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A REMARK ON "HOMOTOPY EQUIVALENCES AND FREE MODULES"

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GIVEN A FINITE group π and an integer r prime to $|\pi|$ (= order of π), define the Swan module (r, N) to be the submodule of $\mathbb{Z}\pi$ generated by $r \cdot e$ and the norm element $N = \sum_{g \in \pi} g$. The isomorphism class of (r, N) depends only on the residue class of r modulo $|\pi|$[3]. In this note we point out that an application of a lemma of R. Swan answers a question of S. Plotnick [2].

THEOREM. *Suppose π acts freely on S^n and $\alpha \in Aut\ \pi$ induces multiplication by r on $H_n(\pi; \mathbb{Z}) = \mathbb{Z}/|\pi|$. Then $(r, N) \oplus \mathbb{Z}\pi \cong \mathbb{Z}\pi \oplus \mathbb{Z}\pi$.*

From the results of [2], we have

COROLLARY. *Let π act freely on S^n. Let Y be the quotient space S^n/π with two or more points removed. Then for every element $\alpha \in Aut\ \pi$ there is a based homotopy equivalence $f: (Y, *) \to (Y, *)$ such that $\alpha = f_*: \pi_1(Y, *) \to \pi_1(Y, *)$.*

A finite group π has a free resolution of period $n + 1$ if there is an exact sequence

$$0 \to \mathbb{Z} \to F_n \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}\pi$ modules where the F_i are finitely generated free. Here π acts trivially on the two $\mathbb{Z}$ terms.

LEMMA 1. *If π acts freely on S^n and n is odd, then π has a free resolution of period $n + 1$.*

Proof. S^n/π is a closed manifold so it has the homotopy type of a finite CW complex X with $\dim X = n$ [1, p. 136]. Let F_i be the cellular i-chains of the universal cover of X.

LEMMA 2. *Suppose π has a free resolution of period $n + 1$. Let $\alpha \in Aut\ \pi$ induce multiplication by r on $H_n(\pi; \mathbb{Z}) = \mathbb{Z}/|\pi|$. Then (r, N) is stably free.*

Proof. Given a $\mathbb{Z}\pi$ module M, let $\alpha^{\#}M$ be the $\mathbb{Z}\pi$ module given by

$$\pi \times M \to M$$

$$(g, m) \to \alpha(g)m.$$

We can compute the effect of α on $H_n(\pi; \mathbb{Z})$ by finding a chain map

$$\begin{array}{ccccccc} 0 \to \mathbb{Z} & \longrightarrow & F_n \to \cdots & \longrightarrow & F_0 & \to \mathbb{Z} \to 0 \\ \downarrow s & & \downarrow & & \downarrow & \downarrow 1 \\ 0 \to \mathbb{Z} & \to & \alpha^{\#} F_n \to \cdots & \to & \alpha^{\#} F_0 & \to \mathbb{Z} \to 0. \end{array}$$

Then $s \equiv r \pmod{|\pi|}$. By Lemma 7.3 of Swan[3],

$$\sum_{i=0}^{n} (-1)^i [F_i] - \sum_{i=0}^{n} (-1)^i [\alpha^* F_i] = (-1)^{n+1} (s, N) \in \tilde{K}_0(\mathbb{Z}\pi).$$

Hence $(r, N) \cong (s, N)$ is stably free.

If P is stably free over $\mathbb{Z}\pi$ (π finite), then $P \oplus \mathbb{Z}\pi$ is free[4]. The theorem now follows.

For a finite group π, $H_n(\pi; \mathbb{Z}) = \mathbb{Z}/|\pi|$ is equivalent to the existence of a projective resolution of period $n + 1$. Does Lemma 2 hold for such a group? This involves checking whether the Euler characteristic of the resolution in $\tilde{K}_0(\mathbb{Z}\pi)$ is invariant under automorphisms of π. Such a question is likely to be quite difficult.

REFERENCES

1. R. C. KIRBY and L. C. SIEBENMANN: Foundational essays on topological manifolds, smoothings and triangulations. *Ann. of Math Studies*, Vol. 88. Princeton University Press (1977).
2. S. PLOTNICK: Homotopy equivalences and free modules. *Topology* **21**, (1982), 91–99.
3. R. G. SWAN: Periodic resolutions for finite groups. *Ann. of Math.* **72**, (1960), 267–291.
4. R. G. SWAN: Strong approximation and locally free modules. *Ring Theory and Algebra III*. Markel Decker, New York (1980).

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