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The p-chain Spectral Sequence

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Abstract. We introduce a new spectral sequence called the p-chain spectral sequence which converges to the (co-)homology of a contravariant $\mathcal{C}$-space with coefficients in a covariant $\mathcal{C}$-spectrum for a small category $\mathcal{C}$. It is different from the corresponding Atiyah–Hirzebruch-type spectral sequence. It can be used in combination with the Isomorphism Conjectures of Baum and Connes and Farrell and Jones to compute algebraic K- and L-groups of group rings and topological K-groups of reduced group C^*-algebras.

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1. Introduction

In [9] we defined Abelian groups $H_n^{\mathcal{C}}(X; \mathbf{E})$ when $X\colon \mathcal{C} \to \mathsf{SPACES}$ is a contravariant functor and $\mathbf{E}\colon \mathcal{C} \to \mathsf{SPECTRA}$ is a covariant functor.* These Abelian groups behave like a generalized homology theory defined on $\mathcal{C}$-spaces, for example, there is a long exact Mayer–Vietoris sequence. They are weak homotopy invariant; given a map $f\colon X \to Y$ of $\mathcal{C}$-spaces (i.e. a natural transformation) which induces a weak homotopy equivalence $f(c)\colon X(c) \to Y(c)$ for all objects $c \in \operatorname{Ob}\mathcal{C}$, then $f_*\colon H_n^{\mathcal{C}}(X; \mathbf{E}) \to H_n^{\mathcal{C}}(Y; \mathbf{E})$ is an isomorphism.

Three special cases will illustrate these groups:

(a) Fixing an object $c \in \operatorname{Ob}\mathcal{C}$, $H_n^{\mathcal{C}}(\operatorname{mor}_{\mathcal{C}}(?, c)) = \pi_n(\mathbf{E}(c))$. This should be thought of as giving the coefficients of the generalized homology theory;
(b) If $X = \star$ is the constant functor, then $H_n^{\mathcal{C}}(\star; \mathbf{E}) = \pi_n(\operatorname{hocolim}_{\mathcal{C}} \mathbf{E})$;
(c) If $\mathcal{C}$ is a category with a single object, all of whose morphisms are isomorphisms, our generalized homology theory reduces to Borel homology. More

*We also defined cohomology groups $H^n_{\mathcal{C}}(X; \mathbf{E})$ when $X\colon \mathcal{C} \to \mathsf{SPACES}$ and $\mathbf{E}\colon \mathcal{C} \to \mathsf{SPECTRA}$ are *both* contravariant functors. In this introduction we will only discuss homology in order to simplify the exposition.

precisely, let G be the group of morphisms and let X be a CW-complex with an action of G by cellular maps, then

$$H_n^{\mathcal{C}}(X; \mathbf{E}) = H_n^G(X; \mathbf{E}) := \pi_n((X \times EG)_+ \wedge_G \mathbf{E}).$$

In [9] we gave a spectral sequence converging to $H_{p+q}^{\mathcal{C}}(X; \mathbf{E})$ whose E^2-term is $E_{p,q}^2 = H_p^{\mathcal{C}}(X; \pi_q(\mathbf{E}))$. This spectral sequence is both quite useful and quite standard. It is related to spectral sequences of Atiyah and Hirzebruch, Bousfield and Kan, and Quinn. The point of the current paper is to introduce a *new* spectral sequence. It converges to $H_{p+q}^{\mathcal{C}}(X; \mathbf{E})$ and has as E^1-term

$$E_{p,q}^1 = \coprod_{\substack{\overline{c}_* \in \prod_{i=0}^p \mathrm{Is}(\mathcal{C}), \\ S(\overline{c}_*) \neq \emptyset}} H_q^{\mathcal{C}}(X(c_p) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} \mathrm{mor}(?, c_0); \mathbf{E}),$$

where $\mathrm{Is}(\mathcal{C})$ is the set of isomorphism classes of objects in $\mathcal{C}$ and $S(\overline{c}_*)$ is a certain $\mathrm{aut}(c_p)$-$\mathrm{aut}(c_0)$-set. In the special case of an orbit category $\mathrm{Or}(G, \mathcal{F})$, where G is a (discrete) group and $\mathcal{F}$ is a family of subgroups satisfying the condition $gHg^{-1} \subseteq H \Rightarrow gHg^{-1} = H$ for $g \in G$ and $H \in \mathcal{F}$, then $E_{p,q}^1$ can be indexed by 'p-chains'

$$(H_0) < (H_1) < \cdots < (H_p),$$

where (H_i) is the conjugacy class of $H_i \in \mathcal{F}$ and $(H_i) < (H_{i+1})$ means that there exists $g \in G$ so that $gH_ig^{-1} \subsetneqq H_{i+1}$. More generally, if $\mathcal{C}$ is a EI-category (endomorphisms are isomorphisms), then $E_{p,q}^1$ is indexed by similar p-chains. The $E_{p,q}^1$-term looks formidable, but we discuss many simplifications and reinterpretations in cases of interest. In general, the E^1-term of the spectral sequence is related to Borel homology of the groups $\{\mathrm{aut}(c)\}_{c \in \mathrm{Ob}\,\mathcal{C}}$ and the differentials are related to assembly maps.

It should be emphasized that the motivation for this abstract-looking spectral sequence comes from geometric topology (the surgery classification of manifolds), geometry (which manifolds admits metrics of positive scalar curvature), and analysis (the study of K-theory of group C^*-algebras), in conjunction with the study of (fundamental) groups which are infinite, but contain torsion. The connection with these various subjects comes through isomorphism conjectures and assembly maps, for more on this see [9] and the end of Section 2 of the current paper.

In Section 2 we review the definition of $H_n^{\mathcal{C}}(X; \mathbf{E})$ and discuss maps connected with a functor $F\colon \mathcal{C} \to \mathcal{D}$. In Section 3 we derive the p-chain spectral sequence and its differentials and discuss simplifications which occur when the category $\mathcal{C}$ is left-free. In Section 4 we discuss the important special case $H_n^{\mathcal{C}}(\star; \mathbf{E})$, where $\mathcal{C}$ is the restricted orbit category. Here, by using the Cofinality theorem and an analogue of Quillen's theorem A, computations can be simplified. We also discuss differentials in the p-chain spectral sequence, which often turn out to be assembly maps themselves. In Section 5 we give examples of groups where the methods of the preceeding sections in combination with the Isomorphism Conjectures of Baum and Connes and Farrell and Jones lead to explicit computations of algebraic K- and L-groups of group rings and topological K-groups of reduced group C^*-algebras.

The p-chain spectral sequence is a generalization of the spectral sequence in [17, Chapter 17] and is related to the paper of Slominska [29].

2. Review of Spaces Over a Category and Assembly Maps

In this section we review some basic facts from [9] for the convenience of the reader.

Let SPACES and SPACES_+ be the categories of topological spaces and pointed topological spaces, respectively. We will always work in the category of compactly generated spaces (see [30; 33, I.4]). A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps $\sigma(n)\colon E(n) \wedge S^1 \to E(n+1)$, called *structure maps*. A *(strong) map* of spectra $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps, i.e. $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$. This should not be confused with the notion of map of spectra in the stable category (see [1, III.2]). A spectrum is called an Ω-*spectrum* if the adjoint of each structure map $E(n) \to \Omega E(n+1)$ is a weak homotopy equivalence. The homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) = \operatorname*{colim}_{n\to\infty} \pi_{i+n}(E(n)),$$

where the system $\pi_{i+n}(E(n))$ is given by the composite

$$\pi_{i+n}(E(n)) \xrightarrow{S} \pi_{i+n+1}(E(n) \wedge S^1) \xrightarrow{\sigma(n)_*} \pi_{i+n+1}(E(n+1))$$

of the suspension homomorphism and the homomorphism induced by the structure map.

Let $\mathcal{C}$ be a small category, i.e. a category such that the objects and the morphisms form sets. A *covariant* (*or contravariant*) $\mathcal{C}$-*space*, *pointed* $\mathcal{C}$-*space*, $\mathcal{C}$-*spectrum*, etc. is a covariant (or contravariant) functor from $\mathcal{C}$ to SPACES, SPACES_+, $\mathsf{SPECTRA}$, etc. and a morphism is a natural transformation.

Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their *tensor product* to be the space

$$X \otimes_{\mathcal{C}} Y = \coprod_{c\in \mathrm{Ob}(\mathcal{C})} X(c) \times Y(c)/\sim,$$

where $\sim$ is the equivalence relation which is generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi\colon c \to d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If X and Y are $\mathcal{C}$-spaces of the same variance, denote by $\hom_{\mathcal{C}}(X, Y)$ the space of maps of $\mathcal{C}$-spaces from X to Y with the subspace topology coming from the obvious inclusion into $\prod_{c \in \mathrm{Ob}(\mathcal{C})} \mathrm{map}(X(c), Y(c))$. If Y is a $\mathcal{C}$-space and Z a space, let $Y \times Z$ be the $\mathcal{C}$-space with the same variance as Y whose value at the object c in $\mathcal{C}$ is $Y(c) \times Z$. For $\mathcal{C}$-spaces X and Y of the same variance, the set of homotopy classes of maps of $\mathcal{C}$-spaces $[X, Y]^{\mathcal{C}}$ is defined using maps of $\mathcal{C}$-spaces $X \times [0, 1] \to Y$. If Y is a $\mathcal{C}$-space and Z a space, let $\mathrm{map}(Y, Z)$ be the $\mathcal{C}$-space with the opposite variance as Y whose value at the object c in $\mathcal{C}$ is $\mathrm{map}(Y(c), Z)$. If X is a contravariant $\mathcal{C}$-space, Y a covariant $\mathcal{C}$-space and Z a space, there is a canonical adjunction homeomorphism [9, Lemma 1.5]

$$\mathrm{map}(X \otimes_{\mathcal{C}} Y, Z) \xrightarrow{\cong} \hom_{\mathcal{C}}(X, \mathrm{map}(Y, Z)).$$

All of the above notions also make sense for pointed spaces; one has to substitute wedges for disjoint unions, smash products for Cartesian products, and pointed mapping spaces for mapping spaces.

A *contravariant $\mathcal{C}$-CW-complex* X is a contravariant $\mathcal{C}$-space X together with a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X = \bigcup_{n \geqslant 0} X_n$$

such that $X = \mathrm{colim}_{n \to \infty} X_n$ and for any $n \geqslant 0$ the *n-skeleton* X_n is obtained from the $(n-1)$-skeleton X_{n-1} by attaching $\mathcal{C}$-n-cells, i.e. there exists a pushout of $\mathcal{C}$-spaces of the form

$$\begin{array}{ccc}
\coprod_{i \in I_n} \mathrm{mor}_{\mathcal{C}}(?, c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\
\big\downarrow & & \big\downarrow \\
\coprod_{i \in I_n} \mathrm{mor}_{\mathcal{C}}(?, c_i) \times D^{n} & \longrightarrow & X_n
\end{array},$$

where the vertical maps are inclusions, I_n is an index set, and the c_i are objects of $\mathcal{C}$. The definition of a *covariant $\mathcal{C}$-CW-complex* is analogous. In [9] these were called contravariant free and covariant free $\mathcal{C}$-CW-complexes; we will omit the word free here. One of the main properties of CW-complexes carries over to $\mathcal{C}$-CW-complexes, namely, a map $f\colon Y \to Z$ of $\mathcal{C}$-spaces is a *weak homotopy equivalence*, i.e. $f(c)$ is a weak homotopy equivalence of spaces for all objects c in $\mathcal{C}$, if and only if for any $\mathcal{C}$-CW-complex X the induced map

$$f_*\colon [X, Y]^{\mathcal{C}} \to [X, Z]^{\mathcal{C}}, \qquad [g] \mapsto [g \circ f]$$

between the homotopy classes of maps of $\mathcal{C}$-spaces is bijective [9, Theorem 3.4]. In particular Whitehead's Theorem carries over: a map of $\mathcal{C}$-CW-complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence. A $\mathcal{C}$-CW-approximation (X, f) of a $\mathcal{C}$-space Y consists of a $\mathcal{C}$-CW-complex X together with a weak homotopy equivalence $f\colon X \to Y$. Such a $\mathcal{C}$-CW-approximation always exists, there is even a functorial construction. If (X, f) and (X', f') are two $\mathcal{C}$-CW-approximations of a $\mathcal{C}$-space Y, there is a homotopy equivalence $h\colon X \to X'$ of $\mathcal{C}$-spaces which is determined uniquely up to homotopy by the property that $f' \circ h$ and f are homotopic.

Let (X, A) be a pair of contravariant pointed $\mathcal{C}$-spaces. Denote the reduced cone of the pointed space A by $\operatorname{cone}(A)$. For a covariant $\mathcal{C}$-spectrum $\mathbf{E}$ define

$$\mathbf{E}_q^{\mathcal{C}}(X, A) = \pi_q(X \cup_A \operatorname{cone}(A) \otimes_{\mathcal{C}} \mathbf{E}).$$

Given a contravariant $\mathcal{C}$-spectrum $\mathbf{E}$, define

$$\mathbf{E}^q_{\mathcal{C}}(X, A) = \pi_{-q}(\hom_{\mathcal{C}}(X \cup_A \operatorname{cone}(A), \mathbf{E})).$$

Define (X_+, A_+) to be the pair of contravariant pointed $\mathcal{C}$-spaces which is obtained from (X, A) by adding a disjoint base point. Let $(u, v)\colon (X', A') \to (X, A)$ be a $\mathcal{C}$-CW-approximation. For a covariant $\mathcal{C}$-spectrum $\mathbf{E}$ define the *homology of* (X, A) *with coefficients in* $\mathbf{E}$ by

$$H_q^{\mathcal{C}}(X, A; \mathbf{E}) = \mathbf{E}_q^{\mathcal{C}}(X'_+, A'_+).$$

Given a contravariant $\mathcal{C}$-spectrum $\mathbf{E}$, define the *cohomology of* (X, A) *with coefficients in* $\mathbf{E}$ by

$$H^q_{\mathcal{C}}(X, A; \mathbf{E}) = \mathbf{E}^q_{\mathcal{C}}(X'_+, A'_+).$$

When A is empty we omit it from the notation.

Then $H_q^{\mathcal{C}}(X, A; \mathbf{E})$ and $H^q_{\mathcal{C}}(X, A; \mathbf{E})$ are unreduced homology and cohomology theories on pairs of $\mathcal{C}$-spaces which satisfy the WHE-axiom, which says that a weak homotopy equivalence induces an isomorphism on (co-)homology. The homology theory satisfies the disjoint union axiom. The cohomology theory satisfies the disjoint union axiom provided that $\mathbf{E}$ is a $\mathcal{C}$-Ω-spectrum. Let $\star_{\mathcal{C}}$, or briefly $\star$, be the $\mathcal{C}$-space which takes each object to a point. Then for $(X, A) = (\star, \emptyset)$ then the above notions reduce to

$$\pi_q(\operatorname{hocolim}_{\mathcal{C}} \mathbf{E}) \quad \text{and} \quad \pi_{-q}(\operatorname{holim}_{\mathcal{C}} \mathbf{E}),$$

respectively. If $\mathcal{C}$ is the category associated to a group G, i.e. $\mathcal{C}$ has a single object, the morphisms in $\mathcal{C}$ are in one-to-one correspondence with G, and the composition law in $\mathcal{C}$ corresponds to multiplication in G, then $H_q^{\mathcal{C}}(X, A; \mathbf{E})$ can be identified with Borel homology $\pi_q(EG \times (X_+, A_+) \wedge_G \mathbf{E})$, and similarly for cohomology, provided that (X, A) is a CW-pair with a G-action by cellular maps. Finally, if $\mathcal{C}$ is the trivial category with precisely one object and morphism, these notions reduce to the standard notions of the homology and cohomology of spaces given by a spectrum.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a $\mathcal{D}$-space (or spectrum) Y, define the *restriction of* Y *with respect to* F to be the $\mathcal{C}$-space (or spectrum) $F^*Y(c) = Y(F(c))$. For X a contravariant, respectively covariant, $\mathcal{C}$-space define the *induction of X with respect to F* to be the $\mathcal{D}$-space

$$F_*X(??) = X(?) \otimes_{\mathcal{C}} \operatorname{mor}_{\mathcal{D}}(??, F(?)),$$

respectively

$$F_*X(??) = \operatorname{mor}_{\mathcal{D}}(F(?), ??) \otimes_{\mathcal{C}} X(?).$$

There are natural adjunction isomorphisms (see [9, Lemma 1.9])

$$X \otimes_{\mathcal{C}} F^*Y \cong F_*X \otimes_{\mathcal{D}} Y, \tag{1}$$

$$\hom_{\mathcal{C}}(X, F^*Y) \cong \hom_{\mathcal{D}}(F_*X, Y). \tag{2}$$

LEMMA 2.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor.*

(a) *If $f\colon X \to Y$ is a fibration of $\mathcal{D}$-spaces, then $F^*f\colon F^*X \to F^*Y$ is a fibration of $\mathcal{C}$-spaces. If $f\colon X \to Y$ is a cofibration of $\mathcal{C}$-spaces, then $F_*f\colon F_*X \to F_*Y$ is a cofibration of $\mathcal{D}$-spaces*;

(b) *If X is a $\mathcal{C}$-CW-complex, then F_*X* is a *$\mathcal{D}$-CW-complex*;

(c) *Let X be a contravariant $\mathcal{C}$-space and $\mathbf{E}$ be a covariant $\mathcal{D}$-spectrum. Then there is a map, natural in X and $\mathbf{E}$,*

$$\Phi_F\colon H_q^{\mathcal{C}}(X; F^*\mathbf{E}) \to H_q^{\mathcal{D}}(F_*X; \mathbf{E}).$$

If either

(i) *X is a $\mathcal{C}$-CW-complex, or,*

(ii) *for all objects d of $\mathcal{D}$ the covariant $\mathcal{C}$*-set $\operatorname{mor}_{\mathcal{D}}(d, F(?))$ *is isomorphic to a disjoint union of covariant $\mathcal{C}$-sets of the form* $\operatorname{mor}_{\mathcal{C}}(c_0, ?)$),

then $\Phi_F\colon H_q^{\mathcal{C}}(X; F^*\mathbf{E}) \to H_q^{\mathcal{D}}(F_*X; \mathbf{E})$ is an isomorphism.

Similar statements hold true for cohomology.

Proof. (a) To show that F^*f is a fibration, one sets up the homotopy lifting problem for $\mathcal{C}$-spaces, solves the adjoint problem for $\mathcal{D}$-spaces, and uses the adjoint property to translate the solution back to $\mathcal{C}$-spaces. The proof for cofibrations is similar.

(b) This follows from two facts. First, for any $c \in \operatorname{Ob} \mathcal{C}$, one can identify $F_* \operatorname{mor}_{\mathcal{C}}(?, c) \cong \operatorname{mor}_{\mathcal{D}}(?, F(c))$. Second, since F_* has a right adjoint, it commutes with colimits, in particular, it commutes with pushouts.

(c) Define the map Φ_F so that when $X' \to X$ is a $\mathcal{C}$-CW-approximation, the following diagram commutes:

$$\begin{array}{ccc}
(F^*\mathbf{E})_q^{\mathcal{C}}(X') & \xrightarrow{\cong} & (\mathbf{E})_q^{\mathcal{D}}(F_*X') \\
\cong\downarrow & & \downarrow\cong \\
H_q^{\mathcal{C}}(X'; F^*\mathbf{E}) & & H_q^{\mathcal{D}}(F_*X'; \mathbf{E}) \\
\cong\downarrow & & \downarrow \\
H_q^{\mathcal{C}}(X; F^*\mathbf{E}) & \xrightarrow{\Phi_F} & H_q^{\mathcal{D}}(F_*X; \mathbf{E}).
\end{array}$$

Uniqueness of $\mathcal{C}$-CW-approximations (up to homotopy) gives that Φ_F is well-defined. If condition (i) holds, then the identity map $X \to X$ is a $\mathcal{C}$-CW-approximation and the claim follows.

If condition (ii) is satisfied, then a weak homotopy equivalence $X' \to X$ gives a weak homotopy equivalence $F_*X' \to F_*X$ and hence the lower right vertical map in the above diagram is an isomorphism. □

For the applications we are most interested in the most important example is the *orbit category* $\mathrm{Or}(G)$ defined for a group G. The objects are homogeneous G-spaces G/H and the morphisms are the G-maps. More generally, for a family $\mathcal{F}$ of subgroups of G, define the *restricted orbit category* $\mathrm{Or}(G, \mathcal{F})$ to be the category whose objects are the homogeneous G-spaces G/H where $H \in \mathcal{F}$ and the morphisms are the G-maps. Some examples for $\mathcal{F}$ are $\mathcal{TR}$, $\mathcal{FIN}$, $\mathcal{VC}$, and $\mathcal{ALL}$, which are the families consists of the trivial group, the finite subgroups, the virtually cyclic subgroups, and all subgroups, respectively.

The remainder of this section is not necessary for the discussion of the p-chain spectral sequence itself in Section 3, but does give the motivation for this paper and is necessary for Section 4. We will review the point of view of [9] concerning assembly maps and the Farrell–Jones and Baum–Connes Isomorphism Conjectures.

DEFINITION 2.2. Let $F\colon \mathcal{B} \to \mathcal{C}$ and $\mathbf{E}\colon \mathcal{C} \to \mathsf{SPECTRA}$ be covariant functors. Then the composite

$$H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) \xrightarrow{\Phi_F} H_q^{\mathcal{C}}(F_*\star; \mathbf{E}) \xrightarrow{H_q^{\mathcal{C}}(\mathrm{pr};\mathbf{E})} H_q^{\mathcal{C}}(\star; \mathbf{E})$$

is called the assembly map induced by F, where Φ_F is the map appearing in Lemma 2.1 (c) and $\mathrm{pr}: F_*\star \to \star$ is the constant map at each object. We sometimes abbreviate $F^*\mathbf{E}$ by $\mathbf{E}$.

Remark 2.3. This map can be identified with the map $\pi_q\,(\mathrm{hocolim}_{\mathcal{B}}\, F^*\mathbf{E}) \to \pi_q\,(\mathrm{hocolim}_{\mathcal{C}}\, \mathbf{E})$ induced by the functor F.

The three $\mathrm{Or}(G)$-spectra which are useful to us are covariant functors

$\mathbf{K}^{\mathrm{alg}}\colon \mathrm{Or}(G) \to \mathsf{SPECTRA}$,
$\mathbf{L}^{\langle j\rangle}\colon \mathrm{Or}(G) \to \mathsf{SPECTRA}$,
$\mathbf{K}^{\mathrm{top}}\colon \mathrm{Or}(G) \to \mathsf{SPECTRA}$.

These functors were constructed in [9, Section 2], but there was a problem with the construction of $\mathbf{K}^{\mathrm{top}}$ connected with the pairing on [9, p. 217]. This problem can easily be fixed and the construction can be replaced by more refined ones (see [14]). The key property of these functors is that $\pi_q(\mathbf{K}^{\mathrm{alg}}(G/H)) = K_q(RH)$ for a fixed ring R, $\pi_q(\mathbf{L}^{\langle j\rangle}(G/H)) = L_q^{\langle j\rangle}(RH)$ for a fixed ring R with involution, and $\pi_q(\mathbf{K}^{\mathrm{top}}(G/H)) = K_q(C_r^*H)$. Here $K_q(C_r^*H)$ is the K-theory of the real or complex reduced C^*-algebra of H. In connection with L-groups we use the involution on RG sending $r \cdot g$ to $\overline{r} \cdot g^{-1}$. The index $j \in \mathbb{Z} \cup -\infty$ on the L-theory is the K-theory decoration; the important cases for us are $j = -\infty$ which arises in the isomorphism conjecture, and the case $j = 2$ which is used in Wall's book [32] to classify manifolds. The Isomorphism Conjecture of Farrell and Jones is not true for the decorations $j = 0, 1, 2$, which correspond to the decorations p, h and s appearing in the literature [13].

The Isomorphism Conjecture of Baum and Connes for a group G is that the assembly map associated to the inclusion functor I: $\mathrm{Or}(G, \mathcal{FIN}) \to \mathrm{Or}(G)$ (see Definition 2.2) yields an isomorphism

$$H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \mathbf{K}^{\mathrm{top}}) \to H_q^{\mathrm{Or}(G)}(\star; \mathbf{K}^{\mathrm{top}}) = K_q(C_r^*(G)).$$

The Isomorphism Conjecture of Farrell and Jones for RG says that

$$H_q^{\mathrm{Or}(G,\mathcal{VC})}(\star; \mathbf{K}^{\mathrm{alg}}) \to H_q^{\mathrm{Or}(G)}(\star; \mathbf{K}^{\mathrm{alg}}) = K_q(RG)$$

and

$$H_q^{\mathrm{Or}(G,\mathcal{VC})}(\star; \mathbf{L}^{\langle -\infty \rangle}) \to H_q^{\mathrm{Or}(G)}(\star; \mathbf{L}^{\langle -\infty \rangle}) = L_q^{\langle -\infty \rangle}(RG)$$

are isomorphisms. We will mainly consider the case $R = \mathbb{Z}$. The point of these conjectures is that they express the target, which is the group one wants to compute, by the source, which only involves the K-theory of the family of finite or virtually cyclic subgroups and is much easier to compute. In the case of the family $\mathcal{FIN}$ the source can rationally be computed by equivariant Chern characters [18, 19]. The p-chain spectral sequence is an important tool for integral computations which are much harder. More information about these conjectures can be found for instance in [2, 12, 22, 31].

3. The p-Chain Spectral Sequence

We establish a spectral sequence converging to the homology (respectively cohomology) of a $\mathcal{C}$-space X with coefficients in a $\mathcal{C}$-spectrum $\mathbf{E}$. In the special case where $X = \star$, this gives a spectral sequence converging to $\pi_*(\mathrm{hocolim}_{\mathcal{C}}\, \mathbf{E})$ (respectively $\pi_*(\mathrm{holim}_{\mathcal{C}}\, \mathbf{E})$). It is different from the standard spectral sequence [9, Theorem 4.7; 26, Theorem 8.7; 6, XII, 5.7, p. 339; XI, 7.1, p. 309] which is an Atiyah–Hirzebruch type spectral sequence and comes from a skeletal filtration. We will need some preliminaries for its construction.

For every nonnegative integer p, define the category $[p]$ whose objects are $\{0, 1, 2, \ldots, p\}$, with precisely one morphism from i to j if $i \leqslant j$ and no morphism otherwise. Let $\mathbf{\Delta}$ be the category of finite ordered sets, i.e. objects are the categories $[0], [1], [2], \ldots$ and morphisms are the functors from $[p]$ to $[q]$. In other words, the morphisms from an object $\{0, 1, 2, \ldots, p\}$ to an object $\{0, 1, 2, \ldots, q\}$ are the monotone increasing functions. A *simplicial set* is a contravariant $\mathbf{\Delta}$-set. There is a covariant $\mathbf{\Delta}$-space $\mathbf{\Delta}_\bullet$ which sends an object $[p]$ to the standard p-simplex. The *geometric realization of a simplicial set* $X_\bullet$ is the space $|X_\bullet| = X_\bullet \otimes_{\mathbf{\Delta}} \mathbf{\Delta}_\bullet$. Recall that the *nerve of a category* $\mathcal{C}$ is the simplicial set

$$N_p\mathcal{C} = \text{functor}([p], \mathcal{C})$$

and its classifying space $B\mathcal{C}$ is the geometric realization $|N_\bullet\mathcal{C}|$ of its nerve. Next we introduce a similar construction. Denote by $\widetilde{\text{functor}}([p], \mathcal{C})$ the equivalence classes of covariant functors from $[p]$ to $\mathcal{C}$, where two such functors are called equivalent if they are related by a natural transformation whose evaluation at any object is an isomorphism. More explicitly, $\text{functor}([p], \mathcal{C})$ consists of the set of diagrams in $\mathcal{C}$ of the shape

$$c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} c_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{p-1}} c_p$$

and $\widetilde{\text{functor}}([p], \mathcal{C})$ is the set of equivalence classes under the following equivalence relation on $\text{functor}([p], \mathcal{C})$: Two such diagrams (c_*, ϕ_*) and (c'_*, ϕ'_*) are equivalent if there is a commutative diagram with isomorphisms as vertical maps

$$\begin{array}{ccccccc} c_0 & \xrightarrow{\phi_0} & c_1 & \xrightarrow{\phi_1} & \cdots & \xrightarrow{\phi_{p-1}} & c_p \\ \cong\downarrow & & \cong\downarrow & & & & \cong\downarrow \\ c'_0 & \xrightarrow{\phi'_0} & c'_1 & \xrightarrow{\phi'_1} & \cdots & \xrightarrow{\phi'_{p-1}} & c'_p \end{array}$$

Define $\widetilde{N}_\bullet\mathcal{C}$ to be the simplicial set given by

$$\widetilde{N}_p\mathcal{C} = \widetilde{\text{functor}}([p], \mathcal{C})$$

and $\widetilde{B}\mathcal{C}$ to be its geometric realization. We shall proceed to develop basic properties of $\widetilde{B}\mathcal{C}$ analogous to those of $B\mathcal{C}$, as discussed in [9, pp. 227–229].

If $\mathcal{C}$ is a groupoid, i.e. all morphisms are isomorphisms, then $\widetilde{B}\mathcal{C}$ is the (discrete) set of isomorphism classes of objects. If $\mathcal{C}$ is a category such that the identity morphisms are the only isomorphisms in $\mathcal{C}$, then $B\mathcal{C}$ is the same as $\widetilde{B}\mathcal{C}$. For instance, $\widetilde{B}[1] = B[1] = [0, 1]$.

LEMMA 3.1. *The projections from* $\mathcal{C}_0 \times \mathcal{C}_1$ *to* $\mathcal{C}_i$ *for* $i = 0, 1$ *induce a homeomorphism*

$$\widetilde{B}(\mathcal{C}_0 \times \mathcal{C}_1) \to \widetilde{B}\mathcal{C}_0 \times \widetilde{B}\mathcal{C}_1.$$

Proof. Given two simplicial sets $A_\bullet$ and $B_\bullet$, define their product $A_\bullet \times B_\bullet$ by sending $[p]$ to the product $A_p \times B_p$. The projections induce a homeomorphism [16, p. 43]

$$|A_\bullet \times B_\bullet| \to |A_\bullet| \times |B_\bullet|.$$

Now the claim follows since the projections induce isomorphisms of simplicial sets:

$$\widetilde{\text{functor}}([p], \mathcal{C}_0 \times \mathcal{C}_1) \to \widetilde{\text{functor}}([p], \mathcal{C}_0) \times \widetilde{\text{functor}}([p], \mathcal{C}_1). \qquad \square$$

Given two objects ? and ?? in $\mathcal{C}$, define the category $?\downarrow \mathcal{C} \downarrow ??$ as follows: An object is a diagram

$$? \xrightarrow{\alpha} c \xrightarrow{\beta} ??$$

in $\mathcal{C}$. A morphism from $? \xrightarrow{\alpha} c \xrightarrow{\beta} ??$ to $? \xrightarrow{\alpha'} c' \xrightarrow{\beta'} ??$ is a commutative diagram in $\mathcal{C}$ of the shape

$$\begin{array}{ccccc} ? & \xrightarrow{\alpha} & c & \xrightarrow{\beta} & ?? \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ ? & \xrightarrow{\alpha'} & c' & \xrightarrow{\beta'} & ?? \end{array}$$

Let $\widetilde{B}_p(\,?\downarrow \mathcal{C} \downarrow ??\,)$ be the p-skeleton of $\widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,)$. We will regard $\widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,)$ as a contravariant $\mathcal{C} \times \mathcal{C}^{op}$-space where ? is the variable in $\mathcal{C}$ and ?? the variable in $\mathcal{C}^{\mathrm{op}}$. Since maps on classifying spaces induced by functors are cellular, we get a filtration of the contravariant $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-space $\widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,)$ by the contravariant $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-spaces $\widetilde{B}_p(\,?\downarrow \mathcal{C} \downarrow ??\,)$ such that

$$\widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,) = \operatorname*{colim}_{p \to \infty} \widetilde{B}_p(\,?\downarrow \mathcal{C} \downarrow ??\,).$$

Let $\overline{\mathrm{mor}(?, ??)}$ be the category whose set of objects is $\mathrm{mor}(?, ??)$ and whose only morphisms are the identity morphisms of objects. Consider the functor

$$\mathrm{pr}\colon\ ?\downarrow \mathcal{C} \downarrow ?? \to \overline{\mathrm{mor}(?, ??)} \qquad \left(? \xrightarrow{\alpha} c \xrightarrow{\beta} ??\right) \mapsto (\beta \circ \alpha\colon ? \to ??)\,.$$

It induces a map of contravariant $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-spaces

$$\widetilde{B}(\mathrm{pr})\colon \widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,) \to \widetilde{B}(\overline{\mathrm{mor}(?, ??)}) = \mathrm{mor}(?, ??).$$

Let X be a contravariant $\mathcal{C}$-space. We obtain contravariant $\mathcal{C}$-spaces $X \otimes_{\mathcal{C}} \widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,)$ and $X \otimes_{\mathcal{C}} \mathrm{mor}(?, ??)$ where the tensor product is taken over the variable ??. Define a map of contravariant $\mathcal{C}$-spaces

$$p\colon X \otimes_{\mathcal{C}} \widetilde{B}(\,?\downarrow \mathcal{C} \downarrow ??\,) \xrightarrow{\mathrm{id} \otimes_{\mathcal{C}} \widetilde{B}(\mathrm{pr})} X \otimes_{\mathcal{C}} \mathrm{mor}(?, ??) \xrightarrow{\cong} X, \tag{3}$$

where the second map is the canonical isomorphism given by $x \otimes \phi \mapsto X(\phi)(x)$.

Let $n.d.\widetilde{N}_p(\,?\downarrow \mathcal{C} \downarrow ??\,)$ denote the set of nondegenerate p-simplices of the simplicial set $\widetilde{N}_\bullet(\,?\downarrow \mathcal{C} \downarrow ??\,)$. Elements are given by classes of diagrams

$$? \xrightarrow{\alpha} c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} c_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{p-1}} c_p \xrightarrow{\beta} ??$$

such that no ϕ_i is an isomorphism.

LEMMA 3.2.
(a) *We have*
$$X \otimes_{\mathcal{C}} \widetilde{B}(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,) = \operatorname*{colim}_{p\to\infty} X \otimes_{\mathcal{C}} \widetilde{B}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,)$$
as contravariant $\mathcal{C}$*-spaces*;
(b) *There is a pushout of contravariant* $\mathcal{C}$*-spaces whose vertical maps are* $(p-1)$*-connected cofibrations of contravariant* $\mathcal{C}$*-spaces*
$$\begin{array}{ccc} \big(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,)\big) \times S^{p-1} & \to & X \otimes_{\mathcal{C}} \widetilde{B}_{p-1}(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,) \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes_{\mathcal{C}}\mathrm{inc}} & & \big\downarrow{\scriptstyle \mathrm{id}\otimes_{\mathcal{C}}\mathrm{inc}} \\ \big(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??)\big) \times D^p & \to & X \otimes_{\mathcal{C}} \widetilde{B}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,); \end{array}$$
(c) *The map* $p\colon X \otimes_{\mathcal{C}} \widetilde{B}(\,?\downarrow\mathcal{C}\downarrow??\,) \to X$ *defined in* (3) *is a weak homotopy equivalence of contravariant* $\mathcal{C}$*-spaces.*

Proof. (a) Since the functor $X\otimes_{\mathcal{C}} -$ from the category of covariant $\mathcal{C}$-spaces to the category of spaces has a right adjoint, it is compatible with colimits.

(b) There is a canonical CW-structure on the geometric realization of a simplicial set whose cells are in bijective correspondence with the nondegenerate simplices [16, p. 39]. Hence, we get the following pushout of contravariant $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-spaces:

$$\begin{array}{ccc} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,) \times S^{p-1} & \to & \widetilde{B}_{p-1}(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,) \\ \big\downarrow & & \big\downarrow \\ n.d.\widetilde{N}_p(\,?\!\downarrow\!!\downarrow\mathcal{C}\,??) \times D^p & \to & \widetilde{B}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,). \end{array}$$

Since the functor $X\otimes_{\mathcal{C}} -$ has a right adjoint, it is compatible with pushouts. For any space Z there is a natural homeomorphism

$$\big(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,)\big) \times Z \to X \otimes_{\mathcal{C}} \big(n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,) \times Z\big)\,.$$

This shows that the diagram appearing in assertion (b) is a pushout of contravariant $\mathcal{C}$-spaces. As the inclusion of S^{p-1} into D^p is a $(p-1)$-connected cofibration of spaces, the left vertical and hence also the right vertical arrow are $(p-1)$-connected cofibrations of contravariant $\mathcal{C}$-spaces.

(c) Fix an object c in $\mathcal{C}$. Define a functor

$$j'\colon \overline{\mathrm{mor}(c,??)} \to c\!\downarrow\!\mathcal{C}\!\downarrow\!??\,, \qquad \Big(c \xrightarrow{\alpha} ??\Big) \mapsto \Big(c \xrightarrow{\mathrm{id}} c \xrightarrow{\alpha} ??\Big)\,.$$

It induces a map of spaces by

$$j\colon X(c) \xrightarrow{\cong} X \otimes_{\mathcal{C}} \widetilde{B}\,\overline{(\mathrm{mor}(c,??))} \xrightarrow{\mathrm{id}_X \otimes_{\mathcal{C}} \widetilde{B}j'} X \otimes_{\mathcal{C}} \widetilde{B}(\,c\!\downarrow\!\mathcal{C}\!\downarrow\!??\,).$$

Define a natural transformation

$$S\colon j' \circ \mathrm{pr}(c,??) \to \mathrm{id}_{c\downarrow\mathcal{C}\downarrow??}$$

by assigning to an object $c \xrightarrow{\alpha} d \xrightarrow{\beta} ??$ in $c\downarrow\mathcal{C}\downarrow ??$ the morphism in $c\downarrow\mathcal{C}\downarrow ??$

$$\begin{array}{ccccc} c & \xrightarrow{\text{id}} & c & \xrightarrow{\beta\circ\alpha} & ?? \\ {\scriptstyle\text{id}}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\text{id}}\big\downarrow \\ c & \xrightarrow{\alpha} & d & \xrightarrow{\beta} & ?? \end{array}$$

Thus we have a homotopy of maps of covariant $\mathcal{C}$-spaces

$$\begin{aligned} h'\colon \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??) \times [0,1] &\to \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??) \times \widetilde{B}[1] \\ &\to \widetilde{B}(c\downarrow\mathcal{C}\downarrow ?? \times [1]) \to \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??), \end{aligned}$$

where the first map comes from the identification $[0,1] = \widetilde{B}[1]$, the second from the homeomorphism of Lemma 2.1 and the third from interpreting S as a functor

$$S\colon c\downarrow\mathcal{C}\downarrow ?? \times [1] \to c\downarrow\mathcal{C}\downarrow ??.$$

It induces a homotopy of maps of spaces

$$h = \text{id}_X \otimes_{\mathcal{C}} h'\colon X \otimes_{\mathcal{C}} \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??) \times [0,1] \to X \otimes_{\mathcal{C}} \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??).$$

One easily checks that $p(c) \circ j$ is the identity on $X(c)$ and h is a homotopy from $j \circ p(c)$ to the identity on $X \otimes_{\mathcal{C}} \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??)$. Hence

$$p(c)\colon X \otimes_{\mathcal{C}} \widetilde{B}(c\downarrow\mathcal{C}\downarrow ??) \to X(c)$$

is a homotopy equivalence and in particular a weak homotopy equivalence for all objects c. This finishes the proof of Lemma 3.2. □

Remark 3.3. Notice that the map $p\colon X \otimes_{\mathcal{C}} \widetilde{B}(?\downarrow\mathcal{C}\downarrow ??) \to X$ is *not* a homotopy equivalence of contravariant $\mathcal{C}$-spaces. In the proof of Lemma 3.2 we have constructed a homotopy inverse and a corresponding homotopy for $p(c)$ for each object c, but they do not fit together to an homotopy inverse of p as a map of contravariant $\mathcal{C}$-spaces. Therefore it is important that we use the (co-)homology $H(X; \mathbf{E})$ which satisfies the WHE-axiom.

We can now apply [9, Theorem 4.7] to the filtration of $X \otimes_{\mathcal{C}} \widetilde{B}(?\downarrow\mathcal{C}\downarrow ??)$ by the subspaces $X \otimes_{\mathcal{C}} \widetilde{B}_p(?\downarrow\mathcal{C}\downarrow ??)$ and obtain the following theorem.

THEOREM 3.4. *Let X be a contravariant $\mathcal{C}$-space and $\mathbf{E}$ a covariant respectively contravariant $\mathcal{C}$-spectrum. Let $H_p^{\mathcal{C}}(X; \mathbf{E})$ respectively $H_{\mathcal{C}}^p(X; \mathbf{E})$ be the associated homology respectively cohomology theories satisfying the WHE-axiom.*

(a) *There is a spectral (homology) sequence $(E^r_{p,q}, d^r_{p,q})$ whose E^1-term is given by*

$$E^1_{p,q} = H_q^{\mathcal{C}}(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(?\downarrow\mathcal{C}\downarrow ??); \mathbf{E}).$$

The first differential

$$d^1_{p,q}\colon\ H^{\mathcal{C}}_q(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,);\mathbf{E}) \to H^{\mathcal{C}}_q(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_{p-1}(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,);\mathbf{E})$$

is $\sum_{i=0}^{p}(-1)^i\cdot H^{\mathcal{C}}_q(\mathrm{id}\otimes_{\mathcal{C}} d^p_i;\mathbf{E})$ *where* $d^p_i\colon n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,) \to n.d.\widetilde{N}_{p-1}(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,)$ *is the ith face map. This spectral sequence converges to* $H^{\mathcal{C}}_{p+q}(X;\mathbf{E})$, *i.e. there is an ascending filtration* $F_{p,m-p}H^{\mathcal{C}}_m(X,\mathbf{E})$ of $H^{\mathcal{C}}_m(X,\mathbf{E})$ *such that*

$$F_{p,q}H^{\mathcal{C}}_{p+q}(X,\mathbf{E})/F_{p-1,q+1}H^{\mathcal{C}}_{p+q}(X,\mathbf{E}) \cong E^{\infty}_{p,q};$$

(b) *Assume that* $\mathbf{E}$ *is a* $\mathcal{C}$-Ω-*spectrum*, *i.e. for each object c the spectrum* $\mathbf{E}(c)$ *is an* Ω-*spectrum. Then there is a spectral* (*cohomology*) *sequence* $(E^{p,q}_r, d^{p,q}_r)$ *whose* E^1-*term is given by*

$$E^{p,q}_1 = H^q_{\mathcal{C}}(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,);\mathbf{E}).$$

The first differential

$$d^{p,q}_1\colon\ H^q_{\mathcal{C}}(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_p(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,);\mathbf{E}) \to H^q_{\mathcal{C}}(X \otimes_{\mathcal{C}} n.d.\widetilde{N}_{p+1}(\,?\!\downarrow\!\mathcal{C}\!\downarrow\!??\,);\mathbf{E})$$

is $\sum_{i=0}^{p+1}(-1)^i \cdot H^q(\mathrm{id}\otimes_{\mathcal{C}} d^p_i;\mathbf{E})$. *If one of the following conditions is satisfied*:

(i) *The filtration is finite*, *i.e. there is an integer* $n > 0$ *such that for any diagram*

$$c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} c_n$$

one of the morphisms ϕ_i *is an isomorphism*;

(ii) *There is* $n \in \mathbb{Z}$ *such that* $\pi_q(\mathbf{E}(c))$ *vanishes for all objects* $c \in \mathrm{Ob}(\mathcal{C})$ *and* $q < n$;

then the spectral sequence converges to $H^{p+q}_{\mathcal{C}}(X;\mathbf{E})$, *i.e. there is a descending filtration* $F^{p,m-p}H^m_{\mathcal{C}}(X,\mathbf{E})$ *of* $H^m_{\mathcal{C}}(X,\mathbf{E})$ *such that*

$$F^{p,q}H^{p+q}_{\mathcal{C}}(X;\mathbf{E})/F^{p+1,q-1}H^{p+q}_{\mathcal{C}}(X;\mathbf{E}) \cong E^{p,q}_{\infty}.$$

Next we want to analyse the E^1-term further. Let $\mathrm{Is}(\mathcal{C})$ be the set of isomorphism classes $\overline{c}$ of objects c in $\mathcal{C}$. Fix for any isomorphism class $\overline{c}$ a representative $c \in \overline{c}$. For two objects c and d let $\mathrm{mor}_{\ncong}(c,d)$ be the subset of $\mathrm{mor}(c,d)$ consisting of all morphisms from c to d which are not isomorphisms. For an element

$$\overline{c}_* = (\overline{c_0}, \overline{c_1}, \ldots, \overline{c_p}) \in \prod_{i=0}^{p}\mathrm{Is}(\mathcal{C})$$

and $p \geqslant 1$ define a left-$\mathrm{aut}(c_p)$-right-$\mathrm{aut}(c_0)$-set

$$S(\overline{c}_*) \;=\; \mathrm{mor}_{\ncong}(c_{p-1},c_p) \times_{\mathrm{aut}(c_{p-1})} \mathrm{mor}_{\ncong}(c_{p-2},c_{p-1}) \times_{\mathrm{aut}(c_{p-2})} \cdots \times_{\mathrm{aut}(c_1)} \mathrm{mor}_{\ncong}(c_0,c_1). \tag{4}$$

If A is a right-$\mathrm{aut}(c_p)$-set and B is a left-$\mathrm{aut}(c_0)$-set, define $A \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} B$ in the obvious way for $p \geqslant 1$ and by $A \times_{\mathrm{aut}(c_0)} B$ for $p = 0$. One easily checks that the map

$$\coprod_{\substack{\overline{c}_* \in \prod_{i=0}^{p} \mathrm{Is}(\mathcal{C}),\\ S(\overline{c}_*)\neq\emptyset}} \mathrm{mor}(c_p, ??) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} \mathrm{mor}(?, c_0)$$
$$\to n.d.\widetilde{N}_p(\,?\downarrow\mathcal{C}\downarrow??\,)$$

which sends the element represented by

$$(\phi_p, \phi_{p-1}, \ldots, \phi_0, \phi) \in \mathrm{mor}(c_p, ??) \times \mathrm{mor}_{\ncong}(c_{p-1}, c_p) \times$$
$$\times \cdots \times \mathrm{mor}_{\ncong}(c_0, c_1) \times \mathrm{mor}(?, c_0)$$

to the class of

$$? \xrightarrow{\phi} c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{p-1}} c_p \xrightarrow{\phi_p} ??$$

is natural in ? and ?? and bijective. Since there is a natural isomorphism of contravariant $\mathcal{C}$-spaces

$$X \otimes_{\mathcal{C}} \mathrm{mor}(c_p, ??) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} \mathrm{mor}(?, c_0)$$
$$\xrightarrow{\cong} X(c_p) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} \mathrm{mor}(?, c_0),$$

we conclude the following lemma.

LEMMA 3.5. *There are identifications for the E^1-terms of the spectral sequences in Theorem* 3.4

$$E^1_{p,q} = \bigoplus_{\substack{\overline{c}_* \in \prod_{i=0}^{p} \mathrm{Is}(\mathcal{C}),\\ S(\overline{c}_*)\neq\emptyset}} H_q^{\mathcal{C}}(X(c_p) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} \mathrm{mor}(?, c_0); \mathbf{E})$$

and

$$E_1^{p,q} = \bigoplus_{\substack{\overline{c}_* \in \prod_{i=0}^{p} \mathrm{Is}(\mathcal{C}),\\ S(\overline{c}_*)\neq\emptyset}} H^q_{\mathcal{C}}(X(c_p) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*) \times_{\mathrm{aut}(c_0)} \mathrm{mor}(?, c_0); \mathbf{E}).$$

If the category satisfies an additional condition, we can do a much better job identifying the E^1-terms.

DEFINITION 3.6. We call $\mathcal{C}$ left-free if for any two objects c and c' the left $\mathrm{aut}(c')$-action on $\mathrm{mor}(c, c')$ given by composition is free.

For any group G and any family $\mathcal{F}$ of subgroups the orbit category $\mathrm{Or}(G, \mathcal{F})$ is left-free since any G-map of homogeneous G-spaces $G/H \to G/K$ is surjective.

Let Z be a (left) G-space and let $\mathbf{F}$ be spectrum with an action of G by maps of spectra. We can interpret $\mathbf{F}$ also as a covariant $\mathrm{Or}(G, \mathcal{TR})$-spectrum, where $\mathcal{TR}$ is the family consisting of the trivial subgroup of G. We write

$$H_q^G(Z; \mathbf{F}) := H_q^{\mathrm{Or}(G,\mathcal{TR})}(Z, \mathbf{F}). \tag{5}$$

Explicitly we get after a choice of a free G-CW-complex Z' together with a G-map $u\colon Z' \to Z$ which is a weak equivalence after forgetting the group action

$$H_q^G(Z;\mathbf{F}) = \pi_q(Z'_+ \wedge_G \mathbf{F}). \tag{6}$$

For instance, $H_q^G(\star;\mathbf{F}) = \pi_q(EG_+ \wedge_G \mathbf{F})$.

LEMMA 3.7. *Suppose $\mathcal{C}$ is left-free. Then there are identifications for the E^1-terms of the spectral sequences in Theorem* 3.4

$$E^1_{p,q} = \bigoplus_{\substack{\overline{c}_* \in \prod_{i=0}^p \operatorname{Is}(\mathcal{C}),\\ S(\overline{c}_*)\neq\emptyset}} H_q^{\operatorname{aut}(c_0)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_*); \mathbf{E}(c_0))$$

and

$$E_1^{p,q} = \bigoplus_{\substack{\overline{c}_* \in \prod_{i=0}^p \operatorname{Is}(\mathcal{C}),\\ S(\overline{c}_*)\neq\emptyset}} H^q_{\operatorname{aut}(c_0)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_*); \mathbf{E}(c_0))$$

where $X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_*)$ means $X(c_0)$ for $p = 0$.

With this identification the first differential can be written as

$$d^1_{p,q} = \sum_{i=0}^{p} (-1)^i (d^1_{p,q})_i$$

for certain maps

$$(d^1_{p,q})_0\colon H_q^{\operatorname{aut}(c_0)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_*); \mathbf{E}(c_0)) \\ \to H_q^{\operatorname{aut}(c_1)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_1,\ldots,\overline{c}_p); \mathbf{E}(c_1)),$$

$$(d^1_{p,q})_i\colon H_q^{\operatorname{aut}(c_0)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_*); \mathbf{E}(c_0)) \\ \to H_q^{\operatorname{aut}(c_0)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_0,\ldots,\overline{c}_{i-1},\overline{c}_{i+1},\ldots,\overline{c}_p); \mathbf{E}(c_0)),$$

for $0 < i < p$, and

$$(d^1_{p,q})_p\colon H_q^{\operatorname{aut}(c_0)}(X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_*); \mathbf{E}(c_0)) \\ \to H_q^{\operatorname{aut}(c_0)}(X(c_{p-1}) \times_{\operatorname{aut}(c_{p-1})} S(\overline{c}_0,\ldots,\overline{c}_{p-1}); \mathbf{E}(c_0)).$$

For $0 < i \leqslant p$, these maps are induced by maps of $\operatorname{aut}(c_0)$*-sets given by concatenation. The description of $(d^1_{p,q})_0$ is more difficult, due to the change of group and coefficients. Let $c_0 \to c_1$ denote the full subcategory of $\mathcal{C}$ with objects $\{c_0, c_1\}$. We label the inclusions of categories $i\colon \operatorname{aut}(c_0) \to (c_0 \to c_1)$, $j\colon \operatorname{aut}(c_1) \to (c_0 \to c_1)$, and $k\colon (c_0 \to c_1) \to \mathcal{C}$. Let Y be the* $\operatorname{aut}(c_1)$*-space*

$$X(c_p) \times_{\operatorname{aut}(c_p)} S(\overline{c}_1,\ldots,\overline{c}_p).$$

Then with the above identification of E^1 the map $(d^1_{p,q})_0$ is given by

$$\begin{aligned} &H_q^{\mathrm{aut}(c_0)}(Y \times_{\mathrm{aut}(c_1)} \mathrm{mor}_{\not\cong}(c_0, c_1); i^*k^*\mathbf{E}) \\ &\quad \to H_q^{c_0 \to c_1}(i_*(Y \times_{\mathrm{aut}(c_1)} \mathrm{mor}_{\not\cong}(c_0, c_1)); k^*\mathbf{E}) \\ &\quad \to H_q^{c_0 \to c_1}(j_*Y; k^*\mathbf{E}) \xrightarrow{\cong} H_q^{\mathrm{aut}(c_1)}(Y; j^*k^*\mathbf{E}), \end{aligned}$$

where the first map is the map Φ_i from Lemma 2.1 (c), *the middle map is given by the map of* $(c_0 \to c_1)$*-spaces given as the adjoint of the inclusion of* $\mathrm{aut}(c_0)$*-spaces*

$$Y \times_{\mathrm{aut}(c_1)} \mathrm{mor}_{\not\cong}(c_0, c_1) \to Y \times_{\mathrm{aut}(c_1)} \mathrm{mor}(c_0, c_1) = i^* j_* Y,$$

and the last map is $(\Phi_j)^{-1}$.

Similar statements are valid in cohomology.

Proof. For an object c_0 of $\mathcal{C}$, let $i_{c_0}\colon\ \mathrm{aut}(c_0) \to \mathcal{C}$ be the corresponding inclusion of categories. Then, according to Lemma 3.5, the E^1-term of the p-chain spectral sequence is identified with

$$E^1_{p,q} = \bigoplus_{\substack{\overline{c}_* \in \prod_{i=0}^p \mathrm{Is}(\mathcal{C}),\\ S(\overline{c}_*) \neq \emptyset}} H_q^{\mathcal{C}}((i_{c_0})_*(X(c_p) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*)); \mathbf{E}).$$

The maps $\Phi_{i_{c_0}}$ send the sum below to the sum above

$$\bigoplus_{\substack{\overline{c}_* \in \prod_{i=0}^p \mathrm{Is}(\mathcal{C}),\\ S(\overline{c}_*) \neq \emptyset}} H_q^{\mathrm{aut}(c_0)}(X(c_p) \times_{\mathrm{aut}(c_p)} S(\overline{c}_*); \mathbf{E}(c_0)).$$

This map between sums is an isomorphism by Lemma 2.1 (c), since $\mathcal{C}$ is left-free. We have thus established the first identification in the lemma. The identifications of the differentials are established similarly. □

Remark 3.8. Lemma 3.7 shows for a left-free category $\mathcal{C}$ what the spectral sequence does. Namely, it reduces the computation of the (co-)homology groups of spaces and spectra over a category to the special case of spaces and spectra over a group. The most important case of the p-chain spectral sequence is where $X = \star$ is the constant functor given by a point at every object. In this case the $\mathrm{aut}(c_0)$-sets are all discrete and hence a disjoint union of homogeneous spaces $\mathrm{aut}(c_0)/H$, and so the differentials $(d^1_{p,q})_i$ for $0 < i \leqslant p$ all involve change of group maps $BH_0 \to BH_1$. The remaining differential $(d^1_{p,q})_0$ is more subtle and should be thought of as some sort of assembly map. We will comment further on this map in the next section.

Remark 3.9. We have put some effort into avoiding the assumption that $\mathcal{C}$ is an *EI-category*, i.e that all endomorphisms are isomorphisms. Otherwise we would have excluded the orbit category $\mathrm{Or}(G, \mathcal{F})$ for $\mathcal{F}$ the family of virtually cyclic subgroups of G (see [17, Example 1.32]). But this category appears in the

Isomorphism Conjecture in algebraic K or L-theory of Farrell–Jones. The Baum–Connes Conjecture, however, uses only the orbit category $\mathrm{Or}(G, \mathcal{FIN})$ for $\mathcal{FIN}$ the family of finite subgroups of G which is an EI-category.

If one has an EI-category, the bookkeeping simplifies a little bit. The EI-property makes it possible to define a partial ordering on $\mathrm{Is}(\mathcal{C})$ by

$$\overline{c} \leqslant \overline{d} \iff \mathrm{mor}(c, d) \neq \emptyset.$$

We write $\overline{c} < \overline{d}$ if $\overline{c} \leqslant \overline{d}$ and $\overline{c} \neq \overline{d}$ holds. A p-chain $\overline{c_*}$ in $\mathcal{C}$ is a sequence

$$\overline{c_0} < \overline{c_1} < \cdots < \overline{c_p}.$$

Let $ch_p(\mathcal{C})$ be the set of p-chains. Now one can replace the index set

$$\left\{ \overline{c}_* \in \prod_{i=0}^{p} \mathrm{Is}(\mathcal{C}) \;\middle|\; S(\overline{c}_*) \neq \emptyset \right\}$$

in Lemmas 3.5 and 3.7 by $ch_p(\mathcal{C})$ and replace in the definition of $S(\overline{c_*})$ the set $\mathrm{mor}_{\not\cong}(c_{i-1}, c_i)$ by $\mathrm{mor}(c_{i-1}, c_i)$.

EXAMPLE 3.10. Let $\mathcal{F}$ be a family of subgroups of the (discrete) group G. We get a bijection

$$\{(H) \mid H \in \mathcal{F}\} \xrightarrow{\cong} \mathrm{Is}(\mathrm{Or}(G, \mathcal{F})), \quad (H) \mapsto \overline{G/H},$$

where (H) denotes the conjugacy class of a subgroup $H \subseteq G$. Let NH be the normalizer and $WH = NH/H$ be the Weyl group of $H \subseteq G$. We obtain a bijection

$$WH \xrightarrow{\cong} \mathrm{aut}(G/H), \quad gH \mapsto \left(R_{g^{-1}} \colon G/H \to G/H\right),$$

where $R_{g^{-1}}$ maps $g'H$ to $g'g^{-1}H$. For

$$\left(\overline{G/H_i} \mid i = 0, 1, \ldots p\right) \in \prod_{i=0}^{p} \mathrm{Is}(\mathrm{Or}(G, \mathcal{F})),$$

we get

$$\begin{aligned} S(\overline{G/H_i} \mid i = 0, 1, \ldots p) \;=\; & \mathrm{map}_{\not\cong}(G/H_{p-1}, G/H_p)^G \times_{WH_{p-1}} \\ & \cdots \times_{WH_1} \mathrm{map}_{\not\cong}(G/H_0, G/H_1)^G, \end{aligned}$$

where $\mathrm{map}_{\not\cong}(G/H_{i-1}, G/H_i)^G$ is the set of G-maps which are not bijective.

Suppose that $\mathrm{Or}(G, \mathcal{F})$ is an EI-category. Then a p-chain $\overline{G/H_0} < \cdots < \overline{G/H_p}$ is the same as a sequence of conjugacy classes of subgroups

$$(H_0) < \cdots < (H_p),$$

where $(H_{i-1}) < (H_i)$ means that H_{i-1} is subconjugated, but not conjugated to (H_i). The WH_p-WH_0-set associated to such a p-chain is

$$S((H_0) < \cdots < (H_p)) \\ = \mathrm{map}(G/H_{p-1}, G/H_p)^G \times_{WH_{p-1}} \ldots \times_{WH_1} \mathrm{map}(G/H_0, G/H_1)^G.$$

The E^1-terms of the spectral sequences in Theorem 3.4 become

$$E^1_{p,q} = \bigoplus_{(H_0)<\cdots<(H_p)} H_q^{WH_0}(X(G/H_p) \times_{WH_p} S((H_0) < \cdots < (H_p)); \mathbf{E}(G/H_0)),$$

and

$$E_1^{p,q} = \prod_{(H_0)<\cdots<(H_p)} H^q_{WH_0}(X(G/H_p) \times_{WH_p} S((H_0) < \cdots < (H_p)); \mathbf{E}(G/H_0)),$$

where $X(G/H_p) \times_{WH_p} S((H_0) < \cdots < (H_p))$ means $X(G/H_0)$ for $p = 0$.

There is a module or chain complex version of the spectral sequence above. In the sequel we use the notation and language of [17]. Let R be a commutative associative ring with unit and $\mathcal{C}$ be a small category. One replaces the contravariant space X by a contravariant $R\mathcal{C}$-chain complex C which satisfies $C_p = 0$ for $p < 0$ and the covariant respectively contravariant spectrum $\mathbf{E}$ by a covariant respectively contravariant $R\mathcal{C}$-chain complex D which may have nontrivial chain modules in negative dimensions. The role of contravariant $\mathcal{C}$-CW-complexes is now played by projective contravariant $R\mathcal{C}$-chain complexes. The tensor product $\otimes_{\mathcal{C}}$ is replaced by the tensor product $\otimes_{R\mathcal{C}}$ of $R\mathcal{C}$-modules and the mapping space is replaced by the R-module of homomorphisms of $R\mathcal{C}$-modules $\hom_{R\mathcal{C}}$. The homology $H_p^{\mathcal{C}}(X; \mathbf{E})$ now becomes $\mathrm{Tor}_p^{R\mathcal{C}}(C, D)$ and the cohomology $H^p_{\mathcal{C}}(X; \mathbf{E})$ now becomes $\mathrm{Ext}^p_{R\mathcal{C}}(C, D)$. Notice that a $R\mathcal{C}$-module can be interpreted as a chain complex concentrated in dimension 0. The proof of the next result is analogous to the results of this section and generalizes the spectral sequence in [17, section 17].

THEOREM 3.11. *Let M be a contravariant $R\mathcal{C}$-module and N be a covariant respectively contravariant $R\mathcal{C}$-module. Suppose that $\mathcal{C}$ is left-free. Then*

(a) *There is a spectral* (*homology*) *sequence* $(E^r_{p,q}, d^r_{p,q})$ *whose E^1-term is given by*

$$E^1_{p,q} = \bigoplus_{\overline{c}_* \in \prod_{i=0}^p \mathrm{Is}(\mathcal{C})} \mathrm{Tor}_q^{R[\mathrm{aut}(c_0)]}(M(c_p) \times_{R[\mathrm{aut}(c_p)]} RS(\overline{c}_*), N(c_0)),$$

where $R[\mathrm{aut}(c_i)]$ is the group ring of $\mathrm{aut}(c_i)$ *with coefficients in R and $RS(\overline{c}_*)$ is the free R-module generated by the set $S(\overline{c}_*)$. The spectral sequence converges to* $\mathrm{Tor}^{R\mathcal{C}}_{p+q}(M, N)$;

(b) *There is a spectral (cohomology) sequence* $(E_r^{p,q}, d_r^{p,q})$ *whose* E^1*-term is given by*

$$E_1^{p,q} = \prod_{\overline{c}_* \in \prod_{i=0}^{p} \mathrm{Is}(\mathcal{C})} \mathrm{Ext}^q_{R[\mathrm{aut}(c_0)]}(M(c_p) \times_{R[\mathrm{aut}(c_p)]} RS(\overline{c}_*); N(c_0)).$$

The spectral sequence converges to $\mathrm{Ext}^{p+q}_{R\mathcal{C}}(M, N)$.

4. Assembly Maps

Let $F\colon \mathcal{B} \to \mathcal{C}$ and $\mathbf{E}\colon \mathcal{C} \to \mathsf{SPECTRA}$ be covariant functors. We introduced their assembly map

$$H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) \to H_q^{\mathcal{C}}(\star; \mathbf{E})$$

in Definition 2.2. Recall that we sometimes write $\mathbf{E}$ instead of $F^*\mathbf{E}$ to simplify notation.

Let $F\colon \mathcal{B} \to \mathcal{C}$ be a covariant functor. For any object $c \in \mathrm{Ob}\, C$, define the *undercategory* $c \downarrow F$ and the *overcategory* $F \downarrow c$ as follows. An object of $c \downarrow F$ is a pair $(b, \phi\colon c \to F(b))$ where b is an object in $\mathcal{B}$ and ϕ a morphism in $\mathcal{C}$. A morphism f from $(b, \phi\colon c \to F(b))$ to $(b', \phi'\colon c \to F(b'))$ is a morphism $f\colon b \to b'$ in $\mathcal{B}$ satisfying $F(f) \circ \phi = \phi'$. An object of $F \downarrow c$ is a pair $(b, \phi\colon F(b) \to c)$. A morphism f from $(b, \phi\colon F(b) \to c)$ to $(b', \phi'\colon F(b') \to c)$ is a morphism $f\colon b \to b'$ in $\mathcal{B}$ satisfying $\phi' \circ F(f) = \phi$. We denote the under and overcategories associated to the identity functor $F\colon \mathcal{C} \to \mathcal{C}$ by $c \downarrow \mathcal{C}$ and $\mathcal{C} \downarrow c$.

A covariant functor $F\colon \mathcal{B} \to \mathcal{C}$ is *cofinal* if for every object c of $\mathcal{C}$, the classifying space $B(c \downarrow F)$ is contractible.

For a category $\mathcal{C}$, let $E\mathcal{C}$ be the contravariant $\mathcal{C}$-space given by

$$E\mathcal{C}(?) = B(? \downarrow \mathcal{C}).$$

The $\mathcal{C}$-map $E\mathcal{C} \to \star$ is a $\mathcal{C}$-CW-approximation by [9, p. 230].

THEOREM 4.1 (Cofinality Theorem). *Let* $F\colon \mathcal{B} \to \mathcal{C}$ *be a cofinal covariant functor. Let* $\mathbf{E}\colon \mathcal{C} \to \mathsf{SPECTRA}$ *be a covariant functor. Then the assembly map*

$$H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) \to H_q^{\mathcal{C}}(\star; \mathbf{E})$$

is an isomorphism.

Proof. The proof can be found in [15, 4.4]. Here is a proof in our language. Notice that $F_*E\mathcal{B}(?)$ is a $\mathcal{C}$-CW-complex by Lemma 2.1 (b). Since there is a natural isomorphism $F_*E\mathcal{B}(?) \cong B(? \downarrow F)$ and $B(? \downarrow F)$ is contractible for each ? by cofinality, the unique map $F_*E\mathcal{B}(?) \to \star$ is a $\mathcal{C}$-CW-approximation. Using Lemma 2.1 (c), we conclude

$$H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) = H_q^{\mathcal{B}}(E\mathcal{B}; F^*\mathbf{E}) = H_q^{\mathcal{C}}(F_*E\mathcal{B}; \mathbf{E}) = H_q^{\mathcal{C}}(\star; \mathbf{E}). \qquad \square$$

For example, let $\mathcal{C}$ be a category with a final object c_0. Let $\mathcal{B}$ be the subcategory with single object c_0 and only the identity morphism. Then the inclusion functor $F\colon \mathcal{B} \to \mathcal{C}$ is cofinal, and so Theorem 4.1 shows that

$$\pi_q(\mathbf{E}(c_0)) = H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) \cong H_q^{\mathcal{C}}(\star; \mathbf{E}),$$

which is a well-known fact about homotopy colimits. This also follows from the observation that $\star_{\mathcal{C}} = \mathrm{mor}_{\mathcal{C}}(?, c_0)$ is a $\mathcal{C}$-CW-complex in this case. Note that $\mathrm{Or}(G)$ has a final object G/G.

Given a functor $F\colon \mathcal{B} \to \mathcal{C}$ and an object c in $\mathcal{C}$, there is a commutatative square of functors

$$\begin{array}{ccc} F \downarrow c & \xrightarrow{F_c} & \mathcal{C} \downarrow c \\ {\scriptstyle P_c}\big\downarrow & & {\scriptstyle Q_c}\big\downarrow \\ \mathcal{B} & \xrightarrow{F} & \mathcal{C} \end{array}$$

where

$$P_c(b, \phi\colon F(b) \to c) = b,$$
$$F_c(b, \phi\colon F(b) \to c) = (F(b), \phi\colon F(b) \to c),$$
$$Q_c(c', \phi\colon c' \to c) = c'.$$

Let $\mathbf{E}$ be a covariant $\mathcal{C}$-spectrum. Since $\mathcal{C} \downarrow c$ has a final object, namely (c, id_c), we get an identification

$$H_q^{\mathcal{C}\downarrow c}(\star; Q_c^*\mathbf{E}) = \pi_q(\mathbf{E}(c)).$$

THEOREM 4.2 (Analogue of Quillen's Theorem A). *Let n be an integer and let $\mathcal{P}$ be a set of primes. Let $F\colon \mathcal{B} \to \mathcal{C}$ and $\mathbf{E}\colon \mathcal{C} \to$SPECTRA be covariant functors. Suppose that for all objects c in $\mathcal{C}$ the assembly map induced by F_c*

$$H_q^{F\downarrow c}(\star; F_c^* Q_c^*\mathbf{E}) \to H_q^{\mathcal{C}\downarrow c}(\star; Q_c^*\mathbf{E}) = \pi_q(\mathbf{E}(c))$$

is a $\mathcal{P}$-isomorphism for all $q \leqslant n$, where a $\mathcal{P}$-isomorphism is a map of Abelian groups which becomes bijective after inverting all elements in $\mathcal{P}$.

Then the assembly map induced by F

$$H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) \to H_q^{\mathcal{C}}(\star; \mathbf{E})$$

is a $\mathcal{P}$-isomorphism for all $q \leqslant n$.

Remark 4.3. Quillen's Theorem A [25] states that for a covariant functor $F\colon \mathcal{B} \to \mathcal{C}$, if for all objects c of $\mathcal{C}$, one has that $BF_c\colon B(F \downarrow c) \to B(C \downarrow c) \simeq \star$ is a homotopy equivalence, then $BF\colon B\mathcal{B} \to B\mathcal{C}$ is a homotopy equivalence.

Theorems similar to and special cases of Theorem 4.2 are well known. Since we lack a reference and for the reader's convenience, we give a proof in our case. We need some preparation.

LEMMA 4.4. *Let X be a contravariant $(F \downarrow c)$-space. Then there is a natural homeomorphism of contravariant $\mathcal{B}$-spaces*

$$v\colon P_{c*}X(?) \xrightarrow{\cong} \coprod_{\phi \in \mathrm{mor}_{\mathcal{C}}(F(?),c)} X(?, \phi\colon F(?) \to c).$$

In particular P_{c*} maps weak equivalences of contravariant $(F \downarrow c)$-spaces to weak equivalences of contravariant $\mathcal{B}$-spaces.

Proof. A typical element in the domain of v is represented by a pair (x, f), where $x \in X(b, \psi\colon F(b) \to c)$ and $f\colon ? \to b$ is a morphism in $\mathcal{B}$. The image of it under v is given by the image of x under the map

$$X(b, \psi\colon F(b) \to c) \to X(?, \psi \circ F(f)\colon F(?) \to c)$$

which is given by X applied to the morphism

$$f\colon (?, \psi \circ F(f)\colon F(?) \to c) \to (b, \psi\colon F(b) \to c)$$

in $F \downarrow c$. One easily checks that this is consistent with the tensor-relations appearing in the definition of $P_{c*}X(?)$ as a tensor product. The map in the other direction sends $x \in X(?, \phi\colon F(?) \to c)$ to the element in $P_{c*}X$ represented by the pair $(x, \mathrm{id}_?)$. One easily checks using the tensor relation that this map is surjective and v composed with it is the identity on $\coprod_{\phi \in \mathrm{mor}_{\mathcal{C}}(F(?),c)} X(?, \phi\colon F(?) \to c)$.

Since a disjoint union of weak equivalences of spaces is again a weak equivalence of spaces, v sends weak equivalences to weak equivalences. □

The contravariant $\mathcal{B}$-space $P_{c*}E(F \downarrow c)$ is actually a contravariant $\mathcal{C}^{\mathrm{op}} \times \mathcal{B}$-space, where the covariant $\mathcal{C}$-structure comes from the functoriality in the object c in $\mathcal{C}$.

By a *homotopy $\mathcal{B}$-CW-approximation of a $\mathcal{B}$-space Y* we mean a weak homotopy equivalence $Y' \to Y$ of $\mathcal{B}$-spaces where Y' has the homotopy type of a $\mathcal{B}$-CW-complex.

LEMMA 4.5. *Let $F\colon \mathcal{B} \to \mathcal{C}$ be a covariant functor and let X be a contravariant $\mathcal{C}$-CW-complex. Then the map*

$$X \otimes_{\mathcal{C}} P_{c*}E(F \downarrow c) \to X \otimes_{\mathcal{C}} P_{c*} = X \otimes_{\mathcal{C}} \mathrm{mor}_{\mathcal{C}}(F(-), c) = F^*X$$

is a homotopy $\mathcal{B}$-CW-approximation.

Proof. Since X is a $\mathcal{C}$-CW-complex and, by Lemma 4.4, $P_{c*}E(F \downarrow c) \to P_{c*}\star$ is a weak homotopy equivalence, Theorem 3.11 of [9] shows that $X \otimes_{\mathcal{C}} P_{c*}E(F \downarrow c) \to X \otimes_{\mathcal{C}} P_{c*}\star$ is weak homotopy equivalence.

To complete the proof we will show that for any contravariant $\mathcal{C}$-CW-complex Y and any covariant functor Z from $\mathcal{C}$ into the category of contravariant $\mathcal{B}$-CW-complexes that the contravariant $\mathcal{B}$-space $Y \otimes_{\mathcal{C}} Z$ has the homotopy type of a contravariant $\mathcal{B}$-CW-complex. We first show by induction that for each $n \geqslant -1$ the $\mathcal{B}$-space $Y_n \otimes_{\mathcal{C}} Z$ has the homotopy type of a contravariant $\mathcal{B}$-CW-complex. The induction beginning $n = -1$ is trivial because of $Y_{-1} \otimes_{\mathcal{C}} Z = \emptyset$. In the induction step we can write Y_n as a pushout of contravariant $\mathcal{C}$-spaces

$$\begin{array}{ccc} \coprod_{i\in I} S^n \times \mathrm{mor}_{\mathcal{C}}(?, c_i) & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^{n+1} \times \mathrm{mor}_{\mathcal{C}}(?, c_i) & \longrightarrow & Y_{n+1} \end{array}$$

Since the functor $- \otimes_{\mathcal{C}} Z$ has a right adjoint, it is compatible with pushouts and we obtain a pushout of contravariant $\mathcal{B}$-spaces

$$\begin{array}{ccc} \coprod_{i\in I} S^n \times Z(c_i) & \longrightarrow & Y_n \otimes_{\mathcal{C}} Z \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^{n+1} \times Z(c_i) & \longrightarrow & Y_{n+1} \otimes_{\mathcal{C}} Z \end{array}$$

whose left vertical arrow is a cofibration of contravariant $\mathcal{B}$-spaces. Since the contravariant spaces occuring in the left upper, right upper and left lower corner have the homotopy type of contravariant $\mathcal{B}$-CW-complexes, the same is true for $Y_{n+1} \otimes_{\mathcal{C}} Z$. Since each inclusion $Y_n \to Y_{n+1}$ is a cofibration of contravariant $\mathcal{C}$-CW-complexes, each inclusion $Y_n \otimes_{\mathcal{C}} Z \to Y_{n+1} \otimes_{!lc} Z$ is a cofibration of contravariant $\mathcal{B}$-spaces which have th e homotopy type of contravariant $\mathcal{B}$-CW-complexes. Since $Y = \mathrm{colim}_{n\to\infty} Y_n$, then $Y \otimes_{\mathcal{C}} Z = \mathrm{colim}_{n\to\infty} Y_n \otimes_{\mathcal{C}} Z$. Hence, $Y \otimes_{\mathcal{C}} Z$ and in particular $X \otimes_{\mathcal{C}} P_{c*}E(F \downarrow c)$ have the homotopy type of a contravariant $\mathcal{B}$-CW-complex. □

Now we can prove Theorem 4.2.

Proof. We will factor the assembly map

$$\begin{array}{rll} H_q^{\mathcal{B}}(\star; F^*\mathbf{E}) & \xleftarrow{\cong} H_q^{\mathcal{B}}(F^*E\mathcal{C}; F^*\mathbf{E}) & \text{w.h. invariance} \\ & \xleftarrow{\cong} \pi_q((E\mathcal{C} \otimes_{\mathcal{C}} P_{c*}E(F \downarrow c)) \otimes_{\mathcal{B}} F^*\mathbf{E}) & \text{Lemma 4.5} \\ & = \pi_q(E\mathcal{C} \otimes_{\mathcal{C}} (P_{c*}E(F \downarrow c) \otimes_{\mathcal{B}} F^*\mathbf{E})) & \\ & = H_q^{\mathcal{C}}(\star; P_{c*}E(F \downarrow c) \otimes_{\mathcal{B}} F^*\mathbf{E}) & \\ & = H_q^{\mathcal{C}}(\star; E(F \downarrow c) \otimes_{F\downarrow c} {P_c}^* F^*\mathbf{E}) & \text{adjunction (1)} \\ & = H_q^{\mathcal{C}}(\star; E(F \downarrow c) \otimes_{F\downarrow c} {F_c}^* {Q_c}^*\mathbf{E}) & \\ & \to H_q^{\mathcal{C}}(\star; \mathbf{E}). & \end{array}$$

By assumption, the map

$$\pi_i(E(F \downarrow c) \otimes_{F\downarrow c} {F_c}^* {Q_c}^*\mathbf{E}) \to \pi_i(\mathbf{E}(c))$$

is a $\mathcal{P}$-isomorphism for each $i \leqslant n$ and each object c in $\mathcal{C}$. Now Theorem 4.2 follows from a standard comparison argument applied to the spectral homology sequence [9, Theorem 4.7 (1)]. □

Our motivation for studying assembly maps, as well as for creating the p-chain spectral sequence, is that the standard conjectures of high-dimensional topology (Farrell–Jones and Baum–Connes) can be expressed in terms of assembly maps. One problem is that applying the p-chain spectral sequence directly to $\mathrm{Or}(G, \mathcal{VC})$ can be quite unpleasant, since there are so many virtually cyclic subgroups. We wish to make several remarks aimed at applying Theorems 4.1 and 4.2 to the case where $\mathcal{B} = \mathrm{Or}(G, \mathcal{F}) \hookrightarrow \mathrm{Or}(G, \mathcal{G}) = \mathcal{C}$ where $\mathcal{F} \subseteq \mathcal{G}$ are families of subgroups of G. The aim is to 'simplify' a family to make computations easier. In the sequel *family* $\mathcal{F}$ of subgroups of G means a set of subgroups of G which is closed under conjugation.

An example is the following corollary which appeared $n = \infty$ and $\mathcal{P} = \emptyset$ in [12, Theorem A.10] and for $\mathcal{P} = \emptyset$ in [20, Theorem 2.3].

COROLLARY 4.6. *Let $\mathcal{F} \subseteq \mathcal{G}$ be families of subgroups of G. Let* $\mathbf{E}$: $\mathrm{Or}(G, \mathcal{G}) \to$ SPECTRA *be a covariant functor. For a subgroup H of G, put $\mathcal{F}_H = \{K \mid K \in \mathcal{F}$ and $K \subseteq H\}$. Let n be an integer and let $\mathcal{P}$ be a set of primes. Suppose that for all $H \in \mathcal{G} - \mathcal{F}$ and for all $q \leqslant n$, the assembly map*

$$H_q^{\mathrm{Or}(H,\mathcal{F}_H)}(\star; \mathbf{E}) \to H_q^{\mathrm{Or}(H,\mathcal{ALL})}(\star; \mathbf{E})$$

is a $\mathcal{P}$-isomorphism. Then the assembly map

$$H_q^{\mathrm{Or}(G,\mathcal{F})}(\star; \mathbf{E}) \to H_q^{\mathrm{Or}(G,\mathcal{G})}(\star; \mathbf{E})$$

is a $\mathcal{P}$-isomorphism for all $q \leqslant n$.

Proof. First note that the condition in the statement of the lemma holds for all $H \in \mathcal{G}$, since for $H \in \mathcal{F}$, $\mathrm{Or}(H, \mathcal{F}_H)$ has a final object H/H.

Then Theorem 4.2 applies where F: $\mathcal{B} = \mathrm{Or}(G, \mathcal{F}) \hookrightarrow \mathcal{C} = \mathrm{Or}(G, \mathcal{G})$, since for all $H \in \mathcal{G}$, there is an equivalence of categories

$$\begin{aligned} &\mathrm{Or}(H, \mathcal{F}_H) \to F \downarrow G/H \\ &H/K \mapsto (G/K, gK \mapsto gH). \end{aligned}$$ □

We next apply the Cofinality theorem.

LEMMA 4.7. *Let $\mathcal{F} \subseteq \mathcal{G}$ be families of subgroups of G. Suppose that for every $H \in \mathcal{G} - \mathcal{F}$ there exists $K_H \in \mathcal{F}$ with the properties that $H \subseteq K_H$ and that for any $L \in \mathcal{F}$ with $H \subseteq L$ we have $L \subseteq K_H$, in other words, K_H is the maximal element of the set $\{L \in \mathcal{F} \mid H \subseteq L\}$.*

Then the inclusion functor I: $\mathrm{Or}(G, \mathcal{F}) \to \mathrm{Or}(G, \mathcal{G})$ is cofinal.

Proof. We have to show for $G/H \in \mathrm{Or}(G, \mathcal{G})$ that $B(G/H \downarrow I)$ is contractible. Recall the well-known fact that $B\mathcal{C}$ is contractible if $\mathcal{C}$ contains an initial or final object c_0. If H belongs to $\mathcal{F}$, the object $(G/H, \mathrm{id}\colon G/H \to G/H)$ is an initial object in $G/H \downarrow I$. It remains to show for $H \in \mathcal{G} - \mathcal{F}$ that there exists a final object in $G/H \downarrow I$.

Let K_H be the maximal element of the set $\{L \in \mathcal{F} \mid H \subseteq L\}$. Let $\mathrm{pr}_H : G/H \to G/K_H$ be the canonical projection. It suffices to show for any $L \in \mathcal{F}$ and G-map $\phi\colon G/H \to G/L$ that there is a G-map $\overline{\phi}\colon G/L \to G/K_H$ satisfying $\overline{\phi} \circ \phi = \mathrm{pr}_H$. Since ϕ is surjective, $\overline{\phi}$ is uniquely determined by this property and one easily checks that the object $(G/H, \mathrm{pr}_H : G/H \to G/K_H)$ is final in $G/H \downarrow I$. Choose $\gamma \in G$ with $\phi(1H) = \gamma L$. Then $H \subseteq \gamma L \gamma^{-1}$. By assumption $\gamma L \gamma^{-1} \subseteq K_H$. Hence we can define a G-map $\overline{\phi}\colon G/L \to G/K_H$ by sending gL to $g\gamma^{-1}K_H$. One easily checks $\overline{\phi} \circ \phi = \mathrm{pr}_H$. □

COROLLARY 4.8. *Let $\mathcal{F} \subseteq \mathcal{G}$ be families of subgroups of G. Suppose that every element of $\mathcal{G} - \mathcal{F}$ is contained in a unique element of $\mathcal{F}$.*

Then the inclusion functor $I\colon$ $\mathrm{Or}(G, \mathcal{F}) \to \mathrm{Or}(G, \mathcal{G})$ is cofinal.

The next result presents a systematically way to replace the family $\mathcal{G}$ by a smaller family $\mathcal{G}'$.

COROLLARY 4.9. *Let $\mathcal{G}$ be a family of subgroups of G. Let $\mathcal{G}' \subset \mathcal{G}$ be the subfamily consisting of all maximal elements of $\mathcal{G}$, together with all elements of $\mathcal{G}$ which are contained in no or in more than one maximal elements of $\mathcal{G}$.*

Then the inclusion functor $I\colon$ $\mathrm{Or}(G, \mathcal{G}') \to \mathrm{Or}(G, \mathcal{G})$ is cofinal.

Let $H \subseteq G$ be a normal subgroup of G. Let $\{H\}$ be the family consisting of a single subgroup, namely H, and let $\mathcal{SUB}(H)$ be the family of subgroups of H. Both are indeed closed under conjugation, since H is normal. Let $\mathbf{E}$ be a covariant $\mathrm{Or}(G, \mathcal{SUB}(H))$-spectrum. The inclusion functor $I\colon$ $\mathrm{Or}(G, \{H\}) \to \mathrm{Or}(G, \mathcal{SUB}(H))$ is cofinal by Corollary 4.8. Hence, the assembly map

$$H_q^{\mathrm{Or}(G,\{H\})}(\star; I^*\mathbf{E}) \xrightarrow{\cong} H_q^{\mathrm{Or}(G,\mathcal{SUB}(H))}(\star; \mathbf{E})$$

is bijective by Theorem 4.1. Let Q be the quotient group G/H. There is an obvious isomorphism $\mathrm{Or}(G, \{H\}) \to \mathrm{Or}(Q, \mathcal{TR})$. It induces an isomorphism

$$H_q^{\mathrm{Or}(G,\{H\})}(\star; I^*\mathbf{E}) \xrightarrow{\cong} H_q^{Q}(\star; \mathbf{E}(G/H)),$$

where we equip $\mathbf{E}(G/H)$ with the obvious Q-action and $H_q^Q(\star; \mathbf{E}(G/H))$ has been introduced in (5). Thus we rediscover the isomorphism from [20, Lemma 2.6]

$$H_q^{\mathrm{Or}(G,\mathcal{SUB}(H))}(\star; \mathbf{E}) \xrightarrow{\cong} H_q^{Q}(\star; \mathbf{E}(G/H)). \tag{7}$$

In the sequel 1 denotes the trivial group.

DEFINITION 4.10. Let H be a normal subgroup of G and let Q be the quotient group. Let $\mathbf{E}$: $\mathrm{Or}(G, SUB(H)) \to$SPECTRA be a covariant functor. Then the assembly map associated to the inclusion $\mathrm{Or}(G, TR) \to \mathrm{Or}(G, SUB(H))$ gives using the identifications (5) and (7) a map

$$H_q^G(\star; \mathbf{E}(G/1)) \to H_q^Q(\star; \mathbf{E}(G/H)).$$

It is called the partial assembly map, assembling along H.

This partial assembly map can be identified with the composition

$$\pi_q\,(EG_+ \wedge_G \mathbf{E}(G/1)) \xrightarrow{\pi_q(\mathrm{id}\wedge_G \mathbf{E}(\mathrm{pr}))} \pi_q\,(EG_+ \wedge_G \mathbf{E}(G/H))$$
$$= \pi_q\left((H\backslash EG)_+ \wedge_Q \mathbf{E}(G/H)\right) \xrightarrow{\pi_q(f_+\wedge_Q \mathrm{id})} \pi_q\left(EQ_+ \wedge_Q \mathbf{E}(G/H)\right),$$

where $\mathrm{pr}\colon G/1 \to G/H$ is the projection and $f\colon H\backslash EG \to EQ$ is the up to Q-homotopy unique Q-map.

We next wish to make an analysis of a few of the terms and differentials in the p-chain spectral sequence. Recall the Weyl group $WH = NH/H$. We leave the elementary proof of the next result to the reader.

THEOREM 4.11. *Let* $\mathbf{E}\colon \mathrm{Or}(G) \to$SPECTRA *be a covariant functor. Consider the p-chain spectral sequence when $X = \star$. Then*

(a) *The summand of $E^1_{0,q}$ corresponding to a* 0*-chain of the form $\overline{G/H}$ is isomorphic to*
$$H_q^{WH}(\star; \mathbf{E}(G/H)) = \pi_q(EWH_+ \wedge_{WH} \mathbf{E}(G/H)),$$
where the WH-action on $\mathbf{E}(G/H)$ comes from the action of the automorphisms of G/H in $\mathrm{Or}(G)$.

(b) *The summand of $E^1_{1,q}$ corresponding to a* 1*-chain of the form $(\overline{G/1}, \overline{G/H})$ is isomorphic to*
$$H_q^{NH}(\star; \mathbf{E}(G/1)) = \pi_q(ENH_+ \wedge_{NH} \mathbf{E}(G/1)),$$
where the NH-action on $\mathbf{E}(G/1)$ is the restriction of the G-action which is given by the identification $G = \mathrm{aut}_{\mathrm{Or}(G)}(G/1)$.

(c) *The component of $d^1_{1,q}$ corresponding to omitting $\overline{G/1}$ from a* 1*-chain of the form $(\overline{G/1}, \overline{G/H})$ can be identified with the partial assembly map* (*see Definition* 4.10)
$$H_q^{NH}(\star; \mathbf{E}(G/1)) \to H_q^{WH}(\star; \mathbf{E}(G/H)),$$
for $H \subset NH$ and $\mathbf{E}$ considered as covariant functor $\mathrm{Or}(NH, \mathcal{SUB}(H)) \to$ SPECTRA *by sending NH/L to $\mathbf{E}(G/L)$.*

If there are no p-chains for $p > 1$, then the p-chain spectral sequence gives a long exact sequence.

COROLLARY 4.12. *Let $\mathcal{F}$ be a family of subgroups of G. Suppose that any G-map $G/H \to G/K$ is bijective, provided $H, K \in \mathcal{F}$, $H \neq 1$, $K \neq 1$.*

Let Λ be the set of conjugacy classes (H) of nontrivial subgroups in $\mathcal{F}$. Let $\mathbf{E}\colon \operatorname{Or}(G,\mathcal{F}) \to \mathsf{SPECTRA}$ *be a functor. Then there is a long exact sequence*

$$\begin{aligned} \ldots \to & \bigoplus_{(H)\in\Lambda} H_q^{NH}(\star;\mathbf{E}(G/1)) \\ \xrightarrow{pa,-i} & \left(\bigoplus_{(H)\in\Lambda} H_q^{WH}(\star;\mathbf{E}(G/H))\right) \bigoplus H_q^{G}(\star;\mathbf{E}(G/1)) \\ \xrightarrow{h\oplus a} & H_q^{\operatorname{Or}(G,\mathcal{F})}(\star;\mathbf{E}) \to \bigoplus_{(H)\in\Lambda} H_{q-1}^{NH}(\star;\mathbf{E}(G/1)) \xrightarrow{pa,-i} \ldots \end{aligned}$$

Here the map pa is a sum of partial assembly maps, i is a change of group map associated to the inclusion $NH \to G$, and a is the assembly map induced by the inclusion of categories $I\colon \operatorname{Or}(G,\mathcal{TR}) \to \operatorname{Or}(G,\mathcal{F})$ under the identification $H_q^{\operatorname{Or}(G,\mathcal{TR})}(\star; I^*\mathbf{E}) = H_q^G(\star;\mathbf{E}(G/1))$. The NH-action on $\mathbf{E}(G/1)$ is the restriction of the $G = \operatorname{aut}_{\operatorname{Or}(G)}(G/1)$-action and the WH-action on $\mathbf{E}(G/H)$ comes from the identification $WH = \operatorname{aut}_{\operatorname{Or}(G)}(G/H)$.

For the reader's convenience we restate the exact sequence appearing Corollary 4.12 in more familiar terms given by homotopy groups

$$\begin{aligned} \ldots \to & \bigoplus_{(H)} \pi_q(ENH_+ \wedge_{NH} \mathbf{E}(G/1)) \\ \xrightarrow{pa,-i} & \left(\bigoplus_{(H)} \pi_q(EWH \wedge_{WH} \mathbf{E}(G/H))\right) \bigoplus \pi_q(EG_+ \wedge_G \mathbf{E}(G/1)) \\ \xrightarrow{h\oplus a} & H_q^{\operatorname{Or}(G,\mathcal{F})}(\star;\mathbf{E}) \to \bigoplus_{(H)} \pi_{q-1}(ENH_+ \wedge_{NH} \mathbf{E}(G/1)) \xrightarrow{pa,-i} \cdots . \end{aligned}$$

An example of family $\mathcal{F}$ appearing in Corollary 4.12 is the family $\mathcal{FIN}$ of finite subgroups of a group G, provided every nontrivial finite subgroup of G has prime order. We will see other examples in Section 5.

5. Examples

In this section we illustrate how the methods of the preceding sections can be used to compute $K_q(\mathbb{Z}G)$, $L_q^{\langle-\infty\rangle}(\mathbb{Z}G)$ and $K_q(C_r^*(G))$ for certain discrete groups G explicitly.

Let $\mathcal{MFIN}$ be the subfamily of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$. Consider the following assertions concerning G:

(M) Every nontrivial finite subgroup of G is contained in a unique maximal finite subgroup;

(NM) $M \in \mathcal{MFIN} \Rightarrow NM = M$;

(VCL) If V is an infinite virtually cyclic subgroup of G, then V is isomorphic to $\mathbb{Z}$;

(BC) The Baum and Connes Conjecture for G is true, i.e. the assembly map
$$H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star;\mathbf{K}^{\mathrm{top}}) \to H_q^{\mathrm{Or}(G)}(\star;\mathbf{K}^{\mathrm{top}}) = K_q(C_r^*(G))$$
is bijective for all $q \in \mathbb{Z}$;

(FJK_N) The Isomorphism Conjecture of Farrell and Jones for algebraic K-theory is true for $\mathbb{Z}G$ in the range $q \leqslant N$ for a fixed element $N \in \mathbb{Z} \coprod \{\infty\}$, i.e. the assembly map
$$H_q^{\mathrm{Or}(G,\mathcal{VC})}(\star;\mathbf{K}^{\mathrm{alg}}) \to H_q^{\mathrm{Or}(G)}(\star;\mathbf{K}^{\mathrm{alg}}) = K_q(\mathbb{Z}G)$$
is bijective for $q \in \mathbb{Z}$ with $q \leqslant N$;

(FJL) The Isomorphism Conjecture of Farrell and Jones for algebraic L-theory is true for $\mathbb{Z}G$, i.e. the assembly map
$$H_q^{\mathrm{Or}(G,\mathcal{VC})}(\star;\mathbf{L}^{\langle-\infty\rangle}) \to H_q^{\mathrm{Or}(G)}(\star;\mathbf{L}^{\langle-\infty\rangle}) = L_q^{\langle-\infty\rangle}(\mathbb{Z}G)$$
is bijective for $q \in \mathbb{Z}$;

(FJL[1/2]) The Isomorphism Conjecture of Farrell and Jones for algebraic L-theory is true for $\mathbb{Z}G$ after inverting 2 , i.e. the assembly map
$$H_q^{\mathrm{Or}(G,\mathcal{VC})}(\star;\mathbf{L}^{\langle-\infty\rangle}) \to H_q^{\mathrm{Or}(G)}(\star;\mathbf{L}^{\langle-\infty\rangle}) = L_q^{\langle-\infty\rangle}(\mathbb{Z}G)$$
is bijective after inverting 2 for all $q \in \mathbb{Z}$.

Denote by $\underline{E}G$ *the classifying space for proper G-actions*. It is a G-CW-complex such that all of its isotropy groups are finite and the fixed point set of each finite subgroup of G is contractible. It has the universal property that for any G-CW-complex X with all isotropy groups finite, there is a G-map from X to $\underline{E}G$, and any two G-maps from X to $\underline{E}G$ are G-homotopic. This property characterizes $\underline{E}G$ up to G-homotopy.

For a group H, define $\widetilde{K}_q(\mathbb{Z}H)$ to be the cokernel of the map $K_q(\mathbb{Z}[1]) \to K_q(\mathbb{Z}H)$ induced by the inclusion of the trivial subgroup 1 into H. Define $\widetilde{L}^{\langle-\infty\rangle}(\mathbb{Z}H)$, $\widetilde{K}_q(C_r^*(H))$ and $\widetilde{K}_q(BH)$ analogously.

THEOREM 5.1. *Let $\mathbb{Z} \subseteq A \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in A. Let Λ be the set of conjugacy classes (H) of non-trivial subgroups belonging to $\mathcal{MFIN}$. Then*

(a) *If G satisfies (M), (NM) and (BC), then there are exact sequences of topological K-groups*
$$\cdots \to \bigoplus_{(H)\in\Lambda} K_q(BH) \to \left(\bigoplus_{(H)\in\Lambda} K_q(C_r^*(H))\right) \bigoplus K_q(BG)$$
$$\to K_q(C_r^*(G)) \to \bigoplus_{(H)\in\Lambda} K_{q-1}(BH) \to \cdots$$
and
$$0 \to \bigoplus_{(H)\in\Lambda} \widetilde{K}_q(C_r^*(H)) \to K_q(C_r^*(G)) \to K_q(G\backslash\underline{E}G) \to 0.$$

The maps $K_q(BH) \to K_q(C_r^*(H))$ and $K_q(BG) \to K_q(C_r^*(G))$ are the assembly maps associated to $\mathcal{TR} \subseteq \mathcal{ALL}$. The maps $K_q(BH) \to K_q(BG)$, $K_q(C_r^*(H)) \to K_q(C_r^*(G))$ and $\widetilde{K}_q(C_r^*(H)) \to K_q(C_r^*(G))$ are induced by the inclusion $H \to G$. The second sequence splits after applying $- \otimes_{\mathbb{Z}} A$.

(b) *If G satisfies* (*M*), (*NM*), (*VCL*) *and* (*FJL*), *then there are exact sequences of algebraic L-groups*

$$\cdots \to \bigoplus_{(H)\in\Lambda} H_q(BH;\mathbf{L}(\mathbb{Z})) \to \left(\bigoplus_{(H)\in\Lambda} L_q^{\langle-\infty\rangle}(\mathbb{Z}H)\right) \bigoplus H_q(BG;\mathbf{L}(\mathbb{Z}))$$
$$\to L_q^{\langle-\infty\rangle}(\mathbb{Z}G) \to \bigoplus_{(H)\in\Lambda} H_{q-1}(BH;\mathbf{L}(\mathbb{Z})) \to \cdots$$

and

$$\cdots \to H_{q+1}(G\backslash\underline{E}G;\mathbf{L}(\mathbb{Z})) \to \bigoplus_{(H)\in\Lambda} \widetilde{L}_q^{\langle-\infty\rangle}(\mathbb{Z}H)$$
$$\to L_q^{\langle-\infty\rangle}(\mathbb{Z}G) \to H_q(G\backslash\underline{E}G;\mathbf{L}(\mathbb{Z})) \to \cdots,$$

where $\mathbf{L}(\mathbb{Z}) = \mathbf{L}^{-\langle\infty\rangle}(\mathbb{Z})$ is the algebraic L-theory-spectrum associated to the ring $\mathbb{Z}$. The maps $H_q(BH;\mathbf{L}(\mathbb{Z})) \to L_q^{\langle-\infty\rangle}(\mathbb{Z}H)$ and $H_q(BG;\mathbf{L}(\mathbb{Z})) \to L_q^{\langle-\infty\rangle}(\mathbb{Z}G)$ are the assembly maps associated to $\mathcal{TR} \subseteq \mathcal{ALL}$. The maps and $H_q(BH;\mathbf{L}(\mathbb{Z})) \to H_q(BG;\mathbf{L}(\mathbb{Z}))$, $L_q^{\langle-\infty\rangle}(\mathbb{Z}H) \to L_q^{\langle-\infty\rangle}(\mathbb{Z}G)$ and $\widetilde{L}_q^{\langle-\infty\rangle}(\mathbb{Z}H) \to L_q^{\langle-\infty\rangle}(\mathbb{Z}G)$ are induced by the inclusion $H \to G$. The second sequence splits after applying $- \otimes_{\mathbb{Z}} A$, more precisely

$$L_q^{\langle-\infty\rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} A \to H_q(G\backslash\underline{E}G;\mathbf{L}(\mathbb{Z})) \otimes_{\mathbb{Z}} A$$

is a split-surjective map of A-modules.

(c) *If G satisfies* (*M*), (*NM*), *and* (*FJL*[1/2]), *then the conclusion of assertion* (*b*) *still holds if we invert* 2 *everywhere. Moreover*, *the second sequence reduces to a short exact sequence*

$$0 \to \bigoplus_{(H)\in\Lambda} \widetilde{L}_q(\mathbb{Z}H)\left[\frac{1}{2}\right] \to L_q(\mathbb{Z}G)\left[\frac{1}{2}\right] \to H_q(G\backslash\underline{E}G;\mathbf{L}(\mathbb{Z}))\left[\frac{1}{2}\right] \to 0.$$

which splits after applying $- \otimes_{\mathbb{Z}[1/2]} A\,[1/2]$.

(*Recall that the decorations in L-theory do not matter after inverting* 2).

(d) *If G satisfies* (*M*), (*NM*), *and* (FJK_N), *then there is for $q \in \mathbb{Z}$, $q \leqslant N$ an isomorphism of Whitehead groups*

$$\bigoplus_{(H)\in\Lambda} \mathrm{Wh}_q(H) \xrightarrow{\cong} \mathrm{Wh}_q(G),$$

where $\mathrm{Wh}_q(H) \to \mathrm{Wh}_q(G)$ is induced by the inclusion $H \to G$.

Proof. (a) The inclusion functor $\mathrm{Or}(G, \mathcal{TR} \cup \mathcal{MFIN}) \to \mathrm{Or}(G, \mathcal{FIN})$ is cofinal because of assumption (M) and Corollary 4.9. The Cofinality Theorem implies that the assembly map associated to the inclusion $I\colon \mathrm{Or}(G, \mathcal{TR} \cup \mathcal{MFIN}) \to \mathrm{Or}(G, \mathcal{FIN})$

$$H_q^{\mathrm{Or}(G,\mathcal{TR}\cup\mathcal{MFIN})}(\star;\mathbf{K}^{\mathrm{top}}) \xrightarrow{\cong} H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star;\mathbf{K}^{\mathrm{top}})$$

is bijective for $q \in \mathbb{Z}$. Now the first long exact sequence follows from assumptions (NM) and (BC) and Corollary 4.12.

From [20, Lemma 2.8 (c)] we obtain an exact sequence

$$\begin{aligned} \cdots &\to K_{q+1}(G\backslash \underline{E}G) \to H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \widetilde{\mathbf{K}}^{\mathrm{top}}) \\ &\to H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \mathbf{K}^{\mathrm{top}}) \to K_q(G\backslash \underline{E}G) \to \cdots \end{aligned}$$

such that the map

$$H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \mathbf{K}^{\mathrm{top}}) \otimes_{\mathbb{Z}} A \to K_q(G\backslash \underline{E}G) \otimes_{\mathbb{Z}} A$$

is a split surjective maps of A-modules. Here $\widetilde{\mathbf{K}}^{\mathrm{top}}$ is a covariant $\mathrm{Or}(G)$-spectrum satisfying $\pi_q(\widetilde{\mathbf{K}}^{\mathrm{top}}(G/H)) = \widetilde{K}_q(C_r^*(H))$. Since $\pi_q(\widetilde{\mathbf{K}}^{\mathrm{top}}(G/1))$ vanishes for all $q \in \mathbb{Z}$, we obtain from Corollaries 4.9 and 4.12 and assumption (NM) an isomorphism

$$\bigoplus_{(H)\in\Lambda} \widetilde{K}_q(C_r^*(H)) \xrightarrow{\cong} H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \widetilde{\mathbf{K}}^{\mathrm{top}}).$$

Thus we obtain a long exact sequence which splits after applying $- \otimes_{\mathbb{Z}} A$

$$\begin{aligned} \cdots &\to K_{q+1}(G\backslash \underline{E}G) \to \bigoplus_{(H)\in\Lambda} \widetilde{K}_q(C_r^*(H)) \\ &\to K_q(C_r^*(G)) \to K_q(G\backslash \underline{E}G) \to \cdots . \end{aligned}$$

Since $\widetilde{K}_q(C_r^*(H))$ is a finitely generated torsionfree Abelian group, assertion (a) follows.

(b) The proof is analogous to the one of assertion (a) except that one additionally has to prove that the assembly map associated to the inclusion $\mathrm{Or}(G, \mathcal{FIN}) \to \mathrm{Or}(G, \mathcal{VC})$ induces an isomorphism

$$H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \mathbf{L}^{\langle -\infty\rangle}) \xrightarrow{\cong} H_q^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \mathbf{L}^{\langle -\infty\rangle}).$$

Because of Corollary 4.6 and assumption (VCL) it suffices to check that the following assembly map is an isomorphism

$$H_q^{\mathrm{Or}(\mathbb{Z},\mathcal{TR})}(\star; \mathbf{L}^{\langle -\infty\rangle}) \xrightarrow{\cong} H_q^{\mathrm{Or}(\mathbb{Z})}(\star; \mathbf{L}^{\langle -\infty\rangle}).$$

This follows from the Shaneson splitting [28, Theorem 5.1], the Rothenberg sequence and the fact that $\widetilde{K}_i(\mathbb{Z}[\mathbb{Z}])$ for $i \leqslant 0$ and $\mathrm{Wh}(\mathbb{Z})$ vanish.

(c) The proof is analogous to the one of assertion (b) using the conclusion from [34] that for any virtually cyclic group V the assembly map

$$H_q^{\mathrm{Or}(V,\mathcal{FIN})}(\star; \mathbf{L}^{\langle -\infty\rangle}) \left[\frac{1}{2}\right] \xrightarrow{\cong} H_q^{\mathrm{Or}(V)}(\star; \mathbf{L}^{\langle -\infty\rangle}) \left[\frac{1}{2}\right]$$

is bijective, and the conclusion from [27, Proposition 22.34, p. 253] that $L_q(\mathbb{Z}H)[1/2]$ is a torsionfree $\mathbb{Z}[1/2]$-module for a finite group H.

(d) There are canonical identifications [20, Lemma 2.4]

$$H_q^{\mathrm{Or}(G)}(\star_{\mathrm{Or}(G)}, \star_{\mathcal{TR}}; \mathbf{K}^{\mathrm{alg}}) = \begin{cases} \mathrm{Wh}_q(G) & q \geqslant 2, \\ \mathrm{Wh}(G) = \mathrm{Wh}_1(G) & q = 1, \\ \widetilde{K}_0(\mathbb{Z}G) = \mathrm{Wh}_0(G) & q = 0, \\ K_q(\mathbb{Z}G) = \mathrm{Wh}_q(G) & q \leqslant -1, \end{cases}$$

where $\star_{\mathcal{TR}} \subseteq \star_{\mathrm{Or}(G)}$ is the contravariant sub-Or(G)-space which sends $G/1$ to the one-point space and G/H for $H \neq 1$ to the empty set. The assembly map

$$H_p^{\mathrm{Or}(V,\mathcal{FIN})}(\star; \mathbf{K}^{\mathrm{alg}}) \xrightarrow{\cong} H_p^{\mathrm{Or}(V)}(\star; \mathbf{K}^{\mathrm{alg}})$$

is an isomorphism for all $q \in \mathbb{Z}$ if V is $\mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$ (see [20, Lemma 2.5]). Let $V \subseteq G$ be an infinite virtually cyclic subgroup. Any infinite virtually cyclic group V admits an epimorphism with finite kernel F to $\mathbb{Z}$ or $\mathbb{Z}/2*\mathbb{Z}/2$. Let M be a maximal finite subgroup containing F. For $g \in NF$ we have $F \subseteq M \cap gMg^{-1}$. If F is non-trivial, assumption (M) and (NM) imply $g \in NM = M$ which contradicts $V \subseteq NF$. This shows that V is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$. Hence the assembly map

$$H_q^{\mathrm{Or}(G,\mathcal{TR}\cup\mathcal{MFIN})}(\star; \mathbf{K}^{\mathrm{alg}}) \xrightarrow{\cong} H_q^{\mathrm{Or}(G)}(\star; \mathbf{K}^{\mathrm{alg}}) = K_q(\mathbb{Z}G)$$

is an isomorphism for $q \leqslant N$ by assumption (FJK$_N$) and Corollary 4.6. This implies by a Five-Lemma argument that there is for $q \leqslant N$ an isomorphism

$$H_q^{\mathrm{Or}(G,\mathcal{TR}\cup\mathcal{MFIN})}(\star_{\mathrm{Or}(G,\mathcal{TR}\cup\mathcal{MFIN})}, \star_{\mathcal{TR}}; \mathbf{K}^{\mathrm{alg}}) \xrightarrow{\cong} \mathrm{Wh}_q(G).$$

Now apply the p-chain spectral sequence (for pairs) to the source of this map. Notice that here the E^1-term consists only of contributions by 0-chains associated to elements $H \in \mathcal{MFIN}$. This finishes the proof of Theorem 5.1. □

Remark 5.2. The claims for the topological K-theory of the complex reduced C^*-algebra carry directly over to the real reduced C^*-algebra.

One can also replace in the computations of algebraic K and L-theory the ring $\mathbb{Z}$ by some commutative ring R with unit. If R is regular and contains $\mathbb{Q}$, then the assembly map

$$H_p^{\mathrm{Or}(V,\mathcal{FIN})}(\star; \mathbf{K}^{\mathrm{alg}}) \to H_p^{\mathrm{Or}(V)}(\star; \mathbf{K}^{\mathrm{alg}}) = K_q(RV)$$

is bijective for any virtually cyclic group V, essentially because RV is a regular ring for any virtually cyclic group V and $\mathbb{Q} \subseteq R$ and hence all Nil-terms vanish. By Corollary 4.6 the assembly map

$$H_p^{\mathrm{Or}(G,\mathcal{FIN})}(\star; \mathbf{K}^{\mathrm{alg}}) \to H_p^{\mathrm{Or}(G,\mathcal{VC})}(\star; \mathbf{K}^{\mathrm{alg}})$$

is bijective and one is for the computation of $K_q(RG)$ in the same fortunate situation as in $K_p(C_r^*(G))$, where it suffices to consider $\mathcal{FIN}$ instead of $\mathcal{VC}$.

Now we give some groups for which Theorem 5.1 applies:

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$.

  The conditions (M), (NM), (BC) and (FJK_1) and (FJL[1/2]) are satisfied by [20, Theorem 1.2, Lemmas 6.1 and 6.3]. Hence the conclusions appearing in assertions (a), (c) and (d) for $N = 1$ of Theorem 5.1 holds for G. This has already been proved in [20, Theorem 0.2], where further information is given. The L-groups of F are computed in [20, Remark 6.4] for all decorations. There a term UNIL appears which has meanwhile been computed by Connolly and Davis [8];
- Fuchsian groups F

  The conditions (M), (NM), (BC) and (FJK_1) and (FJL[1/2]) are satisfied for F (see for instance [20, Theorem 1.2 and Lemma 4.5]). Hence the conclusions appearing in assertions (a), (c) and (d) for $N = 1$ hold for F. The L-groups of F are computed in [20, Remark 4.10] for all decorations. There a term UNIL appears which has meanwhile been computed by Connolly and Davis [8]. The computation of $\mathrm{Wh}_q(F)$ for a Fuchsian groups F for $q \leqslant 1$ was independently carried out in [3, 4] using the p-chain spectral sequence. In [20] the larger class of cocompact planar groups (sometimes also called cocompact NEC-groups) is treated. The p-chain spectral sequence was applied in [11] to prove the Gromov–Lawson–Rosenberg positive scalar curvature conjecture for manifolds whose fundamental group is Fuchsian;
- One-relator groups G

  Let G be a one-relator group. Let $G = \langle (g_i)_{i \in I} \mid r \rangle$ be a presentation with one relation. Then (BC) is satisfied [5]. We do not know whether (FJK_N) or (FJL) hold for G in general, and we will assume in the discussion below that they do.

  We begin with the case, where G is torsionfree. Then there is a 2-dimensional model for BG which is given by the 2-dimensional CW-complex associated to any presentation with one generator [21, Chapter III, §§9–11]. We obtain isomorphisms
$$K_p(BG) \xrightarrow{\cong} K_p(C_r^*(G)),$$
$$H_q(BG; \mathbf{K}(\mathbb{Z})) \xrightarrow{\cong} K_p(\mathbb{Z}G),$$
$$H_q(BG; \mathbf{L}(\mathbb{Z})) \xrightarrow{\cong} L_p(\mathbb{Z}G).$$
  Now suppose that G is not torsionfree. Let F be the free group with basis $\{g_i \mid i \in I\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geqslant 2$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in G$ represented by s has order m, any finite subgroup of G is subconjugated to C and for any $g \in G$ the implication $g^{-1}Cg \cap C \neq 1 \Rightarrow g \in C$ holds. These claims follows from [21, Propositions 5.17–5.19 in II.5, pp. 107, 108]. Hence G satisfies (M) and (NM) and up to conjugation there is precisely one

maximal finite subgroup, namely C. Hence the conclusion in assertion (a) of Theorem 4.1 holds for G (see also [22]). If we assume (FJL[1/2]) or (FJK$_N$) respectively, this is also true for assertions (c), and (d) of Theorem 5.1.

From now on suppose that (FJL) holds and that $m = |C|$ has odd order. Then (b) of Theorem 5.1 is true. Since the $\mathbb{Z}[1/2]$ module $\widetilde{L}_q^{-\langle -\infty\rangle}(\mathbb{Z}C)[1/2]$ is finitely generated free, we get a short exact sequence

$$0 \to \widetilde{L}_q^{\langle -\infty\rangle}(\mathbb{Z}C) \to L_q^{\langle -\infty\rangle}(\mathbb{Z}G) \to H_q(G\backslash\underline{E}G; \mathbf{L}(\mathbb{Z})) \to 0,$$

which splits after inverting m. One easily checks that it induces a short exact sequence

$$0 \to L_q^{\langle -\infty\rangle}(\mathbb{Z}C) \to L_q^{\langle -\infty\rangle}(\mathbb{Z}G) \to H_q(G\backslash\underline{E}G, *; \mathbf{L}(\mathbb{Z})) \to 0,$$

which splits after inverting m.

We mention that there is a 2-dimensional CW-model for $G\backslash\underline{E}G$ such that there is precisely one 0-cell, precisely one 2-cell and there is a bijective correspondence between the 1-cells and the index set I. This follows from [7, Exercise 2 (c), II. 5, p. 44]. The Atiyah–Hirzebruch spectral sequence yields isomorphisms

$$H_q(G\backslash\underline{E}G, *; \mathbf{L}(\mathbb{Z})) \cong \begin{cases} H_2(G\backslash\underline{E}G; \mathbb{Z}/2) & q = 0(4), \\ H_1(G\backslash\underline{E}G; \mathbb{Z}) & q = 1(4), \\ H_2(G\backslash\underline{E}G; \mathbb{Z}) & q = 2(4), \\ H_1(G\backslash\underline{E}G; \mathbb{Z}/2) & q = 3(4). \end{cases}$$

We mention without giving the proofs the following facts. The Abelian groups $H_2(G\backslash\underline{E}G; \mathbb{Z})$ and $H_2(G\backslash\underline{E}G; \mathbb{Z}/2)$ vanish and there is an exact sequence $0 \to H_1(C; \mathbb{Z}) \to H_1(BG; \mathbb{Z}) \to H_1(G\backslash\underline{E}G; \mathbb{Z}) \to 0$, provided that the relation $r \in F$ does not belong to the commutator $[F, F]$. We have $H_2(G\backslash\underline{E}G; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(G\backslash\underline{E}G; \mathbb{Z}/2) \cong \mathbb{Z}/2$, and $H_1(BG; \mathbb{Z})) \cong H_1(G\backslash\underline{E}G; \mathbb{Z})$ is a free Abelian group of rank $|I| - 1$, provided that $r \in [F, F]$.

There are a number of computations where G is a two-dimensional crystallographic group. The groups $\mathrm{Wh}_q(G)$ for $q \leqslant 1$ were computed in [23], the groups $K_0(\mathbb{C}G)$ were computed in [24], and the groups $K_q(C_*^r(G))$ for $q \in \mathbb{Z}$ were computed in [35] (see also [20, 24]). These computations were done in a unified way in [20, Section 5], as well as $\mathrm{Wh}_2(G)$ and $L_q(\mathbb{Z}G)[1/2]$ for $q \in \mathbb{Z}$.

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