# Manifold aspects of the Novikov Conjecture

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Let $L_M \in H^{4*}(M; \mathbb{Q})$ be the Hirzebruch $L$-class of an oriented manifold $M$. Let $B\pi$ (or $K(\pi, 1)$) denote any aspherical space with fundamental group $\pi$. (A space is aspherical if it has a contractible universal cover.) In 1970 Novikov made the following conjecture.

**Novikov Conjecture.** *Let $h : M' \to M$ be an orientation-preserving homotopy equivalence between closed, oriented manifolds.[1] For any discrete group $\pi$ and any map $f : M \to B\pi$,*

$$f_* \circ h_*(L_{M'} \cap [M']) = f_*(L_M \cap [M]) \in H_*(B\pi; \mathbb{Q})$$

Many surveys have been written on the Novikov Conjecture. The goal here is to give an old-fashioned point of view, and emphasize connections with characteristic classes and the topology of manifolds. For more on the topology of manifolds and the Novikov Conjecture see [58], [47], [17]. This article ignores completely connections with $C^*$-algebras (see the articles of Mishchenko, Kasparov, and Rosenberg in [15]), applications of the Novikov conjecture (see [58],[9]), and most sadly, the beautiful work and mathematical ideas uncovered in proving the Novikov Conjecture in special cases (see [14]).

The level of exposition in this survey starts at the level of a reader of Milnor-Stasheff's book *Characteristic Classes*, but by the end demands more topological prerequisites. Here is a table of contents:

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[1] Does this refer to smooth, PL, or topological manifolds? Well, here it doesn't really matter. If the Novikov Conjecture is true for all smooth manifolds mapping to $B\pi$, then it is true for all PL and topological manifolds mapping to $B\pi$. However, the definition of $L$-classes for topological manifolds depends on topological transversality [25], which is orders of magnitude more difficult than transversality for smooth or PL-manifolds. The proper category of manifolds will be a problem of exposition throughout this survey.

# 1   Hirzebruch $L$-classes

The *signature* $\sigma(M)$ of a closed, oriented manifold $M$ of dimension $4k$ is the signature of its intersection form

$$\phi_M : H^{2k}M \times H^{2k}M \to \mathbb{Z}$$
$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, [M] \rangle$$

For a manifold whose dimension is not divisible by 4, we define $\sigma(M) = 0$. The key property of the signature is its bordism invariance: $\sigma(\partial W) = 0$ where $W$ is a compact, oriented manifold. The signature of a manifold[2] can be used to define the Hirzebruch $L$-class, whose main properties are given by the theorem below.

**Theorem 1.1.** *Associated to a linear, PL, or topological $\mathbb{R}^n$-bundle[3] $\xi$ are characteristic classes*

$$L_i(\xi) \in H^{4i}(B(\xi); \mathbb{Q}) \qquad i = 1, 2, 3, \dots$$

*satisfying*

1.  *$L_i = 0$ for a trivial bundle.*

2.  *For a closed, oriented $4k$-manifold $M$,*

    $$\langle L_k(\tau_m), [M] \rangle = \sigma(M)$$

3.  *Properties 1. and 2. are axioms characterizing the $L$-classes.*

4.  *Let $L = 1 + L_1 + L_2 + L_3 + \dots$ be the total $L$-class. Then*

    $$L(\xi \oplus \eta) = L(\xi)L(\eta).$$

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[2]Gromov [17] says that the signature "is not just 'an invariant' but *the invariant* which can be matched in beauty and power only by the Euler characteristic."

[3]We require that these bundles have 0-sections, i.e. the structure group preserves the origin. We also assume that the base spaces have the homotopy type of a $CW$-complex.

5. *Let $B = BO, BPL$, or $BTOP$ be the classifying spaces for stable linear, PL, or topological Euclidean bundles, respectively. Then*

$$H^*(B; \mathbb{Q}) = \mathbb{Q}[L_1, L_2, L_3, \dots]$$

*where $L_i$ denotes the $i$-th $L$-class of the universal bundle.*

We write $L_M = L(\tau_M)$ and call property 2. the *Hirzebruch signature formula*.

Properties 1.-4. are formal consequences of transversality and Serre's theorem on the finiteness of stable homotopy groups; this is due to Thom-Milnor [32], Kahn [22], and Rochlin-Svarc. Property 5 is not formal. One checks that $L_1, L_2, L_3, \dots$ are algebraically independent by applying the Hirzebruch signature formula to products of complex projective spaces and shows that $H^*(BO; \mathbb{Q})$ is a polynomial ring with a generator in every fourth dimension by computing $H^*(BO(n); \mathbb{Q})$ inductively. That $BO$, $BPL$ and $BTOP$ have isomorphic rational cohomology is indicated by Novikov's result [38] that two homeomorphic smooth manifolds have the same rational $L$-class, but also depends on the result of Kervaire-Milnor [24] of the finiteness of exotic spheres and the topological transversality of Kirby and Siebenmann [25].

We indicate briefly how the properties above can be used to define the $L$-classes, because this provides some motivation for the Novikov Conjecture. By approximating a $CW$-complex by its finite skeleta, a finite complex by its regular neighborhood, and a compact manifold by the orientation double cover of its double, it suffices to define $L_M$ for a closed, oriented $n$-manifold. The idea is that this is determined by signatures of submanifolds with trivial normal bundle. Given a map $f : M \to S^{n-4i}$, the meaning of $\sigma(f^{-1}(*))$ is to perturb $f$ so that it is transverse to $* \in S^{n-4i}$ and take the signature of the inverse image. This is independent of the perturbation by cobordism invariance of the signature. Given such a map, one can show

$$\langle L_M \cup f^*u, [M] \rangle = \sigma(f^{-1}(*))$$

where $u \in H^{n-4i}(S^{n-4i})$ is a generator. Using Serre's result that

$$\pi^{n-4i}(M) \otimes \mathbb{Q} \to H^{n-4i}(M; \mathbb{Q})$$

is an isomorphism when $4i < (n-1)/2$, one sees that the above formula defines the $4i$-dimensional component of $L_M$ when $4i$ is small. To define the high-dimensional components of $L_M$ one uses the low-dimensional components of $L_{M \times S^m}$ for $m$ large.

It is more typical to define Pontryagin classes for linear vector bundles, then define the $L$-classes of linear bundles as polynomials in the Pontryagin classes, then prove the Hirzebruch signature theorem, then define the $L$-classes for PL and topological bundles (as above), and finally define the

Pontryagin classes as polynomials in the $L$-classes. But $L$-classes, which are more closely connected with the topology of manifolds, can be defined without mentioning Pontryagin classes. The Pontryagin classes are more closely tied with the group theory of $SO$, and arise in Chern-Weil theory and the Atiyah-Singer index theorem. They are useful for computations, and their integrality can give many subtle properties of smooth manifolds (e.g. the existence of exotic spheres).

We conclude this section with some remarks on the statement of the Novikov Conjecture. Given a map $f : M \to B\pi$, the 0-dimensional component of $f_*(L_M \cap [M])$ is just the signature of $M$, and its homotopy invariance provides some justification of the Novikov Conjecture. If one proves the Novikov Conjecture for a map $f : M \to B\pi_1 M$ inducing an isomorphism on the fundamental group, then one can deduce the Novikov Conjecture for all maps $M \to B\pi$. But the more general statement is useful, because it may be the case that one can prove it for $\pi$ but not for the fundamental group.

**Definition 1.2.** For $f : M \to B\pi$ and for $u \in H^*(B\pi; \mathbb{Q})$, define the *higher signature*

$$\sigma_u(M, f) = \langle L_M \cup f^* u, [M] \rangle \in \mathbb{Q}$$

When $u = 1 \in H^0$, the higher signature is just the signature of $M$. The higher signature can often be given a geometric interpretation. If the Poincaré dual of $f^* u$ can be represented by a submanifold with trivial normal bundle, the higher signature is the signature of that submanifold. Better yet, if $B\pi$ is a closed, oriented manifold and the Poincaré dual of $u$ in $B\pi$ can be represented by a submanifold $K$ with trivial normal bundle, the higher signature is the signature of the transverse inverse image of $K$. The Novikov Conjecture implies that all such signatures are homotopy invariant.

Henceforth, we will assume all homotopy equivalences between oriented manifolds are orientation-preserving and will often leave out mention of the homotopy equivalence. With this convention we give an equivalent formulation of the Novikov Conjecture.

**Novikov Conjecture.** *For a closed, oriented manifold $M$, for a discrete group $\pi$, for any $u \in H^*(B\pi)$, for any map $f : M \to B\pi$, the higher signature $\sigma_u(M, f)$ is an invariant of the oriented homotopy type of $M$.*

# 2   Novikov Conjecture for $\pi = \mathbb{Z}$

We wish to outline the proof (cf. [17]) of the following theorem of Novikov [37].

**Theorem 2.1.** *Let $M$ be a closed, oriented manifold with a map $f : M \to S^1$. Then $f_*(L_M \cap [M]) \in H_*(S^1; \mathbb{Q})$ is homotopy invariant.*

Since $\langle 1, f_*(L_M \cap [M]) \rangle = \sigma(M)$, the degree-zero component of $f_*(L_M \cap [M])$ is homotopy invariant. Let $u \in H^1(S^1)$ be a generator; it suffices to show $\langle u, f_*(L_M \cap [M]) \rangle$ is a homotopy invariant, where dim $M = 4k + 1$. Let $K^{4k} = f^{-1}(*)$ be the transverse inverse image of a point. (Note: any closed, oriented, codimension 1 submanifold of $M$ arises as $f^{-1}(*)$ for some map $f$.) Let $i : K \hookrightarrow M$ be the inclusion. Then the Poincaré dual of $i_*[K]$ is $f^*u$, since the Poincaré dual of an embedded submanifold is the image of the Thom class of its normal bundle. Thus we need to show the homotopy invariance of

$$
\begin{aligned}
\langle u, f_*(L_M \cap [M]) \rangle &= \langle f^*u, L_M \cap [M] \rangle \\
&= \langle L_M, f^*u \cap [M] \rangle \\
&= \langle L_M, i_*[K] \rangle \\
&= \langle i^*L_M, [K] \rangle \\
&= \langle L_K, [K] \rangle \\
&= \sigma(K).
\end{aligned}
$$

**Definition 2.2.** If $K^{4k}$ is a closed, oriented manifold which is a subspace $i : K \hookrightarrow X$ of a topological space $X$, let

$$
\phi_{K \subset X} : H^{2k}(X; \mathbb{Q}) \times H^{2k}(X; \mathbb{Q}) \to \mathbb{Q}
$$

be the symmetric bilinear form defined by

$$
\phi_{K \subset X}(a, b) = \langle a \cup b, i_*[K] \rangle = \langle i^*a \cup i^*b, [K] \rangle .
$$

If $X = K$, we write $\phi_K$.

**Remark 2.3.**    1. Note that

$$
\sigma(\phi_{K \subset X}) = \sigma(\phi_K : i^*H^{2k}(X; \mathbb{Q}) \times i^*H^{2k}(X; \mathbb{Q}) \to \mathbb{Q}) ,
$$

so that the signature is defined even when $X$ is not compact.

2. If $h : X' \to X$ is a proper, orientation-preserving homotopy equivalence between manifolds of the same dimension, and $K'$ is the transverse inverse image of a closed, oriented submanifold $K$ of $X$, then $h_*i'_*[K'] = i_*[K]$, so

$$
\sigma(\phi_{K' \subset X'}) = \sigma(\phi_{K \subset X}) .
$$

The Novikov conjecture for $\pi = \mathbb{Z}$ follows from:

**Theorem 2.4.** *Let $M^{4k+1}$ be a closed, oriented manifold with a map $f :$ $M \to S^1$. Let $K = f^{-1}(*)$ be the transverse inverse image of a point. Then the signature of $K$ is homotopy invariant. In fact, $\sigma(K) = \sigma(\phi_{K_0 \subset M_\infty})$ where $M_\infty \to M$ is the infinite cyclic cover induced by the universal cover of the circle and $K_0 = \tilde{f}^{-1}(\tilde{*}) \subset M_\infty$ is a lift of $K$.*

There are two key lemmas in the proof.

**Lemma 2.5.** *Let $K^{4k}$ be a closed, oriented manifold which is a subspace $i : K \hookrightarrow X$ of a CW-complex $X$. Suppose $X$ is filtered by subcomplexes*

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X = \bigcup_n X_n .$$

*Then there exists an $N$ so that for all $n \geq N$,*

$$\sigma(\phi_{K \subset X}) = \sigma(\phi_{K \subset X_n})$$

*Proof.* Since $K$ is compact, $K \subset X_n$ for $n$ sufficiently large; so without loss of generality $i_n : K \hookrightarrow X_n$ for all $n$. Let $i_{n*}$ and $i_*$ ($i_n{}^*$ and $i^*$) be the maps induced by $i_n$ and $i$ on rational (co)-homology in dimension $2k$. The surjections $J_{n,n+1} : \text{im } i_{n*} \to \text{im } i_{n+1*}$ must be isomorphisms for $n$ sufficiently large (say for $n \geq N$) since $H_{2k}(K; \mathbb{Q})$ is finite dimensional. We claim

$$J_n : \text{im } i_{n*} \to \text{im } i_*$$

is also an isomorphism for $n \geq N$. Surjectivity is clear. To see it is injective, suppose $[\alpha] \in H_{2k}(K; \mathbb{Q})$ and $i_*[\alpha] = 0$. On the singular chain level $\alpha = \partial\beta$, where $\beta \in S_{2k+1}(X; \mathbb{Q}) = \cup_n S_{2k+1}(X_n; \mathbb{Q})$. Thus $i_{n*}[\alpha] = 0$ for $n$ sufficiently large, and hence for $n \geq N$ since the maps $J_{n,n+1}$ are isomorphisms.

Dualizing by applying $\text{Hom}( ; \mathbb{Q})$ we see that

$$\text{im } i^* \to \text{im } i_n{}^*$$

is also an isomorphism for $n \geq N$, and hence

$$\begin{aligned}
\sigma(\phi_{K \subset X_n}) &= \sigma(\text{im } i_n{}^* \times \text{im } i_n{}^* \to \mathbb{Q}) \\
&= \sigma(\text{im } i^* \times \text{im } i^* \to \mathbb{Q}) \\
&= \sigma(\phi_{K \subset X})
\end{aligned}$$

$\square$

**Lemma 2.6.** *Let $X^{4k+1}$ be a manifold with compact boundary. (Note $X$ may be non-compact.)*

  1. *Let $L = (\text{im } i^* : H^{2k}(X;\mathbb{Q}) \to H^{2k}(\partial X;\mathbb{Q}))$ and*

$$L^\perp = \{a \in H^{2k}(\partial X;\mathbb{Q}) \mid \phi_{\partial X}(a,L) = 0\} \ .$$

  *Then $L^\perp \subset L$.*

  2. $\sigma(\phi_{\partial X \subset X}) = \sigma(\partial X)$.

**Remark 2.7.** In the case of $X$ compact, part 2. above gives the cobordism invariance of the signature. Indeed

$$
\begin{aligned}
\phi_{\partial X \subset X}(a,b) &= \langle i^*a \cup i^*b, [\partial X] \rangle \\
&= \langle i^*a, i^*b \cap \partial_*[X] \rangle \\
&= \langle i^*a, \partial_*(b \cap [X]) \rangle \\
&= \langle \delta^* i^* a, b \cap [X] \rangle \\
&= 0
\end{aligned}
$$

*Proof of Lemma.* 1. Poincaré-Lefschetz duality gives a non-singular pairing

$$\phi_X : H_c^{2k+1}(X, \partial X;\mathbb{Q}) \times H^{2k}(X;\mathbb{Q}) \to \mathbb{Q}$$

Let $\delta^*$ be the coboundary map and $\delta_c^*$ be the coboundary with compact supports. Now $0 = \phi_{\partial X}(L^\perp, i^* H^{2k}(X;\mathbb{Q})) = \phi_X(\delta_c^* L^\perp, H^{2k}(X;\mathbb{Q}))$, so $\delta_c^* L^\perp = 0$, and hence $L^\perp \subset \ker \delta_c^* \subset \ker \delta^* = L$.

2. By 1., $\phi_{\partial X}(L^\perp, L^\perp) = 0$. Choose a basis $e_1, \ldots e_n$ for $L^\perp$, then find $f_1, \ldots f_n \in H^{2k}(\partial X;\mathbb{Q})$ so that $\phi_{\partial X}(e_i, f_j) = \delta_{i,j}$ and $\phi_{\partial X}(f_i, f_j) = 0$. The form $\phi_{\partial X}$ restricted to $H(L^\perp) = \text{Span}(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is non-singular and has zero signature. There is an orthogonal direct sum of vector spaces

$$H^{2k}(\partial X;\mathbb{Q}) = H(L^\perp) \oplus H(L^\perp)^\perp$$

The inclusion of $L$ in $H^{2k}(\partial X;\mathbb{Q})$ followed by the projection onto $H(L^\perp)^\perp$ induces an isometry between the form restricted to $H(L^\perp)^\perp$ and

$$\phi_{\partial X} : L/L^\perp \times L/L^\perp \to \mathbb{Q}$$

Thus

$$\sigma(\partial X) = \sigma(L/L^\perp \times L/L^\perp \to \mathbb{Q}) = \sigma(L \times L \to \mathbb{Q}) = \sigma(\phi_{\partial X \subset X}).$$

$\square$

*Proof of Theorem.* One has the pull-back diagram

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\tilde{f}} & \mathbb{R} \\
\downarrow & & \downarrow{\exp} \\
M & \xrightarrow{\;f\;} & S^1
\end{array}
$$

Suppose $f$ is transverse to $* = 1 \in S^1 \subset \mathbb{C}$. Let $K = f^{-1}(*)$ and, abusing notation slightly, also let $K$ denote $\tilde{f}^{-1}(0) \subset M_\infty$. Let $M_+ = \tilde{f}^{-1}[0, \infty)$ and $X_n = \tilde{f}^{-1}[-n, n]$. Let $t : M_\infty \to M_\infty$ be the generator of the deck transformations corresponding to $x \mapsto x + 1$ in $\mathbb{R}$. Then

$$
\begin{aligned}
\sigma(K) &= \sigma(\phi_{K \subset M_+}) & &\text{(by Lemma 2.6)} \\
&= \sigma(\phi_{K \subset t^N X_N}) & &\text{(for } N \gg 0 \text{ by Lemma 2.5)} \\
&= \sigma(\phi_{t^{-N} K \subset X_N}) & &(t^N \text{ is a homeomorphism)} \\
&= \sigma(\phi_{K \subset X_N}) & &\text{(since } K \text{ and } t^{-N} K \text{ are bordant in } X_N) \\
&= \sigma(\phi_{K \subset M_\infty}) & &\text{(for } N \gg 0 \text{ by Lemma 2.5 again)}
\end{aligned}
$$

$$\square$$

This beautiful proof of Novikov should be useful elsewhere in geometric topology, perhaps in the study of signatures of knots. The modern proof of the Novikov Conjecture is by computing the $L$-theory of $\mathbb{Z}[\mathbb{Z}]$ as in [49]; further techniques are indicated by Remark 4.2 in [47]. Later Farrell-Hsiang [19], [12] and Novikov [39] showed that the Novikov Conjecture for $\pi = \mathbb{Z}^n$ is true, although additional techniques are needed to prove it. Lusztig [27] gave an analytic proof of this result.

## 3   Topological rigidity

Given a homotopy equivalence

$$
h : M'^n \to M^n
$$

between two closed manifolds, one could ask if $h$ is homotopic to a homeomorphism or a diffeomorphism. A naive conjecture would be that the answer is always yes, after all $M'$ and $M$ have the same global topology (since $h$ is a homotopy equivalence) and the same local topology (since they are both locally Euclidean). At any rate, any invariant which answers the question in the negative must be subtle indeed.

Here is one idea for attacking this question. Let $K^k \subset M^n$ be a closed submanifold and perturb $h$ so that it is transverse to $K$. Then we have

$$M' \xrightarrow{\ h\ } M$$
$$\uparrow \qquad\qquad \uparrow$$
$$h^{-1}K \longrightarrow K$$

If $k$ is divisible by 4 and $M$ and $K$ are oriented, then the difference of signatures

$$\sigma(h, K) = \sigma(h^{-1}K) - \sigma(K) \in \mathbb{Z}$$

is an invariant of the homotopy class of $h$ which vanishes if $h$ is homotopic to a homeomorphism. If $n - k = 1$, then $\sigma(h, K)$ is always zero; this follows from the Novikov conjecture for $\pi = \mathbb{Z}$.

**Example 3.1.** This is basically taken from [32, Section 20]. Other examples are given in [47] and [17]. Let $\mathbb{R}^5 \hookrightarrow E \to S^4$ be the 5-plane bundle given by the Whitney sum of the quaternionic Hopf bundle and a trivial line bundle. Then it represents the generator of $\pi_4(BSO(5)) = \pi_3(SO(5)) \cong \mathbb{Z}$ and $p_1(E) = 2u$, where $u \in H^4(S^4)$ is a generator. Let $\mathbb{R}^5 \hookrightarrow E' \to S^4$ be a bundle with $p_1(E') = 48u$, say the pullback of $E$ over a degree 24 map $S^4 \to S^4$. Then $E'$ is fiber homotopically trivial (by using the $J$-homomorphism $J : \pi_3(SO(5)) \to \pi_8(S^5) \cong \mathbb{Z}/24$, see [20] for details). Thus there is a homotopy equivalence

$$h : S(E') \to S^4 \times S^4$$

commuting with the bundle map to $S^4$ in the domain and projecting on the second factor in the target. It is left as an exercise to show $\sigma(h, pt \times S^4) = 16$ and hence $h$ is not homotopic to a homeomorphism. It would be interesting to construct $h$ and $h^{-1}(pt \times S^4)$ explicitly (maybe in terms of algebraic varieties and the $K3$ surface).

The Novikov conjecture for a group $\pi$ implies that if $h : M' \to M$ is a homotopy equivalence of closed, oriented manifolds and if $K \subset M$ is a closed, oriented submanifold with trivial normal bundle which is Poincaré dual to $f^*\rho$ for some $f : M \to B\pi$ and some $\rho \in H^*(B\pi)$, then

$$\sigma(h, K) = 0$$

In particular if $M$ is aspherical, then $M = B\pi$ and $\sigma(h, K) = 0$ for all such submanifolds of $M$. Note that any two aspherical manifolds with isomorphic fundamental group are homotopy equivalent. In this case there is a conjecture much stronger that the Novikov conjecture.

**Borel Conjecture.** [4] *Any homotopy equivalence between closed aspherical manifolds is homotopic to a homeomorphism.*

See [1] for more on the Borel Conjecture and [9] for applications to the topology of 4-manifolds.

We conclude this section with a historical discussion of the distinction between closed manifolds being homotopy equivalent, homeomorphic, PL-homeomorphic, or diffeomorphic. It is classical that all the notions coincide for 2-manifolds. For 3-manifolds, homeomorphic is equivalent to diffeomorphic ([33], [35]). Poincaré conjectured that a closed manifold homotopy equivalent to $S^3$ is homeomorphic to $S^3$; this question is still open. In 1935, Reidemeister, Franz, and DeRham showed that there are homotopy equivalent 3-dimensional lens spaces $L(7; 1, 1)$ and $L(7; 1, 2)$ which are not simple homotopy equivalent (see [31], [7]), and hence not diffeomorphic. However, this was behavior based on the algebraic $K$-theory of the fundamental group. Hurewicz asked whether simply-connected homotopy equivalent closed manifolds are homeomorphic. Milnor [29] constructed exotic 7-spheres, smooth manifolds which are (PL-) homeomorphic but not diffeomorphic to the standard 7-sphere. Thom, Tamura, and Shimada gave examples in the spirit of Example 3.1, which together with Novikov's proof of the topological invariance of the rational $L$-classes showed that simply-connected homotopy equivalent manifolds need not be homeomorphic. Finally, Kirby and Siebenmann [25] gave examples of homeomorphic PL-manifolds which are not PL-homeomorphic. Thus all phenomena are realized. We will return frequently to the question of when homotopy equivalent manifolds are homeomorphic.

# 4   Oriented bordism

Let $\Omega_n$ be the oriented bordism group of smooth $n$-manifolds. $\Omega_*$ is a graded ring under disjoint union and cartesian product. Thom [53] combined his own foundational work in differential topology and with then

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[4]A. Borel conjectured this in 1953, long before the Novikov conjecture. The motivation was not from geometric topology, but rather from rigidity theory for discrete subgroups of Lie groups.

The choice of category is important; the connected sum of an $n$-torus and an exotic sphere need not be diffeomorphic to the $n$-torus.

recent work of Serre [48] to show[5]

$$\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \mathbb{C}P^8, \dots]$$

A map $f : M \to X$ where $M$ is a closed, oriented $n$-manifold and $X$ is a topological space is called *a singular $n$-manifold over $X$*. Let $\Omega_n(X)$ be the oriented bordism group of singular $n$-manifolds over $X$, see [8]. Then $\Omega_*(X)$ is a graded module over $\Omega_*$. The map $H : \Omega_n X \otimes \mathbb{Q} \to H_n(X; \mathbb{Q})$ defined by $H(f : M \to X) = f_*[M]$ is onto. $\Omega_* X \otimes \mathbb{Q}$ is a free module over $\Omega_* \otimes \mathbb{Q}$ with a basis given by any set of singular manifolds $\{(f_\alpha : M_\alpha \to X) \otimes 1\}$ such that $\{(f_\alpha)_*[M_\alpha]\}$ is a basis of $H_*(M; \mathbb{Q})$. The *Conner-Floyd map* is

$$\begin{aligned}
\Omega_* X \otimes_{\Omega_*} \mathbb{Q} &\to& H_*(X; \mathbb{Q}) \\
(f : M \to X) \otimes r &\to& r f_*(L_M \cap [M])
\end{aligned}$$

Here $\mathbb{Q}$ is an $\Omega_*$-module via the signature homomorphism. The domain of the Conner-Floyd map is $\mathbb{Z}_4$-graded, and the Conner-Floyd map is a $\mathbb{Z}_4$-graded isomorphism, i.e. for $i = 0, 1, 2$, or $3$

$$\Omega_{4*+i} X \otimes_{\Omega_*} \mathbb{Q} \cong \bigoplus_{n \equiv i(4)} H_n(X; \mathbb{Q}) \ .$$

All of these statements are easy consequences of the material in the previous footnote.

**Definition 4.1.** Let $M$ be a closed, smooth manifold. An element of the *structure set* $\mathbb{S}(M)$ *of homotopy smoothings* is represented by a homotopy equivalence $h : M' \to M$ where $M'$ is also a closed, smooth manifold. Another such homotopy equivalence $g : M'' \to M$ represents the same element of the structure set if there is an $h$-cobordism $W$ from $M'$ to $M''$ and a map $H : W \to M$ which restricts to $h$ and $g$ on the boundary.

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[5]The modern point of view on Thom's work is:

$$\Omega_* \otimes \mathbb{Q} \cong \pi_*(\mathbf{MSO}) \otimes \mathbb{Q} \cong H_*(\mathbf{MSO}; \mathbb{Q}) \cong H_*(BSO; \mathbb{Q}).$$

Here **MSO** is the oriented bordism Thom spectrum, whose $n$-th space is the Thom space of the universal $\mathbb{R}^n$-bundle over $BSO(n)$. The isomorphism $\Omega_* \cong \pi_*(\mathbf{MSO})$ follows from transversality and is called the Pontryagin-Thom construction. From Serre's computations, the Hurewicz map for an Eilenberg-MacLane spectrum is an rational isomorphism in all dimensions. It follows that the Hurewicz map is a rational isomorphism for all spectra, and that the rational localization of any spectrum is a wedge of Eilenberg-MacLane spectrum. The Thom isomorphism theorem shows $H_*(\mathbf{MSO}) \cong H_*(BSO)$. After tracing through the above isomorphisms one obtains a non-singular pairing

$$H^*(BSO; \mathbb{Q}) \times (\Omega_* \otimes \mathbb{Q}) \to \mathbb{Q} \qquad (\alpha, M) \mapsto \alpha[M]$$

A computation of $L$-numbers of even-dimensional complex projective spaces shows that they freely generate $\Omega_* \otimes \mathbb{Q}$ as a polynomial algebra. References on bordism theory include [8], [50], and [28].

In particular, if $h : M' \to M$ and $g : M'' \to M$ are homotopy equivalences, then $[h] = [g] \in \mathbb{S}(M)$ if there is a diffeomorphism $f : M' \to M''$ so that $g \simeq h \circ f$. The converse holds true if all $h$-cobordisms are products, for example if the manifolds are simply-connected and have dimension greater than 4.

Given a map $f : M^n \to B\pi$, there is a function

$$\mathbb{S}(M) \to \Omega_n(B\pi)$$
$$(M' \to M) \mapsto [M' \to B\pi] - [M \to B\pi]$$

It is an interesting question, not unrelated to the Novikov Conjecture, to determine the image of this map, but we will not pursue this.

There are parallel theories for PL and topological manifolds, and all of our above statements are valid for these theories. There are variant bordism theories $\Omega^{PL}$, $\Omega^{TOP}$ and variant structure sets $\mathbb{S}^{PL}$ and $\mathbb{S}^{TOP}$. The bordism theories are rationally the same, but the integrality conditions comparing the theories are quite subtle.

## 5  A crash course in surgery theory

The purpose of surgery theory is the classification of manifolds up to homeomorphism, PL-homeomorphism, or diffeomorphism; perhaps a more descriptive name would be manifold theory. There are two main goals: *existence*, the determination of the homotopy types of manifolds and *uniqueness*, the classification of manifolds up to diffeomorphism (or whatever) within a homotopy type. For the uniqueness question, the technique is due to Kervaire-Milnor [24] for spheres, to Browder [5], Novikov [36], and Sullivan [51] for simply-connected manifolds, to Wall [55] for non-simply-connected manifolds of dimension $\geq 5$, and to Freedman-Quinn [16] for 4-manifolds. Surgery works best for manifolds of dimension $\geq 5$ due to the Whitney trick, but surgery theory also provides information about manifolds of dimension 3 and 4.

The key result of the uniqueness part of surgery theory is the *surgery exact sequence*[6]

$$\cdots \xrightarrow{\theta} L_{n+1}(\mathbb{Z}\pi_1 M) \xrightarrow{\partial} \mathbb{S}(M) \xrightarrow{\eta} \mathcal{N}(M) \xrightarrow{\theta} L_n(\mathbb{Z}\pi_1 M)$$

for a closed, smooth, oriented manifold $M^n$ with $n \geq 5$. The $L$-groups are abelian groups, algebraically defined in terms of the group ring. They are 4-periodic $L_n \cong L_{n+4}$. For the trivial group, $L_n(\mathbb{Z}) \cong \mathbb{Z}, 0, \mathbb{Z}_2, 0$ for $n \equiv 0, 1, 2, 3 \pmod 4$. The $L$-groups are Witt groups of quadratic forms (see

---

[6]This is given in [55, §10], although note that Wall deals with the $L^s$- and $\mathbb{S}^s$-theory, while we work with the $L^h$ and $\mathbb{S}^h$-theory, see [55, §17D].

Wall [55]), or, better yet, bordism groups of algebraic Poincaré complexes (see Ranicki [45]). The structure set $\mathbb{S}(M)$ and the normal invariant set $\mathcal{N}(M)$ are pointed sets, and the surgery exact sequence is an exact sequence of pointed sets. Furthermore, $L_{n+1}(\mathbb{Z}\pi_1 M)$ acts on $\mathbb{S}(M)$ so that two elements are in the same orbit if and only if that have the same image under $\eta$.

Elements of the normal invariant set $\mathcal{N}(M)$ are represented by degree one normal maps

$$
\begin{array}{ccc}
\nu_{M'} & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
M' & \xrightarrow{g} & M
\end{array}
$$

that is, a map $g$ of closed, smooth, oriented[7] manifolds with $g_*[M'] = [M]$, together with normal data: a stable vector bundle $\xi$ over $M$ which pulls back to the stable normal bundle of $M'$; more precisely, the data includes a stable trivialization of $g^*\xi \oplus T(M')$. There is a notion of two such maps to $M$ being bordant; we say the maps are *normally bordant* or that one can do surgery to obtain one map from the other. The normal invariant set $\mathcal{N}(M)$ is the set of normal bordism classes. The map $\eta : \mathbb{S}(M) \to \mathcal{N}(M)$ sends a homotopy equivalence to itself where the bundle $\xi$ is the pullback of the stable normal bundle of the domain under the homotopy inverse. The map $\theta : \mathcal{N}(M) \to L_n(\mathbb{Z}\pi_1 M)$ is called the surgery obstruction map, and is defined for manifolds of any dimension, however when the dimension is greater than or equal to five, $\theta(g) = 0$ if and only if $g$ is normally bordant to a homotopy equivalence.

Sullivan computed the normal invariant set using homotopy theory; it is closely connected with characteristic classes, see [28]. In fact there is a Pontryagin-Thom type construction which identifies

$$
\mathcal{N}(M) \cong [M, G/O]
$$

Here $G(k)$ is the topological monoid of self-homotopy equivalences of $S^{k-1}$ and $G = \operatorname{colim} G(k)$. There is a fibration

$$
G/O \xhookrightarrow{\psi} BO \to BG
$$

$BG(k)$ classifies topological $\mathbb{R}^k$-bundles up to proper fiber homotopy equivalence and so $G/O$ classifies proper fiber homotopy equivalences between vector bundles. A map $M \to G/O$ corresponds to a proper fiber homotopy equivalence between vector bundles over $M$, and the transverse inverse image of the 0-section gives rise to the degree one normal map. If

---

[7]There are variant versions of surgery for non-orientable manifolds.

we take the classifying map $\hat{g} : M \to G/O$ of a degree one normal map $g : (M', \nu_{M'}) \to (M, \xi)$, then $\psi \circ \hat{g}$ is a classifying map for $\xi \oplus \tau_M$. Finally, the homotopy groups of $G$ are the stable homotopy groups of spheres, which are finite (Serre's result again), and so given any vector bundle $M \to BO$, then some non-zero multiple of it can be realized as a "$\xi$" in a degree one normal map.

Later it will be useful to consider the surgery obstruction map

$$\theta : \mathcal{N}(M^n) \to L_n(\mathbb{Z}\pi)$$

associated to a map $M \to B\pi$, which may not induce an isomorphism on the fundamental group. This is covariant in $\pi$. When $\pi$ is the trivial group and $g : (M', \nu_{M'}) \to (M, \xi)$ with $n \equiv 0 \pmod 4$,

$$\theta(g) = (1/8)(\sigma(M') - \sigma(M)) \in L_n(\mathbb{Z}) \cong \mathbb{Z}$$

For $n \equiv 2 \pmod 4$, the simply-connected surgery obstruction is called the Arf invariant of $g$.

## 6  Surgery and characteristic classes

This section is in some sense the core of this survey. We give a converse to the Novikov conjecture and rephrase the Novikov conjecture in terms of injectivity of an assembly homomorphism. The two key tools are the bordism invariance of surgery obstructions and the product formula for surgery obstructions. They are due to Sullivan in the case of simply-connected manifolds and the facts we need are due to Wall in the non-simply-connected case. Both of the tools were proved in greater generality and were given more conceptual interpretations by Ranicki.

The bordism invariance of surgery obstructions is that the surgery obstruction map associated to a $f : M \to B\pi$ factors [55, 13B.3]

$$\mathcal{N}(M^n) \xrightarrow{\beta} \Omega_n(G/O \times B\pi) \xrightarrow{\hat{\theta}} L_n(\mathbb{Z}\pi)$$

Here $\beta(g) = [(\hat{g}, f) : M \to G/O \times B\pi]$ where $\hat{g}$ classifies the degree one normal map. (The geometric interpretation of this result is that surgery obstruction is unchanged by allowing a bordism in both the domain and *range*.)

**Example 6.1.** The $K3$-surface $K^4$ is a smooth, simply-connected spin 4-manifold with signature 16. There is a degree one normal map

$$g : (K^4, \nu_K) \to (S^4, \xi)$$

where $\xi$ is any bundle with $\langle L_1(\xi), [S^4] \rangle = 16$. Then $\theta(g \times \mathrm{Id}_{S^4}) = 0$ since the classifying map $S^4 \times S^4 \to G/O$ is constant on the second factor and

hence bounds. Thus $g \times \mathrm{Id}_{S^4}$ is normally bordant to a homotopy equivalence $h$ with $\sigma(h^{-1}(pt \times S^4)) = 16$. This is the same homotopy equivalence as in Example 3.1.

The second key tool is the product formula. This deals with the following situation. Given a degree one normal map

$$g : (M'^i, \nu_{M'}) \to (M^i, \xi)$$

and a closed, oriented manifold $N^j$, together with reference maps

$$M \to B\pi, \qquad N \to B\pi',$$

one would like a formula for the surgery obstruction

$$\theta(g \times \mathrm{Id}_N) \in L_{i+j}(\mathbb{Z}[\pi \times \pi'])$$

This has been given a nice conceptual answer by Ranicki [44]. There are "symmetric $L$-groups" $L^j(\mathbb{Z}\pi')$. There is the *Mishchenko-Ranicki symmetric signature map*

$$\sigma^* : \Omega_* B\pi' \to L^*(\mathbb{Z}\pi')$$

(sending a manifold to the bordism class of the chain level Poincaré duality map of its $\pi'$-cover), and a product pairing

$$L_i(\mathbb{Z}\pi) \otimes L^j(\mathbb{Z}\pi') \to L_{i+j}(\mathbb{Z}[\pi \times \pi'])$$

so that

$$\theta(g \times \mathrm{Id}_N) = \theta(g) \otimes \sigma^* N$$

Furthermore, for the trivial group, $L^j(\mathbb{Z}) \cong \mathbb{Z}, \mathbb{Z}_2, 0, 0$ for $j \equiv 0, 1, 2, 3$ (mod 4). For $j \equiv 0$ (mod 4), $\sigma^*(N) \in \mathbb{Z}$ is the signature $\sigma(N)$ and for $j \equiv 1$ (mod 4), $\sigma^*(N) \in \mathbb{Z}_2$ is called the De Rham invariant. We only need the following theorem, which follows from the above, but also from earlier work of Wall [55, 17H].

**Theorem 6.2.**  1. *If $\pi = 1$, $\theta(g \times Id_N) \in L_{i+j}(\mathbb{Z}\pi') \otimes \mathbb{Q}$ depends only on the difference $\sigma(M') - \sigma(M)$ and the bordism class $[N \to B\pi'] \in \Omega_j(B\pi') \otimes \mathbb{Q}$.*

2. *If $\pi' = 1$, then*

$$\theta(g \times Id_N) = \theta(g) \cdot \sigma(N) \in L_{i+j}(\mathbb{Z}\pi) \otimes \mathbb{Q}$$

In the above theorem we are sticking with our usual convention that the signature is zero for manifolds whose dimensions are not divisible by 4.

**Example 6.3.** Let

$$g : (K, \nu_K) \to (S^4, \xi)$$

be a degree one normal map where $K$ is the $K3$-surface as in Example 6.1. Another way to see that $\theta(g \times \mathrm{Id}_{S^4}) = 0$ is by the product formula.

As a formal consequence of the bordism invariance of surgery obstructions, the product formula, and the Conner-Floyd isomorphism, it follows that the Novikov conjecture is equivalent to the injectivity of a rational assembly map. This is due to Wall [55, 17H] and Kaminker-Miller [23].

**Theorem 6.4.** *For any group $\pi$ and for any $n \in \mathbb{Z}_4$, there is a map*

$$A_n : \bigoplus_{i \equiv n(4)} H_i(\pi; \mathbb{Q}) \to L_n(\mathbb{Z}\pi) \otimes \mathbb{Q}$$

*so that*

1. *For a degree one normal map $g : (M', \nu_{M'}) \to (M, \xi)$ and a map $f : M \to B\pi$,*

$$\theta(g) = A_*((f \circ g)_*(L_{M'} \cap [M']) - f_*(L_M \cap [M])) \in L_*(\mathbb{Z}\pi) \otimes \mathbb{Q}$$

2. *If $A_n$ is injective then the Novikov conjecture is true for all closed, oriented manifolds mapping to $B\pi$ whose dimension is congruent to $n$ modulo 4. If the Novikov conjecture is true for all closed, oriented manifolds with fundamental group isomorphic to $\pi$ and whose dimension is congruent to $n$ modulo 4, then $A_n$ is injective.*

*Proof.* Bordism invariance, the Conner-Floyd isomorphism, and the product formula show that the surgery obstruction factors through a $\mathbb{Z}_4$-graded map

$$\widetilde{A}_n : \bigoplus_{i \equiv n(4)} H_i(G/O \times B\pi; \mathbb{Q}) \to L_n(\mathbb{Z}\pi) \otimes \mathbb{Q}$$

with

$$\widetilde{A}_n((\hat{g}, f)_*(L_M \cap [M])) = \theta(g) \in L_n(\mathbb{Z}\pi) \otimes \mathbb{Q}$$

When $\pi$ is trivial, the map $\widetilde{A}_n$ is given by Kronecker pairing $\hat{g}_*(L_M \cap [M])$ with some class $\ell \in H^*(G/O; \mathbb{Q})$. In the fibration

$$G/O \overset{\psi}{\hookrightarrow} BO \to BG$$

$\psi^*$ gives a rational isomorphism in cohomology, and it is not difficult to show that

$$\ell = \psi^*(\frac{1}{8}(\overline{L} - 1))$$

where $\overline{L}$ is the multiplicative inverse of the Hirzebruch $L$-class.[8] The key equation is that if $g : (M', \nu_{M'}) \to (M, \xi)$ is a degree one normal map with classifying map $\hat{g}$, then

$$8\hat{g}^* \ell \cup L_M = \overline{L}(\xi) - L_M$$

The homology of $G/O \times B\pi$ is rationally generated by cross products $\hat{g}_*(L_M \cap [M]) \times f_*(L_N \cap [N])$ where $\hat{g} : M \to G/O$ and $f : N \to B\pi$; these correspond to surgery problems of the form $M' \times N \to M \times N$ The product formula for surgery obstructions shows that

$$\widetilde{A}_*(\hat{g}_*(L_M \cap [M]) \times f_*(L_N \cap [N])) = 0$$

whenever $\langle \ell, \hat{g}_*(L_M \cap [M]) \rangle = 0$. It follows that $\widetilde{A}_n$ factors through the surjection given by the slant product

$$\ell \backslash : \bigoplus_{i \equiv n(4)} H_i(G/O \times B\pi; \mathbb{Q}) \to \bigoplus_{i \equiv n(4)} H_i(B\pi; \mathbb{Q}),$$

giving the rational assembly map

$$A_n : \bigoplus_{i \equiv n(4)} H_i(B\pi; \mathbb{Q}) \to L_i(\mathbb{Z}\pi) \otimes \mathbb{Q}$$

Tracing through the definition of $A_*$ gives the characteristic class formula in part 2. of the theorem. (I have suppressed a good deal of manipulation of cup and cap products here, partly because I believe the reader may be able to find a more efficient way than the author.)

If $A_n$ is injective, the Novikov conjecture immediately follows for $n$-manifolds equipped with a map to $B\pi$, since the surgery obstruction of a homotopy equivalence is zero.

Suppose $A_n$ is not injective; then there exists an non-zero element $a \in H_{4*+n}(B\pi; \mathbb{Q})$ so that $A_n(a) = 0 \in L_n(\mathbb{Z}\pi) \otimes \mathbb{Q}$. There is a $b \in H_{4*+n}(G/O \times B\pi; \mathbb{Q})$ so that $a = \ell \backslash b$. Next note for some $i$, there is an element $c = [(\hat{g}, f) : M \to G/O \times B\pi] \in \Omega_{4i+n}(G/O \times B\pi)$, so that $(\hat{g}, f)_*(L_M \cap [M]) = kb$ where $k \neq 0$. (Note the $i$ here. To get $c \in \Omega_{4*+n}$ one simply uses the Conner-Floyd isomorphism, but one might have to multiply the various components of $c$ by products of $\mathbb{C}P^2$ to guarantee that $c$ is homogeneous). By multiplying $c$ by a non-zero multiple, find a new $c = [(\hat{g}, f) : M \to G/O \times B\pi] \in \Omega_{4i+n}(G/O \times B\pi)$ so that $\theta(c) = 0 \in L_{4i+n}(\mathbb{Z}\pi)$. We may assume that $4i + n > 4$. By (very) low

---

[8]This class $\ell$ has a lift to $H^*(G/O; \mathbb{Z}_{(2)})$ which is quite important for the characteristic class formula for the surgery obstruction of a normal map of closed manifolds [28], [34], [52].

dimensional surgeries [55, Chapter 1], we may assume $M$ has fundamental group $\pi$. Since $\theta(c) = 0$, one may do surgery to obtain a homotopy equivalence $h : M' \to M$, where the difference of the higher signatures in $H_*(B\pi; \mathbb{Q})$ is a multiple of $a$ and hence non-zero.   □

Our next result is folklore (although seldom stated correctly) and should be considered as a converse to and a motivation for the Novikov conjecture. It generalizes Kahn's result [21] that the only possible linear combinations of $L$-classes (equivalently rational Pontrjagin classes) which can be a homotopy invariant of simply-connected manifolds is the top $L$-class of a manifold whose dimension is divisible by 4. The non-simply connected case requires a different proof; in particular one must leave the realm of smooth manifolds.

**Theorem 6.5.** *Let $M$ be a closed, oriented, smooth manifold of dimension $n > 4$, together with a map $f : M \to B\pi$ to the classifying space of a discrete group, inducing an isomorphism on fundamental group. Given any cohomology classes*

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \cdots \in H^{4*}(M; \mathbb{Q}), \qquad \mathcal{L}_i \in H^{4i}(M; \mathbb{Q}),$$

*so that $f_*(\mathcal{L} \cap [M]) = 0 \in H_{n-4*}(B\pi; \mathbb{Q})$, there is a non-zero integer $R$, so that for any multiple $r$ of $R$, there is a homotopy equivalence*

$$h : M' \to M$$

*of closed, smooth manifolds so that*

$$h^*(L_M + r\mathcal{L}) = L_{M'}$$

*Proof.* As motivation suppose that there is a map $\hat{g} : M \to G/O$ so that $\hat{g}^*\ell = (1/8)\mathcal{L}\overline{L_M}$, where $\ell \in H^*(G/O; \mathbb{Q})$ is as above and $\overline{L_M}$ is the multiplicative inverse of the total Hirzebruch $L$-class of $M$. Then if $g : N \to M$ is the corresponding surgery problem, a short computation shows that

$$g^*(L_M + \mathcal{L}) = L_N$$

and, by using Theorem 6.4, Part 2, that $\theta(\hat{g}) = 0 \in L_n(\mathbb{Z}\pi) \otimes \mathbb{Q}$. So the idea is to clear denominators and replace $\hat{g}$ by a multiple. Unfortunately, this is nonsense, since the surgery obstruction map is not a homomorphism of abelian groups.

To proceed we need two things. First that $G/TOP$ is an infinite loop

space[9] and

$$
\begin{array}{ccc}
[M, G/TOP] & \xrightarrow{\ \theta\ } & L_n(\mathbb{Z}\pi) \\
\downarrow & & \downarrow \\
\bigoplus_{i \equiv n(4)} H_i(\pi; \mathbb{Q}) & \xrightarrow{\ A_*\ } & L_n(\mathbb{Z}\pi) \otimes \mathbb{Q}
\end{array}
$$

is a commutative diagram of abelian groups[10] where the left vertical map is given by $\hat{g} \mapsto f_*(\hat{g}^*\ell \cap [M])$. (See [52].)

The second thing we need is a lemma of Weinberger's [57]. Let $j : G/O \to G/TOP$ be the natural map.

**Lemma 6.6.** *For any $n$-dimensional CW-complex $M$, there is a non-zero integer $t = t(n)$ so that for any $[f] \in [M, G/TOP]$, $t[f] \in \operatorname{im} j_* : [M, G/O] \to [M, G/TOP]$.*

Now suppose we are given $\mathcal{L} \in H^{4*}(M; \mathbb{Q})$ as in the statement of the theorem. The cohomology class $\ell$ gives a localization of $H$-spaces at 0

$$
\ell : G/TOP \to \prod_{i>0} K(\mathbb{Q}, 4i)
$$

and hence a localization of abelian groups

$$
[M, G/TOP] \to \prod_{i>0} H^{4i}(M; \mathbb{Q})
$$

In particular, there is an non-zero integer $R_1$, so that for any multiple $r_1$ of $R_1$, there is a map $\hat{g} : M \to G/TOP$ so that

$$
\hat{g}^*\ell = r_1 \frac{1}{8} \mathcal{L} \overline{L_M}
$$

Then $\theta(\hat{g}) = 0 \in L_n(\mathbb{Z}\pi)_{(0)}$, so there is a non-zero $R_2$ so that $R_2\theta(\hat{g}) = 0 \in L_n(\mathbb{Z}\pi)$. Finally, by Weinberger's Lemma, there is a non-zero $R_3$, so that $R_3[R_2\hat{g}]$ factors through $G/O$. Then $R = R_1 R_2 R_3$ works. Surgery theory giving the homotopy equivalence. $\qquad\square$

Weinberger's Lemma follows from the following; applied where $s : Y \to Z$ is the map $G/TOP \to B(TOP/O)$. Note that $B(TOP/O)$ has an infinite loop space structure coming from Whitney sum.

---

[9]$G/O$, $G/PL$, and $G/TOP$ are all infinite loop spaces with the $H$-space structure corresponding to Whitney sum. However, the surgery obstruction map is not a homomorphism. Instead, we use the infinite loop space structure on $G/TOP$ induced by periodicity $\Omega^4(\mathbb{Z} \times G/TOP) \simeq \mathbb{Z} \times G/TOP$. This will be discussed further in the next section.

[10]We can avoid references to topological surgery by using the weaker fact that the surgery obstruction map $\theta : [M, G/PL] \to L_n(\mathbb{Z}\pi)$ is a homomorphism where $G/PL$ is given the $H$-space structure provided by the Characteristic Variety Theorem [51].

**Lemma 6.7.** *Let $s : Y \to Z$ be a map of simply-connected spaces of finite type, where $Y$ is an H-space and $Z$ is an infinite loop space with all homotopy groups finite. Then for any $k$ there is a non-zero integer $t$ so that for any map $f : X \to Y$ whose domain is a $k$-dimensional CW-complex, then $s_*(t[f]) = 0$, where*

$$s_* : [X, Y] \to [X, Z]$$

*is the induced map on based homotopy.*

*Proof.* It suffices to prove the above when $f$ is the inclusion $i_k : Y^k \hookrightarrow Y$ of the $k$-skeleton of $Y$; in other words, we must show $s_*[i_k] \in [Y^k, Z]$ has finite order. Let $Y = Y_n$, $n = 1, 2, 3, \ldots$. Let $\alpha_n : Y_n \to Y_{n+1}$ be a cellular map so that $[\alpha_n] = n![\mathrm{Id}]$. Then $\alpha_n$ induced $\cdot n!$ on homotopy groups, since the co-$H$-group structure equals the $H$-group structure. Let $\mathrm{hocolim}_n Y_n$ denote the infinite mapping telescope of the maps $\alpha_n$. Then $Y \to \mathrm{hocolim}_n Y_n$ induces a localization at 0 on homotopy groups.

Consider the following commutative diagram of abelian groups.

$$
\begin{array}{ccccc}
[\mathrm{hocolim}_n Y_n^{k+2}, Z] & \xrightarrow{\ \Phi\ } & \lim_n [Y_n^{k+2}, Z] & \xrightarrow{\ \mathrm{pr}_1\ } & [Y_1^{k+2}, Z] \\
\ \ \downarrow{\scriptstyle A} & & \ \ \downarrow{\scriptstyle B} & & \ \ \downarrow{\scriptstyle C} \\
[\mathrm{hocolim}_n Y_n^k, Z] & \xrightarrow{\ \Phi\ } & \lim_n [Y_n^k, Z] & \xrightarrow{\ \mathrm{pr}_1\ } & [Y_1^k, Z]
\end{array}
$$

To prove the lemma it suffices to show that some multiple of $s_*[i_k] \in [Y_1^k, Z]$ is in the image of $C \circ \mathrm{pr}_1 \circ \Phi$ and that $A$ is the zero map. First note $C s_*[i_{k+2}] = s_*[i_k]$. Now

$$\mathrm{im}([Y, Z] \to [Y^{k+2}, Z])$$

is a finite set by obstruction theory, so

$$\{s_*(n![i_{k+2}])\}_{n=1,2,3,\ldots}$$

sits in a finite set, hence there exists an $N$ so that $s_*(N![i_{k+2}])$ pulls back arbitrarily far in the inverse sequence. Hence by compactness of the inverse limit of finite sets, there is an $[a] \in \lim_n[Y_n^{k+2}, Z]$ so that $\mathrm{pr}_1[a] = s_*(N![i_{k+2}])$. By Milnor's $\lim^1$ result [30], $\Phi$ is onto, so that $[a] = \Phi[b]$ for some $[b]$.

We next claim that $A$ is the zero map. Indeed the homology groups of $\mathrm{hocolim}_n Y_n^{k+2}$ are rational vector spaces in dimensions less than $k + 2$. By obstruction theory any map $\mathrm{hocolim}_n Y_n^{k+2} \to Z$ is zero when restricted to the $(k + 1)$-st skeleton, and thus when restricted to $\mathrm{hocolim}_n Y_n^k$. Thus we have shown that $s_*(N![i_k])$ is zero by tracing around the outside of the diagram. Let $t = N!$. $\qquad\square$

**Remark 6.8.** For a closed, oriented, topological $n$-manifold $M$, define

$$\Gamma^{TOP}(M) : \mathbb{S}^{TOP}(M) \to H^{4*}(M;\mathbb{Q})$$
$$[h : M' \to M] \mapsto \mathcal{L} \qquad \text{where } h^*(L_M + \mathcal{L}) = L_{M'}$$

If $M$ is smooth, define an analogous map $\Gamma^{DIFF}(M)$ from the smooth structure set. The above discussion shows that the image of $\Gamma^{TOP}(M)$ is a finitely generated, free abelian group whose intersection with the kernel of the map $H^{4*}(M;\mathbb{Q}) \to H_{n-4*}(B\pi_1 M;\mathbb{Q})$ is a lattice (i.e a finitely generated subgroup of full rank). Furthermore if the Novikov Conjecture is valid for $\pi_1 M$, the image of $\Gamma^{TOP}(M)$ is a precisely a lattice in the above kernel.

Similar things are true for smooth manifolds up to finite index. If $t = t(n)$ is the integer from Lemma 6.7, then

$$\frac{1}{t}\text{im } \Gamma^{TOP}(M) \subset \text{im } \Gamma^{DIFF}(M) \subset \text{im } \Gamma^{TOP}(M)$$

However, a recent computation of Weinberger's [56] shows that the image of $\Gamma^{DIFF}(M)$ is not a group when $M$ is a high-dimensional torus. It follows that $\mathbb{S}^{DIFF}(M)$ cannot be given a group structure compatible with that of $\mathbb{S}^{TOP}(M)$ and that $G/O$ cannot be given an $H$-space structure so that the surgery obstruction map is a homomorphism.

# 7 Assembly maps

The notion of an assembly map is central to modern surgery theory, and to most current attacks on the Novikov Conjecture. Assembly maps are useful for both conceptual and computation reasons. We discuss assembly maps to state some of the many generalizations of the Novikov conjecture. Assembly can be viewed as gluing together surgery problems, as a passage from local to global information, as the process of forgetting control in controlled topology, as taking the index of an elliptic operator, or as a map defined via homological algebra. There are parallel theories of assembly maps in algebraic $K$-theory and in the $K$-theory of $C^*$-algebras, but the term assembly map originated in surgery with the basic theory due to Quinn and Ranicki.

With so many different points of view on the assembly map, it is a bit difficult to pin down the concept, and it is perhaps best for the neophyte to view it as a black box and concentrate on its key properties. We refer the reader to the papers [42], [43], [46], [59], [11] for further details.

The classifying space for topological surgery problems is $G/TOP$. The generalized Poincaré conjecture and the surgery exact sequence show that

$\pi_n(G/TOP) = L_n(\mathbb{Z})$ for $n > 0$. There is a homotopy equivalence

$$\Omega^4(\mathbb{Z} \times G/TOP) \simeq \mathbb{Z} \times G/TOP$$

(Perhaps the 4-fold periodicity is halfway between real and complex Bott periodicity.) Let $\mathbf{L}$ denote the corresponding spectrum. Oriented manifolds are oriented with respect to the generalized homology theory defined by $\mathbf{L}$. When localized at 2, $\mathbf{L}$ is a wedge of Eilenberg-MacLane spectra, and after 2 is inverted, $\mathbf{L}$ is homotopy equivalent to inverting 2 in the spectrum resulting from real Bott periodicity. The assembly map

$$A_n : H_n(X; \mathbf{L}) \to L_n(\mathbb{Z}\pi_1 X)$$

is defined for all integers $n$, is 4-fold periodic, and natural in $X$. By naturality, the assembly map for a space $X$ factors through $B\pi_1 X$.

The surgery obstruction map can interpreted in terms of the assembly map. Let $\mathbf{G}/\mathbf{TOP}$ be the connective cover of $\mathbf{L}$, i.e. there is a map $\mathbf{G}/\mathbf{TOP} \to \mathbf{L}$ which is an isomorphism on $\pi_i$ for $i > 0$, but $\pi_i(\mathbf{G}/\mathbf{TOP}) = 0$ for $i \leq 0$. Here $\mathbf{G}/\mathbf{TOP}$ is an $\Omega$-spectrum whose 0-th space is $G/TOP$. The composite

$$H_n(X; \mathbf{G}/\mathbf{TOP}) \to H_n(X; \mathbf{L}) \xrightarrow{A} L_n(\mathbb{Z}\pi_1 X)$$

is also called the assembly map. When $X = B\pi$, this assembly map tensored with $\mathrm{Id}_{\mathbb{Q}}$ is the same as the assembly map from the last section. In particular the Novikov conjecture is equivalent to the rational injectivity of either of the assembly maps when $X = B\pi$.

The surgery obstruction map for a closed, oriented $n$-manifold $M$

$$\theta : [M, G/TOP] \to L_n(\mathbb{Z}\pi_1 M)$$

factors as the composite of Poincaré duality and the assembly map

$$[M, G/TOP] = H^0(M; \mathbf{G}/\mathbf{TOP}) \simeq H_n(M; \mathbf{G}/\mathbf{TOP}) \xrightarrow{A} L_n(\mathbb{Z}\pi_1 M)$$

There is a surgery obstruction map and a structure set for manifolds with boundary (the $L$-groups remain the same however). The basic idea is that all maps are assumed to be homeomorphisms on the boundary throughout. The surgery exact sequence extends to a half-infinite sequence

$$\cdots \to \mathbb{S}^{TOP}(M \times I, \partial) \to [(M \times I)/\partial, G/TOP] \to L_{n+1}(\mathbb{Z}\pi_1 M)$$
$$\to \mathbb{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}\pi_1 M)$$

Assembly maps are induced by maps of spectra; we denote the fiber of

$$X_+ \wedge \mathbf{G}/\mathbf{TOP} \xrightarrow{\mathbf{A}} \mathbf{L}(\mathbb{Z}\boldsymbol{\pi_1}\mathbf{X})$$

by $\mathbb{S}^{\textbf{TOP}}(\textbf{X})$. When $X$ is a manifold, the corresponding long exact sequence in homotopy can be identified with the surgery exact sequence. In particular, the structure set $\mathbb{S}^{TOP}(M) = \pi_n \mathbb{S}^{\textbf{TOP}}(\textbf{M})$ is naturally an abelian group, a fact which is not geometrically clear. Thus computing assembly maps is tantamount to classifying manifolds up to $h$-cobordism.

There is a parallel theory in algebraic $K$-theory. For a ring $A$, there are abelian groups $K_n(A)$ defined for all integers $n$; they are related by the fundamental theorem of $K$-theory which gives a split exact sequence

$$0 \to K_n(A) \to K_n(A[t]) \oplus K_n(A[t^{-1}]) \to K_n(A[t, t^{-1}]) \to K_{n-1}A \to 0$$

There is an $\Omega$-spectrum $\textbf{K}(\textbf{A})$ whose homotopy groups are $K_*(A)$. Abbreviate $\textbf{K}(\mathbb{Z})$ by $\textbf{K}$; its homotopy groups are zero in negative dimensions, $\mathbb{Z}$ in dimension 0, and $\mathbb{Z}/2$ in dimension 1. There is the $K$-theory assembly map [26], [11]

$$A_n : H_n(X; \textbf{K}) \to K_n(\mathbb{Z}\pi_1 X).$$

Computing with the Atiyah-Hirzebruch spectral sequence and the test case where $X$ is the circle shows that $A_n$ is injective for $i < 2$, and the cokernels of $A_n$ are $K_n(\mathbb{Z}\pi_1 X), \tilde{K}_0(\mathbb{Z}\pi_1 X), Wh(\pi_1 X)$ for $n < 0, n = 0, n = 1$.

The analogue of the Novikov conjecture in $K$-theory has been proven!

**Theorem 7.1 (Bökstedt-Hsiang-Madsen [4]).** *Suppose $\pi$ is any group such that $H_n(B\pi)$ is finitely generated for all $n$. Then $A_* \otimes Id_\mathbb{Q}$ is injective.*

Hopefully this result will shed light on the Novikov conjecture in $L$-theory (which has more direct geometric consequences), but so far this has been elusive.

# 8   Isomorphism conjectures

A strong version of Borel's conjecture is:

**Conjecture 8.1.** *Let $h : M' \to M$ be a homotopy equivalence between compact aspherical manifolds so that $h(M' - \partial M') \subset h(M - \partial M)$ and $h|_{\partial M'} : \partial M' \to \partial M$ is a homeomorphism. Then $h$ is homotopic rel $\partial$ to a homeomorphism.*

Applying this to $h$-cobordisms implies that $Wh(\pi_1 M) = 0$, and by crossing with tori, that $\tilde{K}_0(\mathbb{Z}\pi_1 M) = 0$ and $K_{-i}(\mathbb{Z}\pi_1 M) = 0$ for $i > 0$. Similarly, the structure groups $\mathbb{S}^{TOP}(M \times I^i, \partial) = 0$, so the assembly maps are isomorphisms. The following conjecture is motivated by these considerations.

**Borel-Novikov Isomorphism Conjecture.** [11] *For $\pi$ torsion free, the assembly maps*

$$A_* : H_*(B\pi; \mathbf{K}) \to K_*(\mathbb{Z}\pi)$$
$$A_* : H_*(B\pi; \mathbf{L}) \to L_*(\mathbb{Z}\pi)$$

*are isomorphisms.*

This implies the geometric Borel conjecture for manifolds of dimension greater than 4, and for torsion-free groups the vanishing of the Whitehead group, that all finitely generated projective modules are stably free, and the Novikov Conjecture.

What can be said for more general groups, in other words how does one compute the $L$-groups and the surgery obstruction groups? Well, for finite groups, the assembly map has been largely computed, starting with the work of Wall [55] and ending with the work of Hambleton, Milgram, Taylor, and Williams [18]. The techniques here are a mix of number theory, quadratic forms, and topology. The assembly maps are not injective, and not even rational surjective, in $K$-theory, because the Whitehead groups may be infinite, and in $L$-theory, because the multisignature (or $\rho$-invariant) show that the $L$-groups of the group ring of a finite group may be infinite.

There are also analyses of the $K$- and $L$-theory of products $\mathbb{Z} \times G$ and amalgamated free products $A *_B C$ (see [2], [49], [41], [54], [6]). While these give evidence for the Borel-Novikov Isomorphism Conjecture for torsion-free groups, for infinite groups with torsion the "nil" phenomena showed that the non-homological behavior of $L$-groups could not all be blamed on the finite subgroups. In particular there are groups $\pi$ where the assembly map is not an isomorphism, but where the assembly map is an isomorphism for all finite subgroups. To account for this, Farrell and Jones laid the blame on the following class of subgroups.

**Definition 8.2.** *A group $H$ is* virtually cyclic *if it has a cyclic subgroup of finite index.*

For example, a finite group $G$ is virtually cyclic and so is $\mathbb{Z} \times G$. Farrell-Jones have made a conjecture [13] which computes $K_*(\mathbb{Z}\pi)$ and $L_*(\mathbb{Z}\pi)$ in terms of the assembly maps for virtually cyclic subgroups and homological information concerning the group $\pi$ and the lattice of virtually cyclic subgroups of $\pi$. We give the rather complicated statement of the conjecture

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[11] Neither Borel nor Novikov made this conjecture, but rather made weaker conjectures whose statements did not involve assembly maps.

It is a fairly bold conjecture; there exist a lot of torsion-free groups. A more conservative conjecture would be to conjecture this when $B\pi$ is a finite complex and perhaps only that the assembly map is a split injection. One might also wish to restrict the conjecture in $K$-theory to $* < 2$, where there is a geometric interpretation of the results.

below, but for now we note that the Farrell-Jones isomorphism conjecture implies:

1. The Novikov conjecture for a general group $\pi$.

2. The Borel-Novikov isomorphism conjecture for a torsion-free group $\pi$.

3. For any group $\pi$ and for any $N \in \mathbb{Z} \cup \{\infty\}$, if for all virtually cyclic subgroups $H$ of $\pi$, the assembly map

$$A_* : H_*(BH; \mathbf{K}) \to K_*(\mathbb{Z}H)$$

is an isomorphism for $* < N$ and a surjection for $* = N$, then the assembly map for $\pi$ is an isomorphism for $* < N$, and similarly for $L$-theory.

We proceed to the statement of the isomorphism conjecture, as formulated in [11]. We work in $K$-theory, although there is an analogous conjecture in $L$-theory.[12] One can show that an inner automorphism of $\pi$ induces the identity on $K_*(\mathbb{Z}\pi)$, but not necessarily on the associated spectrum. (One needs to worry about such details to make sure that constructions don't depend on the choice of the base point of the fundamental group.) To account for this, one uses the *orbit category* $\mathrm{Or}(\pi)$, whose objects are left $\pi$-sets $\{\pi/H\}_{H \subset \pi}$ and whose morphisms are $\pi$-maps. For a family of subgroups $\mathcal{F}$ of $\pi$ (e.g. the trivial family $\mathcal{F} = 1$ or the family $\mathcal{F} = \mathcal{VC}$ of virtually cyclic subgroups of $\pi$), one defines the *restricted orbit category* $\mathrm{Or}(\pi, \mathcal{F})$ to be the full subcategory of $\mathrm{Or}(\pi)$ with objects $\{\pi/H\}_{H \in \mathcal{F}}$. A functor

$$\mathbf{K} : \mathrm{Or}(\pi) \to \mathrm{SPECTRA}$$

is constructed in [11], with $\pi_*(\mathbf{K}(\pi/H)) = K_*(\mathbb{Z}H)$. The (classical) assembly map is then given by applying homotopy groups to the map

$$\mathbf{A} : \underset{\mathrm{Or}(\pi,1)}{\mathrm{hocolim}}\, \mathbf{K} \to \underset{\mathrm{Or}(\pi)}{\mathrm{hocolim}}\, \mathbf{K}$$

induced on homotopy colimits by the inclusion of the restricted orbit category in the full orbit category.

**Farrell-Jones Isomorphism Conjecture.** *For any group $\pi$,*

$$\underset{\mathrm{Or}(\pi,\mathcal{VC})}{\mathrm{hocolim}}\, \mathbf{K} \to \underset{\mathrm{Or}(\pi)}{\mathrm{hocolim}}\, \mathbf{K}$$

*induces an isomorphism on homotopy groups.*

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[12]In $L$-theory it is necessary to work with a variant theory, $L = L^{\langle -\infty \rangle}$.

This gives a theoretical computation of the $K$-groups, the (classical) assembly map, and in $L$-theory, the classification of manifolds with fundamental group $\pi$. For proofs of the conjecture in special cases see [13]. For applications in some special cases see [40] and [10]. The Farrell-Jones Isomorphism Conjecture is parallel to the $C^*$-algebra conjecture of Baum and Connes [3], with the family of virtually cyclic subgroups replaced by the family of finite subgroups.

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