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Relative semicharacteristic classes

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§0. *Introduction*

In 1973 Ronnie Lee introduced the notion of semicharacteristic classes, which are invariants of the bordism group $\mathfrak{N}_*(B\pi)$ of closed manifolds equipped with a free action of a finite group π. In this paper we relativize his theory. Associated to a homomorphism $G \to \pi$ of finite groups, there is the relative bordism group $\mathfrak{N}_*(BG \to B\pi)$, which is the bordism group of compact manifolds M with a free π-action, so that the action on ∂M is induced from a free G-action, i.e. $\partial M = \pi \times_G N$ for some manifold N with a free G-action. The invariants defined here are invariants of this relative group.

Lee introduced his invariants to give homological restrictions on closed manifolds equipped with a free action of the dihedral group D_{2r}. We will apply our theory to study certain non-free actions. In §2 relative semicharacteristic classes are applied to prove:

THEOREM A. *Let D_{2r} ($r > 1$, r odd) act smoothly on a closed manifold M^{2m+1}, so that the generator g of $\mathbb{Z}_r \subset D_{2r}$ has no fixed points. Then if*

$$\sum_{i=0}^{m} \operatorname{trace}(g_*: H_i(M; \mathbb{F}_2) \to H_i(M; \mathbb{F}_2)) = 1 \in \mathbb{F}_2,$$

the singular set of the action has dimension $\geqslant m$. More precisely, if $\mathscr{S}$ denotes the singular set, then $H_m(\mathscr{S}; \mathbb{F}_2) \neq 0$.

COROLLARY B. *If D_{2r} ($r > 1$, r odd) acts smoothly on a closed manifold M^{2m+1} which has the* mod 2 *homology of a sphere S^{2m+1}, projective space $\mathbb{R}P^{4k+1}$, or a quaternionic space form $S^{8k+3}/Q(2^n)$ ($n \geqslant 3$), so that $\mathbb{Z}_r \subset D_{2r}$ acts freely, then the fixed set of the involution T ($\langle T \rangle = \mathbb{Z}_2 \subset D_{2r}$) has a component of dimension $\geqslant m$.*

Corollary B implies that there are no free dihedral actions on mod 2 homology spheres, a fact which was first proved by J. Milnor. R. Lee proved the general case of Theorem A for *free* dihedral actions. In the case where M is a mod 2 homology sphere, Smith theory implies that the fixed set M^T is a mod 2 homology sphere; since the singular set $\mathscr{S} = D_{2r} \times_{\mathbb{Z}_2} M^T$, all components of the singular set have the same dimension k. Thus the condition $H_m(\mathscr{S}; \mathbb{F}_2) \neq 0$ implies that $k = m$, and we are led to a new proof of a 'regularity' result of D. Montgomery and C. T. Yang:

COROLLARY C. *If D_{2r} (r odd, $r > 1$) acts smoothly on a* mod 2 *homology sphere M^{2m+1} such that $\mathbb{Z}_r \subset D_{2r}$ acts freely, then the singular set is m-dimensional.*

§1. *Relative semicharacteristic classes*

Let π be a finite group. Fix a field K of characteristic 2. Let $K\pi$ denote the group ring.

Definition 1·1. A symmetric K-bilinear form ϕ: $V \times V \to K$ on a finitely generated $K\pi$-module V is *Lee even* if

(i) $\phi(gx, gy) = \phi(x, y)$ for all $x, y \in V$, $g \in \pi$,

(ii) $\phi(tx, x) = 0$ for all $x \in V$ and for all elements t of π of order 2 (i.e. $t^2 = e$, $t \neq e$).

If ϕ is non-singular, we say the module V is *Lee even.*

Example (see [**1**]). If π acts freely on a compact manifold W^{2n}, then the intersection pairing $\phi: H_n(W; K) \times H_n(W; K) \to K$ is Lee even.

R. Lee defined the following Grothendieck group.

Definition 1·2. $\tilde{R}_{\mathrm{GL,\,ev}}(\pi)$ is the abelian group with:

Generators: $[V]$, isomorphism classes of finitely generated $K\pi$-modules;

Relations: (*a*) if $0 \to V' \to V \to V'' \to 0$ is an exact sequence of finitely generated $K\pi$-modules, then $[V]-[V']-[V''] = 0$,

(*b*) $[K\pi] = 0$,

(*c*) if V is Lee even, then $[V] = 0$.

Definition 1·3. If M^{2m+1} is a compact manifold with a π-action, the *Lee semi-characteristic* is

$$\chi_{\frac{1}{2}}(M; K) = \sum_{i=0}^{m} (-1)^i [H_i(M; K)] \in \tilde{R}_{\mathrm{GL,\,ev}}(\pi).$$

Let $\mathfrak{N}_n(B\pi)$ denote the unoriented bordism group of pairs $(M^n, f\colon M^n \to B\pi)$ where M^n is a closed manifold. Let $\tilde{M}$ denote the induced π-cover of M. Equivalently $\mathfrak{N}_n(B\pi)$ is the unoriented bordism group of closed manifolds $\tilde{M}$ equipped with a free π-action.

THEOREM 1·4 (Lee). *The map* $(M, f) \mapsto \chi_{\frac{1}{2}}(\tilde{M}; K)$ *gives a well-defined homomorphism* $\chi_{\frac{1}{2}}\colon \mathfrak{N}_{2m+1}(B\pi) \to \tilde{R}_{\mathrm{GL,\,ev}}(\pi)$.

Now for the relative version of the theory.

Definition 1·5. Let $\varphi\colon G \to \pi$ be a homomorphism of finite groups. Then

$$\tilde{R}_{\mathrm{GL,\,ev}}(G \to \pi) = \tilde{R}_{\mathrm{GL,\,ev}}(\pi)/\langle [K\pi \otimes_{KG} V]\colon V \text{ is a } f.g.\ KG\text{-module}\rangle.$$

If φ is an injection, then $\tilde{R}_{\mathrm{GL,\,ev}}(G \to \pi) = \operatorname{coker}(\tilde{\varphi})$ where $\tilde{\varphi}\colon R_{\mathrm{GL,\,ev}}(G) \to R_{\mathrm{GL,\,ev}}(\pi)$ is defined by $\tilde{\varphi}[V] = [K\pi \otimes_{KG} V]$. In particular, if G is the trivial group then the relative group reduces to $R_{\mathrm{GL,\,ev}}(\pi)$.

Let $\mathfrak{N}_n(BG \to B\pi)$ denote the bordism group of triples

$$(M^n, f\colon M \to B\pi, f_\partial\colon \partial M \to BG)$$

where M is a compact n-dimensional manifold with boundary ∂M and

$$\begin{array}{ccc} \partial M & \hookrightarrow & M \\ \downarrow{\scriptstyle f_\partial} & & \downarrow{\scriptstyle f} \\ BG & \longrightarrow & B\pi \end{array}$$

commutes up to homotopy. Equivalently, we are given a compact manifold $\tilde{M}$ with a free π-action, so that the action on $\partial\tilde{M}$ is induced from a G-action, i.e. $\partial\tilde{M} = \pi \times_G N$

for some manifold N with a free G-action. (Here $\pi \times_G N = \pi \times N/(\rho, gn) \sim (\rho\varphi(g), n)$.) A triple (M, f, f_∂) is zero in $\mathfrak{N}_n(BG \to B\pi)$ if there exists a quadruple $(W^{n+1}, \partial', g\colon W \to B\pi, g_{\partial'}\colon \partial' \to BG)$ where W and ∂' are compact manifolds, $\partial W = \partial' \cup M, \partial\partial' = \partial M = \partial' \cap M$, and the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} \partial M & \hookrightarrow & & \longrightarrow & M \\ \downarrow & \searrow & & \swarrow & \downarrow \\ & BG & \longrightarrow & B\pi & \\ \downarrow & \nearrow & & \nwarrow & \downarrow \\ \partial' & \hookrightarrow & & \longrightarrow & W. \end{array}$$

There is the familiar long exact sequence

$$\ldots \to \mathfrak{N}_n(BG) \to \mathfrak{N}_n(B\pi) \to \mathfrak{N}_n(BG \to B\pi) \to \mathfrak{N}_{n-1}(BG) \to \ldots$$

Definition 1·6. A triple $(M^{2m+1}, f\colon M \to B\pi, f_\partial\colon \partial M \to BG)$ is *induced* if

$$\ker(i_*\colon H_m(\partial\tilde{M}; K) \to H_m(\tilde{M}; K)) = K\pi \otimes_{KG} V$$

for some KG-submodule V of $H_m(\overline{\partial M}; K)$. Here $\tilde{M}$ denotes the π-cover induced by f and $\overline{\partial M}$ denotes the G-cover induced by f_∂. The equality above is interpreted by identifying $H_m(\partial\tilde{M}; K)$ with the induced module $K\pi \otimes_{KG} H_m(\overline{\partial M}; K)$. Let $\mathfrak{N}I_{2m+1}(BG \to B\pi)$ denote the subgroup of $\mathfrak{N}_{2m+1}(BG \to B\pi)$ generated by induced triples.

The most common example of an induced triple would be when i_* is injective.

THEOREM 1·7. *There is a homomorphism*

$$\chi_{\frac{1}{2}}\colon \mathfrak{N}I_{2m+1}(BG \to B\pi) \to \tilde{R}_{\mathrm{GL,\,ev}}(G \to \pi)$$

defined by sending any induced triple (M, f, f_∂) *representing a bordism class to* $\chi_{\frac{1}{2}}(\tilde{M}; K)$.

Proof. It suffices to show that $\chi_{\frac{1}{2}}(\tilde{M}; K) = 0 \in \tilde{R}_{\mathrm{GL,\,ev}}(G \to \pi)$ for an induced triple (M, f, f_∂) which is the boundary of a quadruple $(W, \partial', g, g_{\partial'})$. By Lee's result $\chi_{\frac{1}{2}}(\partial\tilde{W}; K) = 0 \in \tilde{R}_{\mathrm{GL,\,ev}}(\pi)$. Consider the Mayer–Vietoris exact sequence

$$0 \to \ker\alpha \cap \ker\beta \to H_m(\partial\tilde{M}; K) \xrightarrow{\alpha\oplus\beta} H_m(\tilde{M}; K) \oplus H_m(\tilde{\partial}'; K)$$
$$\to H_m(\partial\tilde{W}; K) \to \ldots \to H_0(\partial\tilde{W}; K) \to 0.$$

(Here the tilde always represents the induced π-cover.) The Euler characteristic of an exact sequence is zero, hence $\chi_{\frac{1}{2}}(\tilde{M}; K)$ equals

$$-\chi_{\frac{1}{2}}(\tilde{\partial}'; K) + \sum(-1)^i [H_i(\partial\tilde{M}; K)] + (-1)^{m+1}[\ker\alpha \cap \ker\beta] + \chi_{\frac{1}{2}}(\partial\tilde{W}; K),$$

which is zero in $\tilde{R}_{\mathrm{GL,\,ev}}(G \to \pi)$ since all the modules involved in the first three terms are induced.

Remark. An analogous theorem could be proved for oriented bordism groups with the target of the homomorphism being a relative version of Lee's orthogonal or symplectic Grothendieck groups.

§2. *Proof of Theorem A*

Let $D_{2r} = \langle g, T \mid g^r = T^2 = e, TgT = g^{-1}\rangle$ denote the dihedral group. It is a semi-direct product $D_{2r} = \mathbb{Z}_r \rtimes \mathbb{Z}_2 = \langle g\rangle \rtimes \langle T\rangle$.

Given an action of π on X we define the singular set

$$\mathscr{S} = \mathscr{S}(X,\pi) = \{x \in X \mid \quad gx = x \quad \text{for some} \quad g \in \pi - \{e\}\}.$$

LEMMA 2·1. *If D_{2r} (r odd) acts on a space M so that $\mathbb{Z}_r$ acts freely, then $\mathscr{S}(M, D_{2r}) = D_{2r} \times_{\mathbb{Z}_2} \mathscr{S}(M, \mathbb{Z}_2)$. If in addition the action is smooth on a closed manifold M and $T(\mathscr{S})$ is a D_{2r}-invariant closed tubular neighbourhood of $\mathscr{S}(M, D_{2r})$, then $(M - \operatorname{int} T(\mathscr{S}))/D_{2r}$ represents an element of $\mathfrak{N}_n(B\mathbb{Z}_2 \to BD_{2r})$.*

Proof. A map $f\colon D_{2r} \times_{\mathbb{Z}_2} \mathscr{S}(M, \mathbb{Z}_2) \to \mathscr{S}(M, D_{2r})$ is given by $f(\rho, x) = \rho x$. We define f^{-1} as follows. For any $x \in \mathscr{S}(M, D_{2r})$, $g^i Tx = x$ for some $g^i \in \mathbb{Z}_r$. Furthermore, this element g^i is unique, since $g^i Tx = g^k Tx$ implies that $g^i = g^k$ since $\mathbb{Z}_r$ acts freely. Choosing j so that $2j \equiv i (\operatorname{mod} r)$ one easily checks that $Tg^{-j}x = g^{-j}x$, and we define $f^{-1}(x) = (g^j, g^{-j}x)$. One sees directly that f and f^{-1} are inverses. If the action is smooth on a manifold then

$$f\colon D_{2r} \times_{\mathbb{Z}_2} T(\mathscr{S}(M, \mathbb{Z}_2)) \to T(\mathscr{S}(M, D_{2r}))$$

is also a homeomorphism. The lemma follows.

LEMMA 2·2. *If $\pi = H \rtimes G$ with H of odd order then $\mathfrak{N}_*(BG \to B\pi) = 0$.*

Proof. If $\pi_1 \to \pi_2$ induces an isomorphism $H_*(\pi_1; \mathbb{Z}_{(2)}) \xrightarrow{\sim} H_*(\pi_2; \mathbb{Z}_{(2)})$ and $\mathscr{H}$ is a generalized homology theory, then $\mathscr{H}_*(B\pi_1) \otimes \mathbb{Z}_{(2)} \to \mathscr{H}_*(B\pi_2) \otimes \mathbb{Z}_{(2)}$ is an isomorphism. Indeed the map induces a $\mathbb{Z}_{(2)}$-isomorphism on the E_2-term of the Atiyah–Hirzebruch spectral sequence and hence on E_∞. Unoriented bordism $\mathfrak{N}_*$ is a generalized homology theory and $\mathfrak{N}_* = \mathfrak{N}_* \otimes \mathbb{Z}_{(2)}$ since it is 2-torsion.

Thus we must show $i_*\colon H_*(G; \mathbb{Z}_{(2)}) \to H_*(\pi; \mathbb{Z}_{(2)})$ is an isomorphism. Since, in the exact sequence

$$1 \to H \to \pi \xrightarrow{p} G \to 1,$$

p is split by $i\colon G \to \pi$, i_* is an injection. The order of H is odd, so a transfer argument shows $H_*(H; \mathbb{Z}_{(2)}) = 0$. The Lyndon–Hochschild–Serre spectral sequence then shows $p_*\colon H_*(\pi; \mathbb{Z}_{(2)}) \to H_*(G; \mathbb{Z}_{(2)})$ is an isomorphism, so i_* is surjective.

So the plan of attack on Theorem A is clear; one needs to compute $\tilde{R}_{\mathrm{Gl,ev}}(\mathbb{Z}_2 \to D_{2r})$ and $\chi_{\frac{1}{2}}(M - \operatorname{int} T(\mathscr{S}); K)$.

LEMMA 2·3. *Let V be a finitely generated KD_{2r}-module with r odd. Then $[V] = 0 \in \tilde{R}_{\mathrm{Gl,ev}}(\mathbb{Z}_2 \to D_{2r})$ if and only if $\operatorname{tr}(h\colon V \to V) = 0 \in K$ for all $h \in \mathbb{Z}_r - \{e\}$.*

Remark. Although this is true for any field K of characteristic 2, for simplicity we will only prove and use this in the case where K contains a primitive r-th root of unity ζ.

Proof. There is one 1-dimensional representation of D_{2r}, the trivial representation, denoted V_0. There are $(r-1)/2$ irreducible representations V_i $(1 \leqslant i \leqslant (r-1)/2)$ defined by

$$g \to \begin{bmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{bmatrix}$$
$$T \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The computations of Lee then show

$$\tilde{R}_{\mathrm{GL,ev}}(D_{2r}) = (\mathbb{Z}_2)^{(r+1)/2}$$

with generators $[V_0], [V_1], \ldots, [V_{(r-1)/2}]$. A composition series for any $K\mathbb{Z}_2$-module has all factors isomorphic to K and we know $KD_{2r} \otimes_{K\mathbb{Z}_2} K \cong V_0 \oplus V_1 \oplus \ldots \oplus V_{(r-1)/2}$. It follows that

$$\tilde{R}_{\mathrm{GL,ev}}(\mathbb{Z}_2 \to D_{2r}) = (\mathbb{Z}_2)^{(r+1)/2}/\Delta(\mathbb{Z}_2),$$

where $\Delta: \mathbb{Z}_2 \to (\mathbb{Z}_2)^{(r+1)/2}$ is the diagonal map.

Let V be a finitely generated KD_{2r}-module, choose a composition series for V, and let a_i be the number of times V_i occurs as a factor module. Let

$$\bar{a} = (a_0, a_1, a_2, \ldots, a_{(r-1)/2}, a_{(r-1)/2}, \ldots, a_2, a_1).$$

Then $\operatorname{tr}(g^k: V \to V)$ is the kth component of $M\bar{a}$, where

$$M = \begin{bmatrix} 1 & \zeta & \ldots & \zeta^{r-1} \\ 1 & \zeta^2 & & \zeta^{2(r-1)} \\ \vdots & & & \\ 1 & \zeta^r & & \zeta^{r(r-1)} \end{bmatrix}.$$

M is a van der Monde determinant; in particular it is non-singular. The matrix obtained by deleting the last row thus has 1-dimensional kernel, generated by $(1, 1, \ldots, 1)$. So $\operatorname{tr}(g^k: V \to V) = 0 \in K$ for all $1 \leqslant k \leqslant r-1$ if and only if $\bar{a} \equiv (1, 1, \ldots, 1)$, which occurs if and only if $[V] = 0 \in \tilde{R}_{\mathrm{Gl,ev}}(\mathbb{Z}_2 \to D_{2r})$.

COROLLARY 2·4. *Let D_{2r} $(r > 1, r$ odd$)$ act freely and topologically on a compact topological manifold M^{2m+1}, so that the action on ∂M is induced from a $\mathbb{Z}_2$-action and so that $i_*: H_m(\partial\tilde{M}; \mathbb{F}_2) \to H_m(\tilde{M}; \mathbb{F}_2)$ is injective. Then*

$$\sum_{i=0}^{m} \operatorname{trace}(g_*: H_i(M; \mathbb{F}_2) \to H_i(M; \mathbb{F}_2)) = 0 \in \mathbb{F}_2.$$

Proof. By 1·7, 2·2, and 2·3 the above sum must be zero with $\mathbb{F}_2$ replaced by $\mathbb{F}_2[\zeta]$. But since trace is invariant under extension of fields it must also be zero with $\mathbb{F}_2$-coefficients.

Proof of Theorem A. Throughout the proof homology with K-coefficients is understood. Suppose D_{2r} acts smoothly on M^{2m+1} so that $\mathbb{Z}_r$ acts freely. We will assume that $H_m(\mathscr{S}) = 0$ and will show

$$\chi_{\frac{1}{2}}(M) = 0 \in \tilde{R}_{\mathrm{Gl,ev}}(\mathbb{Z}_2 \to D_{2r}), \tag{$*$}$$

which implies by 2·3 that

$$\sum_{i=0}^{m} \operatorname{trace}(g_*: H_i(M) \to H_i(M)) = 0 \in K$$

and Theorem A follows.

Let $\mathscr{S}^k$ denote the k-dimensional component of the singular set. The boundary of $T(\mathscr{S}^k)$ is an S^{2m-k}-bundle over $\mathscr{S}^k$; from the Gysin sequence it follows that $H_m(\partial T(\mathscr{S}^k)) = 0$ since $H_m(\mathscr{S}^k) \cong H_{k-m}(\mathscr{S}^k) = 0$. By this and 2·1 $M - \operatorname{int} T(\mathscr{S})$ is induced, so by 1·7 and 2·2

$$\chi_{\frac{1}{2}}(M - \mathscr{S}) = 0 \in \tilde{R}_{\mathrm{Gl,ev}}(\mathbb{Z}_2 \to D_{2r}).$$

By the assumption on $\mathscr{S}$, $H^m(\mathscr{S}) = 0$, so there is an exact sequence

$$0 \to H^{m+1}(M, \mathscr{S}) \to H^{m+1}(M) \to H^{m+1}(\mathscr{S}) \to \ldots \to H^{2m+1}(M) \to 0.$$

Thus

$$\sum_{i=m+1}^{2m+1} (-1)^i [H^i(M, \mathscr{S})] - \sum_{i=m+1}^{2m+1} (-1)^i [H^i(M)] + \sum_{i=m+1}^{2m+1} (-1)^i [H^i(\mathscr{S})] = 0.$$

Since the action on $\mathscr{S}$ is induced from a $\mathbb{Z}_2$-action, the last term vanishes. By Alexander–Poincaré duality the first two terms are equal to $\chi_{\frac{1}{2}}(M-\mathscr{S})$ and $\chi_{\frac{1}{2}}(M)$ respectively. Thus (*) holds and Theorem A follows.

Example. Let D_{2r} act on S^1 as symmetries of the regular n-gon, and act diagonally on the join

$$S^{2m+1} = S^1 * S^1 * \ldots * S^1.$$

The fixed set of an involution T is

$$S^m = S^0 * S^0 * \ldots * S^0.$$

It is easily verified that $M = S^{2m+1} - \operatorname{int} T(\mathscr{S})$ is not induced and $\chi_{\frac{1}{2}}(M; K) \neq 0 \in \tilde{R}_{\mathrm{Gl,\,ev}}(\mathbb{Z}_2 \to D_{2r})$ despite the fact that $\mathfrak{N}_*(\mathbb{Z}_2 \to D_{2r}) = 0$.

Remark. With considerably more difficulty, the hypothesis of a smooth action in Theorem A could be replaced by the hypothesis of a locally smooth action on a topological manifold. One needs a substitute for the $\mathbb{Z}_2$-invariant tubular neighbourhood of $\mathscr{S}(M, \mathbb{Z}_2)$ in Lemma 2·1. This is given by constructing a mapping cylinder neighbourhood of $\mathscr{S}(M, \mathbb{Z}_2)$ in $M/\mathbb{Z}_2$ by using Quinn's end theorem [3] (the point here is that $\tilde{K}_i(\mathbb{Z}[\mathbb{Z}_2]) = 0$ for $i \leqslant 0$, so the obstructions vanish), and then lifting to M to get an equivariant mapping cylinder neighbourhood of $\mathscr{S}(M, \mathbb{Z}_2)$ in M. The map from the boundary of the equivariant mapping cylinder neighbourhood to $\mathscr{S}(M, \mathbb{Z}_2)$ is not necessarily a sphere bundle, but can be shown to have the homotopy type of a spherical fibration. This allows one to use the Gysin sequence as in the proof of Theorem A.

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