Ranicki [32, props. 22.14 and 22.34]. □

When  $G$  is cyclic and odd order, according to [36, theorem 13A.4],  $\widetilde{\text{sign}}_G: \tilde{L}_{2d}^s(\mathbb{Z}G) \rightarrow 4\tilde{R}(G)^{(-1)^d}$  is an isomorphism.

THEOREM 10.2 (Geometric structure set of  $M$ ). *Let  $d = (n + l + 1)/2$ . There are injective maps*

$$\sigma \times \rho: \mathcal{S}_{n+l+1}^{\text{per},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L})^{\mathbb{Z}/p} \oplus \left( \bigoplus_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p] \right),$$

and

$$\sigma^{\text{geo}} \times \rho^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}(1))^{\mathbb{Z}/p} \oplus \left( \bigoplus_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p] \right).$$

*The cokernels of these maps are trivial after tensoring with  $\mathbb{Z}[1/2p]$ .*

PROOF. We have already defined  $\sigma$  and  $\sigma^{\text{geo}}$  in Theorems 8.1 and 9.2 respectively.

Let  $\hat{\rho}$  be the composite

$$(10.2) \quad \hat{\rho}: \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma) \xrightarrow{\left( \prod_{(P) \in \mathcal{P}} \text{res}_{\Gamma_{\text{ab}}}^{P'} \right) \circ \mathcal{S}_{n+l+1}^{\text{per},s}(B\text{pr})} \prod_{(P) \in \mathcal{P}} \mathcal{S}_{n+l+1}^{\text{per},s}(BP')$$

$$\xleftarrow{\cong} \prod_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P')[1/p] \xrightarrow{\prod_{(P) \in \mathcal{P}} \widetilde{\text{sign}}_{P'}} \prod_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p],$$

where for  $(P) \in \mathcal{P}$  the subgroup  $P' \subset \Gamma_{\text{ab}}$  is the image of  $P$  under the projection  $\text{pr}: \Gamma \rightarrow \Gamma_{\text{ab}}$ , the first map is the isomorphism taken from Theorem 7.1(2), the second map is given by the product of the inverse of the isomorphism appearing in Theorem 6.1(2), and the third map is the product of the homomorphisms defined in (10.1). Define  $\rho$  to be the composite

$$\rho: \mathcal{S}_{n+l+1}^{\text{per},s}(M) \xrightarrow{\mathcal{S}_{n+l+1}^{\text{per},s}(f)} \mathcal{S}_{n+l+1}^{\text{per},s}(B\Gamma) \xrightarrow{\hat{\rho}} \prod_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p].$$

Note that all these maps are isomorphisms after tensoring with  $\mathbb{Z}[1/2p]$ .

We thus see that  $\hat{\rho}[1/2]$  is an isomorphism and, thus, since the domain of  $\hat{\rho}$  is torsionfree by Theorem 7.1(1),  $\hat{\rho}$  is injective.

The map  $\sigma \times \mathcal{S}_{n+l+1}^{\text{per},s}(f)$  is injective and an isomorphism after tensoring with  $\mathbb{Z}[1/p]$  by Theorem 8.1, so it follows that  $\sigma \times \rho = (\text{id} \times \hat{\rho}) \circ (\sigma \times \mathcal{S}_{n+l+1}^{\text{per},s}(f))$  is injective and is an isomorphism after tensoring with  $\mathbb{Z}[1/2p]$ .

Define  $\rho^{\text{geo}} = \rho \circ j(M)$  and note that  $j(M)$  is injective by Lemma 9.1:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{S}^{\text{geo},s}(M) & \xrightarrow{\sigma^{\text{geo}} \times \rho^{\text{geo}}} & \left( \bigoplus_{i=0}^{n-1} L_{n-i}(\mathbb{Z}) \binom{n}{i} \right)^{\mathbb{Z}/p} \oplus \left( \bigoplus_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p] \right) \\ \downarrow j(M) & & \downarrow \text{inc} \times \text{id} \\ \mathcal{S}_{n+l+1}^{\text{per},s}(M) & \xrightarrow{\sigma \times \rho} & \left( \bigoplus_{i=0}^n L_{n-i}(\mathbb{Z}) \binom{n}{i} \right)^{\mathbb{Z}/p} \oplus \left( \bigoplus_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p] \right) \\ \downarrow & & \downarrow \\ \text{cok}(j(M)) & \xrightarrow{\overline{\sigma \times \rho}} & \text{cok}(\text{inc} \times \text{id}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

The map  $\overline{\sigma \times \rho}$  is the map induced on cokernels by the commutative square above it. The columns are short exact sequences so the snake lemma applies and yields the following exact sequence:

$$0 \rightarrow \ker(\sigma^{\text{geo}} \times \rho^{\text{geo}}) \rightarrow \ker(\sigma \times \rho) \rightarrow \ker(\overline{\sigma \times \rho}) \rightarrow \text{cok}(\sigma^{\text{geo}} \times \rho^{\text{geo}}) \rightarrow \text{cok}(\sigma \times \rho) \rightarrow \text{cok}(\overline{\sigma \times \rho}) \rightarrow 0.$$

Since we have already shown that  $\ker(\sigma \times \rho) = 0$  it follows that  $\sigma^{\text{geo}} \times \rho^{\text{geo}}$  is injective. Since we have already shown that  $(\sigma \times \rho)[1/p]$  is an isomorphism, it follows that both  $\text{cok}(\sigma \times \rho)[1/p]$  and  $\text{cok}(\overline{\sigma \times \rho})[1/p]$  vanish. Note that  $\text{cok}(j(M))$  is an infinite cyclic group by Lemma 9.1 and Theorem 9.2 (6). Since the domain and codomain of  $\overline{\sigma \times \rho}$  are infinite cyclic and the cokernel is  $p$ -torsion, its kernel

is trivial. Hence there is a short exact sequence

$$0 \rightarrow \text{cok}(\sigma^{\text{geo}} \times \rho^{\text{geo}}) \rightarrow \text{cok}(\sigma \times \rho) \rightarrow \text{cok}(\overline{\sigma \times \rho}) \rightarrow 0$$

where the middle term is  $p$ -torsion. Hence all the groups are  $p$ -torsion and  $(\sigma^{\text{geo}} \times \rho^{\text{geo}})[1/p]$  is an isomorphism as desired.  $\square$

We next interpret the detecting maps from Theorem 10.2 in terms of geometric invariants. The first map  $\sigma^{\text{geo}}$  will be given by *splitting invariants*, coming from surgery obstructions along submanifolds and the second map  $\rho^{\text{geo}}$  will be given by  $\rho$ -*invariants* that arise both in index theory for manifolds with boundary as well as in the homeomorphism classification of homotopy lens spaces with odd order fundamental group.

### 10.1 Splitting Invariants

Let  $X$  be a closed oriented manifold with dimension  $d \geq 5$ . Recall that in Section 5 we mentioned an identification of the geometric surgery exact sequence with the 1-connective algebraic surgery exact sequence. To discuss splitting invariants we say precisely what this identification is. In particular, we discuss the bijection  $\mathcal{N}(X) \rightarrow H_d(X; \mathbf{L}\langle 1 \rangle)$ .

Recall that a degree one normal map  $(f, \tilde{f})$  is a degree one map  $f : N^d \rightarrow X^d$  and a trivialization  $\tilde{f} : TN \oplus f^*\xi \cong \mathbb{R}^l$  for stable some bundle  $\xi$  over  $X$ . Then  $\mathcal{N}(X)$  is the set of normal bordism classes of degree one normal maps to  $X$ . The map  $\eta : \mathcal{S}^{\text{geo},s}(X) \rightarrow \mathcal{N}(X)$  is given by considering a simple homotopy equivalence  $h : X' \rightarrow X$  as a degree one map and taking  $\xi = h^{-1*} \nu_{X'}$ , where  $h^{-1}$  is a homotopy inverse of  $h$  and  $\nu_{X'}$  is the normal bundle of  $X'$  with respect to some embedding in Euclidean space. The map  $\sigma : \mathcal{N}(X) \rightarrow L_d^s(\mathbb{Z}[\pi_1(X)])$  is given by the surgery obstruction. In particular, a normal bordism class maps to zero if and only if it has a representative given by a simple homotopy equivalence.

A Pontryagin-Thom construction gives a bijection

$$PT : \mathcal{N}(X) \xrightarrow{\cong} [X, G/TOP].$$

Sullivan/Quinn/Ranicki give a 4-fold periodicity,  $\Omega^4(\mathbb{Z} \times G/TOP) \simeq \mathbb{Z} \times G/TOP$ . The corresponding spectrum is homotopy equivalent to  $\mathbf{L}(\mathbb{Z})$  (which we have been abbreviating  $\mathbf{L}$ ). It follows formally that the 0-space of the 1-connective cover  $\mathbf{L}\langle 1 \rangle$  is homotopy equivalent to  $G/TOP$  and that  $[X, G/TOP] = H^0(X; \mathbf{L}\langle 1 \rangle)$ . Furthermore, Ranicki shows that closed, oriented, topological manifolds are oriented with respect to these two spectra; hence there are Poincaré duality isomorphisms

$$PD : H^i(X; \mathbf{L}) \xrightarrow{\cong} H_{d-i}(X; \mathbf{L}) \quad \text{and} \quad PD : H^i(X; \mathbf{L}\langle 1 \rangle) \xrightarrow{\cong} H_{d-i}(X; \mathbf{L}\langle 1 \rangle).$$

The following foundational theorem is due to Quinn and Ranicki; references are [32, theorem 18.5] and [20, prop. 14.8].



**THEOREM 10.3.** *There is a commutative diagram, with vertical bijections,*

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & L_{d+1}^s(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{\partial} & \mathcal{S}^{\text{geo},s}(X) & \xrightarrow{\eta} & \mathcal{N}(X) & \xrightarrow{\sigma} & L_d^s(\mathbb{Z}[\pi_1(X)]) \\
 & & \parallel & & \downarrow s \cong & & \downarrow t \cong & & \parallel \\
 \cdots & \longrightarrow & L_{d+1}^s(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{\xi_{d+1}^{(1)}} & \mathcal{S}_{d+1}^{(1),s}(X) & \longrightarrow & H_d(X; \mathbf{L}\langle 1 \rangle) & \xrightarrow{A_d^{(1)}(X)} & L_d^s(\mathbb{Z}[\pi_1(X)]),
 \end{array}$$

where  $t$  is the composite

$$\mathcal{N}(X) \xrightarrow{PT} [X, G/TOP] = H^0(X; \mathbf{L}\langle 1 \rangle) \xrightarrow{PD} H_d(X; \mathbf{L}\langle 1 \rangle).$$

The philosophy of splitting invariants is to detect  $\mathcal{N}(X)$  by submanifolds. Let  $Y^j$  be a closed, oriented submanifold of  $X^d$ . (We assume our submanifolds have a normal bundle so we can apply transversality.) The restriction map

$$\text{res}_X^Y: \mathcal{N}(X^d) \rightarrow \mathcal{N}(Y^j)$$

is defined by representing a normal bordism class by a normal map  $(f, \tilde{f})$  where  $f: N \rightarrow X$  is transverse to  $Y$ . Then  $\text{res}[f, \tilde{f}]$  is represented by the restricted map  $f|: f^{-1}Y \rightarrow Y$  with the trivialization  $Tf^{-1}Y \oplus f|^*(\nu(Y \hookrightarrow X) \oplus \xi|_Y) = TN|_{f^{-1}Y} \oplus f|^*\xi \cong \mathbb{R}^l$ .

It follows from the Pontragin-Thom construction that the map  $\text{res}_X^Y$  is the composite

$$\mathcal{N}(X^d) \xrightarrow{PT} [X, G/TOP] \xrightarrow{[-, G/TOP]} [Y, G/TOP] \xrightarrow{PT^{-1}} \mathcal{N}(Y),$$

but we state this result slightly differently:

**LEMMA 10.4.**  *$\text{res}_X^Y$  is the composite*

$$\begin{aligned}
 \mathcal{N}(X^d) &\xrightarrow{PT} [X, G/TOP] = H^0(X; \mathbf{L}\langle 1 \rangle) \\
 &\xrightarrow{\text{inc}^*} H^0(Y; \mathbf{L}\langle 1 \rangle) = [Y, G/TOP] \xrightarrow{PT^{-1}} \mathcal{N}(Y^j).
 \end{aligned}$$

**DEFINITION 10.5** (Sullivan [33, 34]). A *characteristic variety* of  $X^d$  is a collection of closed, oriented, connected, submanifolds  $\{Y_i^j\}$  of  $X^d$  so that any simple homotopy equivalence  $h: X' \rightarrow X$  that vanishes under the composite

$$\mathcal{N}(X^d) \xrightarrow{\text{res}} \prod_{i,j} \mathcal{N}(Y_i^j) \xrightarrow{\sigma} \prod_{i,j} L_j(\mathbb{Z}[\pi_1 Y_i^j]) \xrightarrow{\varepsilon} \prod_{i,j} L_j(\mathbb{Z})$$

is homotopic to a homeomorphism.

Sullivan’s original example is that a characteristic variety for  $\mathbb{C}P^n$  is given by  $\{\mathbb{C}P^j\}$  with  $0 < j < n$ . If a manifold satisfies topological rigidity [19] (for example a sphere or a torus), then the empty set is a characteristic variety.

**THEOREM 10.6.** *Choose a point  $\bullet \in \mathbb{S}^l$  in the sphere.*

(1) *The following map is an isomorphism*

$$\text{inc}^* \times (\text{pr}_{\mathbb{T}^n} \circ PD):$$

$$H^0(\mathbb{T}^n \times \mathbb{S}^l; \mathbf{L}\langle 1 \rangle) \rightarrow H^0(\mathbb{T}^n \times \bullet; \mathbf{L}\langle 1 \rangle) \oplus H_{n+l}(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle).$$

(2) *For a subset  $J \subset \{1, 2, \dots, n\}$ , let  $T^J \times \bullet \subset \mathbb{T}^n \times \mathbb{S}^l$  be the obvious  $|J|$ -dimensional submanifold. Then  $\{T^J \times \bullet \mid \emptyset \neq J \subset \{1, 2, \dots, n\}\}$  is a characteristic variety for  $\mathbb{T}^n \times \mathbb{S}^l$ .*

(3) *Let  $\sigma^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p}$  be the map defined in Theorem 9.2. Then  $\sigma^{\text{geo}}(h : N \rightarrow M) = 0$  if and only if the  $p$ -fold cover  $\bar{h}: \bar{N} \rightarrow \mathbb{T}^n \times \mathbb{S}^l$  is homotopic to a homeomorphism.*

PROOF.

(1) It is an exercise to show the analogue in ordinary homology

$$\text{inc}^* \times (\text{pr}_{\mathbb{T}^n} \circ PD): H^*(\mathbb{T}^n \times \mathbb{S}^l) \xrightarrow{\cong} H^*(\mathbb{T}^n \times \bullet) \oplus H_{n+l-*}(\mathbb{T}^n).$$

The general case follows since the Atiyah-Hirzebruch spectral sequence collapses and the  $\mathbf{L}$ -spectrum Poincaré duality is compatible with classical Poincaré duality since the fundamental class  $[\mathbb{T}^n \times \mathbb{S}^l]_{\mathbf{L}\bullet} \in H_{n+l}(\mathbb{T}^n \times \mathbb{S}^l; \mathbf{L}\bullet)$  maps to the fundamental class  $[\mathbb{T}^n \times \mathbb{S}^l] \in H_{n+l}(\mathbb{T}^n \times \mathbb{S}^l) \subset \bigoplus H_{n+l-*}(\mathbb{T}^n \times \mathbb{S}^l; L^*(\mathbb{Z}))$ . (See section 16, “The  $L$ -theory orientation of topology” in [32].)

(2) Let  $h: X' \rightarrow \mathbb{T}^n \times \mathbb{S}^l$  be a simple homotopy equivalence. We first claim that  $h$  is homotopic to a homeomorphism if and only if

$$PT(\eta(h)) = 0 \in H^0(\mathbb{T}^n \times \mathbb{S}^l; \mathbf{L}\langle 1 \rangle).$$

Since  $H_{n+l+1}(\mathbb{T}^n \times \mathbb{S}^l; \mathbf{L}\langle 1 \rangle) \rightarrow H_{n+l+1}(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)$  is a (split) surjection, and the Farrell-Jones conjecture for  $\mathbb{Z}^n$ , which is due to Shaneson in this case, implies that  $A_{n+l+1}^{(1)}(\mathbb{T}^n): H_{n+l+1}(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle) \rightarrow L_{n+l+1}(\mathbb{Z}[\mathbb{Z}^n])$  is an isomorphism, the composite

$$A_{n+l+1}^{(1)}(\mathbb{T}^n) \circ H_{n+l+1}(\text{pr}_{\mathbb{T}^n}; \mathbf{L}\langle 1 \rangle) = A_{n+l+1}^{(1)}(\mathbb{T}^n \times \mathbb{S}^l)$$

is surjective. Hence the boundary map  $\partial$  in the surgery exact sequence (see Theorem 10.3) is the trivial map and thus  $\eta$  is injective. Our first claim follows.

Our second claim is that the map

$$\prod_J \varepsilon \circ \sigma \circ \text{res}_{\mathbb{T}^n}^{T^J} \circ (PT)^{-1} : H^0(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle) \rightarrow \bigoplus_J L_{|J|}(\mathbb{Z})$$

is an isomorphism where  $J$  runs over all nonempty subsets of  $\{1, 2, \dots, n\}$ . To see that this is an isomorphism, we exhibit its inverse map, which sends the generator  $E^{|J|} \rightarrow S^{|J|}$  of  $L_{|J|}(\mathbb{Z})$  to  $PT((T^J \# E^{|J|}) \times T^{J'} \rightarrow \mathbb{T}^n)$  where  $J'$  is the complement of  $J$ .

Our third claim is that for a simple homotopy equivalence  $h : X' \rightarrow \mathbb{T}^n \times \mathbb{S}^l$ ,

$$(10.3) \quad \text{pr}_{\mathbb{T}^n} \circ (PD(PT(\eta(h)))) = 0 \in H_{n+l}(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle).$$

Indeed,

$$\begin{aligned} 0 &= \sigma(\eta(h)) && \text{by exactness of the surgery} \\ & && \text{exact sequence} \\ &= A_{n+l}^{(1)}(\mathbb{T}^n \times \mathbb{S}^l)(PD(PT(\eta(h)))) && \text{by Theorem 10.3} \\ &= A_{n+l}^{(1)}(\mathbb{T}^n) \operatorname{pr}_{\mathbb{T}^n*}(PD(PT(\eta(h)))) && \text{by naturality of the assembly map.} \end{aligned}$$

Since  $A_{n+l}^{(1)}(\mathbb{T}^n)$  is an isomorphism, (10.3) follows.

Now back to the proof of (2). Let  $h: X' \rightarrow \mathbb{T}^n \times \mathbb{S}^l$  be a simple homotopy equivalence so that all splitting invariants along  $T^J \times \bullet$  vanish. Our goal is to show that  $h$  is homotopic to a homeomorphism. By our first claim, part (1), and our third claim, it suffices to show that

$$\operatorname{inc}^*(PT(\eta(h))) = 0 \in H^0(\mathbb{T}^n \times \bullet; \mathbf{L}\langle 1 \rangle).$$

By Lemma 10.4, this is equivalent to showing

$$PT(\operatorname{res}_{\mathbb{T}^n \times \mathbb{S}^l}^{\mathbb{T}^n \times \bullet}(\eta(h))) = 0 \in H^0(\mathbb{T}^n \times \bullet; \mathbf{L}\langle 1 \rangle)$$

By our second claim, this is equivalent to showing that for any nonempty subset  $J$  of  $\{1, 2, \dots, n\}$ ,

$$\varepsilon(\sigma(\operatorname{res}_{\mathbb{T}^n \times \mathbb{S}^l}^{T^J \times \bullet}(\eta(h)))) = 0 \in L_{|J|}(\mathbb{Z}).$$

It follows that  $\{T^J \times \bullet\}$  is a characteristic variety for  $\mathbb{T}^n \times \mathbb{S}^l$ .

(3) The composite  $\sigma^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p}$  with the inclusion  $H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \hookrightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)$  can be identified with the composite

$$\begin{aligned} \mathcal{S}^{\text{geo},s}(M) &\xrightarrow{\operatorname{tr}} \mathcal{S}^{\text{geo},s}(\mathbb{T}^n \times \mathbb{S}^l) \xrightarrow{\eta} \mathcal{N}(\mathbb{T}^n \times \mathbb{S}^l) \\ &\xrightarrow{\operatorname{res}} \mathcal{N}(\mathbb{T}^n \times \bullet) \xrightarrow{PT} H^0(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle) \xrightarrow{PD} H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle), \end{aligned}$$

where the transfer map  $\operatorname{tr}$  is passing to the  $p$ -fold cover. By Theorem 10.6(1) and Lemma 10.4, the result follows.  $\square$

**DEFINITION 10.7.** A map  $f: N \rightarrow X$  splits along a submanifold  $Y$  if  $f$  is homotopic to a map  $g$ , transverse to  $Y$  so that  $g^{-1}Y \rightarrow Y$  is a homotopy equivalence. The map  $f$   $s$ -splits along  $Y$ , if, in addition,  $g^{-1}Y \rightarrow Y$  is a simple homotopy equivalence.

Thus if  $\{Y_i^j\}$  is a characteristic variety for  $X$ , then a simple homotopy equivalence is homotopic to a homeomorphism if and only if it splits along  $\{Y_i^j\}$ .

We now mention the relationship between restriction and splitting invariants. The key result is due to Browder (see [37, sec. 4.3]).

**THEOREM 10.8.** Suppose  $h: X' \rightarrow X$  is a simple homotopy equivalence and that  $Y^j$  is a submanifold of  $X^d$  with codimension  $d - j \geq 3$  and dimension  $j \geq 5$ . Then  $h$  is splittable along  $Y$  if and only if  $\sigma(\operatorname{res}_X^Y(h)) = 0 \in L_j^s(\mathbb{Z}[\pi_1 Y])$ .

### 10.2 Rho Invariants

DEFINITION 10.9. Let  $N$  be a closed, oriented,  $(2d - 1)$ -dimensional manifold mapping to  $BG$  for a finite  $G$ . The  $\rho$ -invariant

$$\rho(N \rightarrow BG) \in \tilde{R}(G)^{(-1)^d} [1/|G|]$$

is defined by

$$\rho(N \rightarrow BG) = \frac{1}{k} \cdot \text{sign}_G(W) \in \tilde{R}(G)^{(-1)^d} [1/|G|],$$

where  $k$  is a power of  $|G|$  and  $W$  is a compact, oriented,  $2d$ -dimensional manifold with orientation-preserving free  $G$ -action, whose boundary is  $k$  disjoint copies of the  $\bar{N}$ , the induced  $G$ -cover of  $N$ . This was given an analytic interpretation by Atiyah-Patodi-Singer. The  $\rho'$ -function

$$\rho'(N \rightarrow BG): \mathcal{S}^{\text{geo},s}(N) \rightarrow R(G)^{(-1)^d} [1/|G|]$$

is defined by

$$\rho'(N \rightarrow BG)(N' \rightarrow N) = \rho(N' \rightarrow N \rightarrow BG) - \rho(N \rightarrow BG).$$

The  $\rho$ -invariant and the  $\rho'$ -function only depend on the induced homomorphism  $\pi_1 N \rightarrow G$ , or, equivalently, on the induced  $G$ -cover  $\bar{N} \rightarrow N$ . When  $\pi_1 N = G$ , we will write  $\rho(N)$  and  $\rho'$ .

The passage from  $R(G)^{(-1)^d} [1/|G|]$  to  $\tilde{R}(G)^{(-1)^d} [1/|G|]$  ensures that the  $\rho$ -invariant is independent of the choices of  $k$  and  $W$ . For the definition of the  $\rho$ -invariant, see [2, remark after cor. 7.5] or [36, sec. 13B].

Here are two fundamental properties of the  $\rho'$ -function.

THEOREM 10.10. *Let  $N \rightarrow BG$  be a map from a closed, oriented,  $(2d - 1)$ -dimensional manifold to the classifying space of a finite group. Let  $\phi: \pi_1 N \rightarrow G$  be the induced map on fundamental groups. Recall that there is an identification  $\mathcal{S}^{\text{geo},s}(N) = \mathcal{L}_{2d}^{(1),s}(N)$  (see (5.4)). In particular, the geometric structure set is an abelian group.*

- (1) *The map  $\rho'(N \rightarrow BG): \mathcal{S}^{\text{geo},s}(N) \rightarrow \tilde{R}(G)^{(-1)^d} [1/|G|]$  is a homomorphism of abelian groups.*
- (2) *Recall that  $L_{2d}^s(\mathbb{Z}[\pi_1 N])$  acts on  $\mathcal{S}^{\text{geo},s}(N)$  (see [36, theorem 10.4]). For  $x \in L_{2d}^s(\mathbb{Z}\pi_1 N)$  and  $y \in \mathcal{S}^{\text{geo},s}(N)$ ,*

$$\begin{aligned} &\rho'(N \rightarrow BG)(x + y) \\ &= \widetilde{\text{sign}}_G(\tilde{L}_{2d}^s(\phi)(x)) + \rho'(N \rightarrow BG)(y) \in \tilde{R}(G)^{(-1)^d} [1/|G|]. \end{aligned}$$

*In particular, by taking  $y = \text{id}_N$ , we have an equality of maps*

$$\rho'(N \rightarrow BG) \circ \partial = \widetilde{\text{sign}}_G \circ \tilde{L}_{2d}^s(\phi): \tilde{L}_{2d}^s(\mathbb{Z}\pi_1 N) \rightarrow \tilde{R}(G)^{(-1)^d} [1/|G|].$$

PROOF.

(1) This is the main result of the paper [10] by Crowley and Macko.

(2) The following is an easy consequence of the definition of the  $\rho$ -invariant: If  $W^{2d}$  is a compact oriented manifold with a map to  $BG$  and if  $(\partial W \rightarrow BG) = (N' \sqcup -N'' \rightarrow BG)$ , then  $\rho(N' \rightarrow BG) = \text{sign}_G(\overline{W}) + \rho(N'' \rightarrow BG)$ .

The action is implemented by such a  $W$  and the result follows. □

Let  $L^l$  be a homotopy lens space with fundamental group  $P \cong \mathbb{Z}/p$  for  $p$  an odd prime. Let  $d = (l + 1)/2$ . A homotopy lens space is the orbit space of a free action of  $\mathbb{Z}/p$  on  $\mathbb{S}^l$ ; equivalently, it is a closed manifold having the homotopy type of a lens space.

**THEOREM 10.11.**  $\rho' : \mathcal{S}^{\text{geo},s}(L^l) \rightarrow \tilde{R}(P)^{(-1)^d}[1/p] \cong \mathbb{Z}[1/p]^{(p-1)/2}$  is an injection, and is an isomorphism after tensoring with  $\mathbb{Z}[1/2p]$ .

**PROOF.** We first show that  $\mathcal{S}^{\text{geo},s}(L^l)$  is isomorphic to  $\mathbb{Z}^{(p-1)/2}$  and is in particular torsion-free. The Atiyah-Hirzebruch spectral sequence in equivariant homology (see [11, theorem 4.7]) shows that  $H_{l+1}^P(\mathbb{S}^l \rightarrow S^\infty; \mathbf{L}\langle 1 \rangle)$  is zero. The long exact sequence of the triple  $\mathbb{S}^l \rightarrow S^\infty \rightarrow \bullet$  then shows that  $\mathcal{S}_{l+1}^{(1),s}(L^l) \rightarrow \mathcal{S}_{l+1}^{(1),s}(BP)$  is injective. But the domain is  $\mathcal{S}^{\text{geo},s}(L^l)$ , which is finitely generated and has rank  $(p - 1)/2$  by the surgery exact sequence, and the codomain is isomorphic to  $\mathbb{Z}[1/p]^{(p-1)/2}$  by Theorem 6.1.

The multisignature map  $\text{sign}_G : \tilde{L}_{l+1}^s(\mathbb{Z}P) \rightarrow \tilde{R}(P)^{(-1)^d}[1/p]$  is a  $\mathbb{Z}[1/2p]$ -isomorphism by Lemma 10.1. We conclude from Theorem 10.10(2) that the composite  $\tilde{L}_{l+1}^s(\mathbb{Z}P) \xrightarrow{\partial} \mathcal{S}^{\text{geo},s}(L^l) \xrightarrow{\rho'} \tilde{R}(P)^{(-1)^d}[1/p]$  is a  $\mathbb{Z}[1/2p]$ -isomorphism. Since  $\mathcal{S}^{\text{geo},s}(L^l)$  is torsion-free of rank  $(p - 1)/2$ , the result follows. □

*Remark 10.12.* This gives a new proof of Wall’s result [36, chap. 14E] that the structure set of a homotopy lens space is detected by the  $\rho'$  invariant. He showed that if  $L^l$  is a homotopy lens space with fundamental group  $\mathbb{Z}/k$  for  $k$  odd, that  $\rho' : \mathcal{S}^{\text{geo},s}(L^l) \rightarrow \tilde{R}(\mathbb{Z}/k)^{(-1)^d}[1/k]$  is injective. This follows for  $k$  a prime power by our argument above, and, we believe, for all  $k$  odd by the techniques in our paper. In fact, Wall proved much more and identified the image of the map  $\rho'$ , which takes much more work.

Wall’s result that a structure set of lens spaces is detected by the  $\rho'$ -invariant and the fact that splitting invariants detect the structure set of  $\mathbb{T}^n \times \mathbb{S}^l$  provided a major impetus for our paper.

Recall that  $\mathcal{P}$  is the set of conjugacy classes of subgroups of order  $p$  of  $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$ . Recall  $|\mathcal{P}| = p^k$  where  $n = k(p - 1)$ . Let  $\text{pr} : \Gamma \rightarrow \Gamma_{\text{ab}} \cong (\mathbb{Z}/p)^{k+1}$  be the quotient map. For each  $(P) \in \mathcal{P}$ , let  $P' = \text{pr}(P)$ , let  $\Gamma_P = \text{pr}^{-1}(\text{pr}(P))$ , and note that  $\ker \text{pr} \subsetneq \Gamma_P \subsetneq \Gamma$ . Let  $M_P \rightarrow M$  be the cover corresponding to the

subgroup  $\Gamma_P$ . Consider the transfer map

$$\text{res}_\Gamma^{\Gamma_P}: \mathcal{S}^{\text{geo},s}(M) \rightarrow \mathcal{S}^{\text{geo},s}(M_P),$$

obtained by sending a simple homotopy equivalence  $N \rightarrow M$  to the covering simple homotopy equivalence  $N_P \rightarrow M_P$ .

**THEOREM 10.13 (Detection Theorem).** *An element  $h \in \mathcal{S}^{\text{geo},s}(M)$  is the trivial element if and only if  $\sigma^{\text{geo}}(h) = 0$  and  $\rho'(M_P \rightarrow BP')(\text{res}_\Gamma^{\Gamma_P}(h)) = 0$  holds for all  $(P) \in \mathcal{P}$ .*

**PROOF.** Obviously it suffices to show that the restriction of the homomorphism of abelian groups

$$\rho^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow \prod_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p]$$

appearing in Theorem 10.2 to the kernel of homomorphism  $\sigma^{\text{geo}}$  appearing in Theorem 10.2 sends  $h$  to  $(\rho'(M_P \rightarrow BP')(\text{res}_\Gamma^{\Gamma_P}(h)))_{(P) \in \mathcal{P}}$ . The image of the map

$$\nu^{\text{geo}}: \bigoplus_{(P) \in \mathcal{P}} \tilde{L}_{n+l+1}^s(\mathbb{Z}P) \rightarrow \tilde{L}_{n+l+1}^s(\mathbb{Z}\Gamma) \xrightarrow{\tilde{\xi}_{n+l+1}^{(1)}(M)} \mathcal{S}^{\text{geo},s}(M)$$

defined in Theorem 9.2(3) is contained in the kernel of  $\sigma^{\text{geo}}$  and has finite  $p$ -power index. The map  $\rho^{\text{geo}}$  is a homomorphism of abelian groups by Theorem 10.10 (1). Since the image of  $\rho^{\text{geo}}$  is a  $\mathbb{Z}[1/p]$ -module, we conclude that it suffices to show that the restriction of the homomorphism  $\rho^{\text{geo}}$  to the image of the map  $\nu^{\text{geo}}$  sends  $h$  to  $(\rho'(\text{res}_\Gamma^{\Gamma_P}(h)))_{(P) \in \mathcal{P}}$ . Since  $\text{im}(\nu^{\text{geo}})$  is contained in the image of  $\xi_{n+l+1}^{(1)}(M) = \partial$ , it suffices to show that restriction of the homomorphism  $\rho^{\text{geo}}$  to the image of  $\xi_{n+l+1}^{(1)}(M) = \partial$  sends  $h$  to  $(\rho'(M_P \rightarrow BP')(\text{res}_\Gamma^{\Gamma_P}(h)))_{(P) \in \mathcal{P}}$ . This follows from Theorem 10.10 (2).  $\square$

Now Theorem 0.2 follows directly from Theorems 10.2, 10.6, and 10.13.

**EXAMPLE 10.14.** *Take  $p = 3, k = 1, n = 2$ , and the  $\mathbb{Z}/3$ -action in  $\mathbb{Z}^2$  given by the  $(n_1, n_2) \mapsto (-n_2, n_1 - n_2)$ . Then  $\mathbb{Z}_\rho^2$  is  $\mathbb{Z}[\exp(2\pi/3)]$ , and we have*

$$\mathcal{S}^{\text{geo},s}(M) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2.$$

*There is one nontrivial splitting obstruction taking values in  $L_2(\mathbb{Z}) \cong \mathbb{Z}/2$ , defined by making the corresponding map transversal to  $T^2 \times \bullet \subset T^2 \times S^3$ .*

*There are three conjugacy classes of subgroups of order  $p$  in  $\Gamma$ , and each yields a  $\rho$ -invariant type obstruction.*

## 11 Appendix: Open Questions

As mentioned earlier, our inspiration, our muse, for this paper is Wall's classification [36, chap. 14E] of homotopy lens spaces  $L^l$  with odd order fundamental group. This classification is complete. It consists of the classification of homotopy types of homotopy lens spaces, the classification of the simple homotopy types within a homotopy type, the computation of the geometric structure set, and the computation of the moduli space  $\mathcal{M}(L^L)$  of homeomorphism classes of the closed manifolds within a simple homotopy type. In the case of homotopy lens spaces, it is a bit easier to state if one considers the polarized homotopy type, fixing an orientation and an identification of the fundamental group with  $\mathbb{Z}/k$ . Then the polarized homotopy type is given by the first  $k$ -invariant, lying in the group  $(\mathbb{Z}/k)^\times \subset \mathbb{Z}/k = H^{l+1}(B(\mathbb{Z}/k))$ . The simple homotopy types of polarized lens spaces are given by the Reidemeister torsion  $\Delta(L^l)$  and Wall determines the set of possible values which occur. Fixing a simple homotopy type he showed, as mentioned earlier, that

$$\rho': \mathcal{S}^{\text{geo},s}(L^l) \rightarrow \tilde{R}(\mathbb{Z}/k)^{(-1)^{(l+1)/2}}$$

is injective and he computed the image.

From this, it is not difficult to compute the moduli space. For any closed manifold  $X$ , let  $\text{sAut}(X)$  be the group of homotopy classes of simple self-homotopy equivalences. Then  $\text{sAut}(X)$  acts on  $\mathcal{S}^{\text{geo},s}(X)$  with orbit space the moduli space. In the case of a homotopy lens space, then  $\text{sAut}(X)$  can be computed directly since  $L^l$  is a skeleton of  $K(\mathbb{Z}/k, 1)$ , or better yet, one can compute the action of  $\text{sAut}(X)$  on the set of  $k$ -invariants, Reidemeister torsions, and  $\rho'$ -invariants. We omit the details.

The discussion above leads to several questions:

- (1) Can one describe the image of the injective map of Theorem 10.2

$$\sigma^{\text{geo}} \times \rho^{\text{geo}}: \mathcal{S}^{\text{geo},s}(M) \rightarrow H_n(\mathbb{T}^n; \mathbf{L}\langle 1 \rangle)^{\mathbb{Z}/p} \oplus \left( \bigoplus_{(P) \in \mathcal{P}} \tilde{R}(P')^{(-1)^d} [1/p] \right)?$$

A first step is to compute the  $\rho$ -invariant of  $M_P \rightarrow BP'$ .

- (2) In the proof of the detection theorem 10.13 we show that  $\rho^{\text{geo}}$  restricted to the kernel of  $\sigma^{\text{geo}}$  is given by differences of  $\rho$ -invariants. Is this also true for  $\rho^{\text{geo}}$  itself? In other words, does  $\rho^{\text{geo}} = \prod_{(P) \in \mathcal{P}} \rho'(M_P \rightarrow BP') \circ \text{res}_{\Gamma}^P$ ?
- (3) What can we say about the moduli space of homeomorphism classes of manifolds (simple) homotopy equivalent to  $M$ . Is it infinite? Can we compute action of  $\text{sAut}(M)$  on the structure set? How does this group act on the splitting and  $\rho$ -invariants? When is a self-homotopy equivalence of  $M$  homotopic to a homeomorphism? Can we classify all self-homotopy equivalences of  $M$ , perhaps up to finite ambiguity? Does the subgroup of self-homotopy equivalences of  $M$  that are homotopic to a homeomorphism have finite index in the group of self-homotopy equivalences of  $M$ ? Is a

self-homotopy equivalence of  $M$  determined by the induced map on the fundamental group up to finite ambiguity? Can we show that within the homotopy type of  $M$  there are infinitely many mutually different homeomorphism types?

- (4) Is a homotopy equivalence  $h: N \rightarrow M$  splittable along

$$p^k\{\bullet\} \times L^l = (\mathbb{T}^n)^{\mathbb{Z}/p} \times L^l \subset \mathbb{T}^n \times_{\mathbb{Z}/p} \mathbb{S}^l = M$$

if and only if  $h$  is a simple homotopy equivalence? (The if direction follows from Theorem 10.8 and equation (4.11).)

- (5) Is a simple homotopy equivalence  $h: N \rightarrow M = \mathbb{T}^n \times_{\mathbb{Z}/p} \mathbb{S}^l$  homotopic to a homeomorphism if and only if  $\bar{\mp}h: \bar{\mp}N \rightarrow \mathbb{T}^n \times \mathbb{S}^l$  is splittable along  $T^J \times \bullet$  for all nonempty  $J \subset \{1, 2, \dots, n\}$  and if  $h \simeq k$  where  $k|: k^{-1}((\mathbb{T}^n)^{\mathbb{Z}/p} \times L^l) \rightarrow (\mathbb{T}^n)^{\mathbb{Z}/k} \times L^l$  is a homeomorphism? (An alternate conjecture is that a simple homotopy equivalence  $h: N \rightarrow M$  is homotopic to a homeomorphism if and only if  $h \simeq k$  where  $k|: k^{-1}((\mathbb{T}^n)^{\mathbb{Z}/p} \times L^l) \rightarrow (\mathbb{T}^n)^{\mathbb{Z}/p} \times L^l$  and  $\bar{k}|: \bar{k}^{-1}(T^J \times \bullet) \rightarrow T^J \times \bullet$  are homeomorphisms for all  $J$ .)

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## Bibliography

- [1] Atiyah, M. F.; Segal, G. B. Equivariant  $K$ -theory and completion. *J. Differential Geometry* **3** (1969), 1–18.
- [2] Atiyah, M. F.; Singer, I. M. The index of elliptic operators. III. *Ann. of Math. (2)* **87** (1968), 546–604. doi:10.2307/1970717
- [3] Bak, A. The computation of surgery groups of odd torsion groups. *Bull. Amer. Math. Soc.* **80** (1974), no. 6, 1113–1116. doi:10.1090/S0002-9904-1974-13634-7
- [4] Bak, A. Odd dimension surgery groups of odd torsion groups vanish. *Topology* **14** (1975), no. 4, 367–374. doi:10.1016/0040-9383(75)90021-X
- [5] Bak, A. The computation of even dimension surgery groups of odd torsion groups. *Comm. Algebra* **6** (1978), no. 14, 1393–1458. doi:10.1080/00927877808822298
- [6] Bartels, A.; Echterhoff, S.; Lück, W. Inheritance of isomorphism conjectures under colimits. *K-Theory and noncommutative geometry*, 41–70. EMS-Series of Congress Reports, European Mathematical Society, 2008. doi:10.4171/060-1/2
- [7] Bartels, A.; Lück, W. The Borel conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math. (2)* **175** (2012), no. 2, 631–689. doi:10.4007/annals.2012.175.2.5
- [8] Carter, D. W. Lower  $K$ -theory of finite groups. *Comm. Algebra* **8** (1980), no. 20, 1927–1937. doi:10.1080/00927878008822554
- [9] Connolly, F.; Davis, J. F.; Khan, Q. Topological rigidity and  $H_1$ -negative involutions on tori. *Geom. Topol.* **18** (2014), no. 3, 1719–1768. doi:10.2140/gt.2014.18.1719



- [10] Crowley, D.; Macko, T. The additivity of the  $\rho$ -invariant and periodicity in topological surgery. *Algebr. Geom. Topol.* **11** (2011), no. 4, 1915–1959. doi:10.2140/agt.2011.11.1915
- [11] Davis, J. F.; Lück, W. Spaces over a category and assembly maps in isomorphism conjectures in  $K$ - and  $L$ -theory. *K-Theory* **15** (1998), no. 3, 201–252. doi:10.1023/A:1007784106877
- [12] Davis, J. F.; Lück, W. The  $p$ -chain spectral sequence. *K-Theory* **30** (2003), no. 1, 71–104. doi:10.1023/B:KTHE.0000015339.41953.04
- [13] Davis, J. F.; Lück, W. The topological  $K$ -theory of certain crystallographic groups. *J. Noncommut. Geom.* **7** (2013), no. 2, 373–431. doi:10.4171/JNCG/121
- [14] Farrell, F. T.; Jones, L. E. Isomorphism conjectures in algebraic  $K$ -theory. *J. Amer. Math. Soc.* **6** (1993), no. 2, 249–297. doi:10.2307/2152801
- [15] Farrell, T.; Jones, L.; Lück, W. A caveat on the isomorphism conjecture in  $L$ -theory. *Forum Math.* **14** (2002), no. 3, 413–418. doi:10.1515/form.2002.019
- [16] Hambleton, I.; Taylor, L. R. A guide to the calculation of the surgery obstruction groups for finite groups. *Surveys on surgery theory, Vol. 1*, 225–274. Princeton University Press, Princeton, N.J., 2000.
- [17] Higson, N.; Kasparov, G.  $E$ -theory and  $KK$ -theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.* **144** (2001), no. 1, 23–74. doi:10.1007/s002220000118
- [18] Joachim, M.; Lück, W. Topological  $K$ -(co)homology of classifying spaces of discrete groups. *Algebr. Geom. Topol.* **13** (2013), no. 1, 1–34. doi:10.2140/agt.2013.13.1
- [19] Kreck, M.; Lück, W. Topological rigidity for non-aspherical manifolds. *Pure Appl. Math. Q.* **5** (2009), no. 3, 873–914. doi:10.4310/PAMQ.2009.v5.n3.a1
- [20] Kühl, P.; Macko, T.; Mole, A. The total surgery obstruction revisited. *Münster J. Math.* **6** (2013), no. 1, 181–269.
- [21] Land, M.; Nikolaus, T. On the relation between  $K$ - and  $L$ -theory of  $C^*$ -algebras. *Math. Ann.* **371** (2018), no. 1–2, 517–563. doi:10.1007/s00208-017-1617-0
- [22] Lück, W. A basic introduction to surgery theory. *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), ICTP Lect. Notes*, vol. 9, 1–224. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. doi:10.4310/PAMQ.2012.v8.n2.a6
- [23] Lück, W. Chern characters for proper equivariant homology theories and applications to  $K$ - and  $L$ -theory. *J. Reine Angew. Math.* **543** (2002), 193–234. doi:10.1515/crll.2002.015
- [24] Lück, W. Survey on classifying spaces for families of subgroups. *Infinite groups: geometric, combinatorial and dynamical aspects*, 269–322. Progress in Mathematics, 248. Birkhäuser, Basel, 2005. doi:10.1007/3-7643-7447-0\_7
- [25] Lück, W.; Reich, H. The Baum-Connes and the Farrell-Jones conjectures in  $K$ - and  $L$ -theory. *Handbook of K-theory. Vol. 1, 2*, 703–842. Springer, Berlin, 2005. doi:10.1007/978-3-540-27855-9\_15
- [26] Lück, W.; Rosenthal, D. On the  $K$ - and  $L$ -theory of hyperbolic and virtually finitely generated abelian groups. *Forum Math.* **26** (2014), no. 5, 1565–1609. doi:10.1515/forum-2011-0146
- [27] Lück, W.; Stamm, R. Computations of  $K$ - and  $L$ -theory of cocompact planar groups. *K-Theory* **21** (2000), no. 3, 249–292. doi:10.1023/A:1026539221644
- [28] Lück, W.; Weiermann, M. On the classifying space of the family of virtually cyclic subgroups. *Pure Appl. Math. Q.* **8** (2012), no. 2, 497–555. doi:10.4310/PAMQ.2012.v8.n2.a6
- [29] Milnor, J. *Introduction to algebraic K-theory*. Annals of Mathematics Studies, 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971.
- [30] Oliver, R. *Whitehead groups of finite groups*. London Mathematical Society Lecture Note Series, 132. Cambridge University Press, Cambridge, 1988. doi:10.1017/CBO9780511600654
- [31] Ranicki, A. *Lower K- and L-theory*. London Mathematical Society Lecture Note Series, 178. Cambridge University Press, Cambridge, 1992. doi:10.1017/CBO9780511526329
- [32] Ranicki, A. A. *Algebraic L-theory and topological manifolds*. Cambridge Tracts in Mathematics, 102. Cambridge University Press, Cambridge, 1992.
- [33] Sullivan, D. P. Triangulating homotopy equivalences. Ph. D. thesis, Princeton University, 1966.

- [34] Sullivan, D. P. Triangulating and smoothing homotopy equivalences and homeomorphisms. Geometric Topology Seminar Notes. *The Hauptvermutung book*, 69–103. *K-Monographs in Mathematics*, 1. Kluwer Academic Publishers. Dordrecht, 1996. doi:10.1007/978-94-017-3343-4\_3
- [35] Wall, C. T. C. Classification of Hermitian Forms. VI. Group rings. *Ann. of Math. (2)* **103** (1976), no. 1, 1–80. doi:10.2307/1971019
- [36] Wall, C. T. C. *Surgery on compact manifolds*. Second edition. Mathematical Surveys and Monographs, 69. American Mathematical Society, Providence, R.I., 1999. doi:10.1090/surv/069
- [37] Weinberger, S. *The topological classification of stratified spaces*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, 1994.

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