Higher Diagonal Approximations and Skeletons of $K(\pi,1)$'s

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The cup product is graded commutative on the cohomomology lvel, but not on the cochain level. The failure of commutativity is measured by the higher diagonal approximations underlying such invariants as the Steenrod squares [6] and the symmetric signature associated to a Poincare duality space [5].

An n-skeleton of a $K(\pi,1)$ is a CW complex X of dimension n with $\pi_i(X) = 0$ for $1 < i < n$ and $\pi_1 X = \pi$. For example, X could be a space form, a manifold whose universal cover is the sphere or Euclidean space. This paper shows how the geometric higher diagonal approximations of X can be calculated purely algebraically from the cellular chains of the universal cover $\tilde{X}$.

This work was motivated by certain questions of John Jones and R. James Milgram concerning the Cappell-Shaneson detection [1] of a non-zero element $\sigma(S^3/Q_8)$ in the symmetric L-group $L^3(\mathbb{Z} Q_8)$. I wish to thank Andrew Ranicki for repeatedly bringing these questions to my attention.

Using the results of this paper one can compute the symmetric signature $\sigma(S^n/G) \in L^n(\mathbb{Z} G)$ for any free action of a finite group G on S^n. The symmetric signature appears in Ranicki's product formula for surgery obstructions. However, algebraic quadratic surgery shows that the product formula depends only on the chain level Poincare duality map (depending on Δ_0 defined below) and not on the higher diagonal approximations.

1. Preliminaries.

Let W be the standard free $\mathbb{Z}[\mathbb{Z}/2]$ - resolution of $\mathbb{Z}$

$$W: \quad \ldots \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+2} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] .$$

Here $\mathbb{Z}/2 = \langle T \rangle$. Let e_i denote the generator of the i-chains of W. Then $\partial(e_i) = (1+(-1)^i T)e_{i-1}$.

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Let C be a chain complex. Then $\mathbb{Z}/2$ acts on $C \otimes C$ $(= C \otimes_{\mathbb{Z}} C)$ via the interchange map

$$T: \; C \otimes C \longrightarrow C \otimes C$$

$$T(a \otimes b) = (-1)^{\deg a \deg b} \; b \otimes a \; .$$

We will consider $\mathbb{Z}[\mathbb{Z}/2]$ - module chain maps

$$\Delta: W \otimes C \longrightarrow C \otimes C \; .$$

Define $\Delta_i: \; C \longrightarrow C \otimes C$ by $\Delta_i(c) = \Delta(e_i \otimes c)$. Since Δ is a chain map the Δ_i satisfy relations

1.1 $$\partial\Delta_i - (-1)^i \Delta_i \partial = \Delta_{i-1} + (-1)^i \, T\Delta_{i-1} \; .$$

Thus Δ_0 is a chain map, Δ_1 is a chain homotopy between Δ_0 and $T\Delta_0$, Δ_2 is a chain homotopy between Δ_1 and $T\Delta_1$, etc. Conversely given a sequence of maps $\{\Delta_i\}$ satisfying 1.1, they give rise to a $\mathbb{Z}[\mathbb{Z}/2]$-module chain map Δ .

Let $S(X)$ denote the singular chain complex of a topological space X .

<u>Theorem 1.2</u>.

There exist functorial $\mathbb{Z}[\mathbb{Z}/2]$ -module chain maps

$$\Delta: \; W \otimes S(X) \longrightarrow S(X) \otimes S(X)$$

such that $\Delta_0(c) = c \otimes c$ for any singular 0-simplex c .

<u>Proof</u>. Method of acyclic models. □

If a group π acts on a space X , then functoriality implies that Δ is a $\mathbb{Z}[\mathbb{Z}/2 \times \pi]$ -module chain map.

<u>Preposition 1.3</u>.

Let π act freely and cellularly on a connected CW complex X . There is a splitting of $\mathbb{Z}\pi$-module chain complexes $S(X) = A \oplus B$ where A is isomorphic to the cellular chain complex $C(X)$ and $H_*(B) = 0$.

<u>Proof</u>.

Following Wall [7] let

$$D_i(X) = \ker\,(\partial\colon S_i(X^i) \longrightarrow S_{i-1}(X^i)/S_{i-1}(X^{i-1}))$$

Let $E_i(X) = \ker(D_i(X) \longrightarrow C_i(X))$. Then we have an exact sequence of $\mathbb{Z}\pi$-chain complexes

$$0 \longrightarrow E(X) \longrightarrow D(X) \longrightarrow C(X) \longrightarrow 0 \; .$$

Since $D(X) \longrightarrow C(X)$ induces an isomorphism in homology, $H_*(E(X)) = 0$, and hence $E(X)$ is chain contractible. It then follows that $D(X) = C(X) \oplus E(X)$ as chain complexes. Likewise the inclusion $D(X) \longrightarrow S(X)$ splits as a map of chain complexes. □

Let $f: C(X) \longrightarrow S(X)$ and $g: S(X) \longrightarrow C(X)$ be splitting maps in 1.3. Any $\mathbb{Z}[\mathbb{Z}/2 \times \pi]$-module chain map

$$\psi: W \otimes C(X) \longrightarrow C(X) \otimes C(X)$$

chain homotopic to

$$(g \otimes g) \circ \Delta \circ (1 \otimes f): W \otimes C(X) \longrightarrow C(X) \otimes C(X)$$

is called a geometric π-higher diagonal approximation. The mod π reduction of ψ gives a geometric 1-higher diagonal approximation on X/π. If $\pi = 1$, ψ can be used to compute cup products and Steenrod squares. If X is simply connected and X/π is a Poincare complex, then ψ can be used to compute the symmetric signature [5]

$$\sigma(X/\pi) \in L^n(\mathbb{Z}\pi)$$

occuring in the product formula for surgery obstructions.

Let $C = \{C_i, \partial\}_{i \geq 0}$ be a chain complex of $\mathbb{Z}$-modules with augmentation $\varepsilon: C_0 \longrightarrow \mathbb{Z}$. Let $\varepsilon \otimes 1: C \otimes C \longrightarrow \mathbb{Z} \otimes C = C$. Let

$$(C \otimes C)_k = \bigoplus_{i+j=k} C_i \otimes C_j$$

and

$$(C \otimes C)^k = \bigoplus_{\substack{i \leq k \\ j \leq k}} C_i \otimes C_j .$$

Consider $\mathbb{Z}[\mathbb{Z}/2]$-module chain maps $\Delta: W \otimes C \longrightarrow C \otimes C$ satisfying

(i) $\Delta(W \otimes C_i) \subset (C \otimes C)^i$ for all i.

(ii) $(\varepsilon \otimes 1) \circ \Delta_0 = 1$
$(1 \otimes \varepsilon) \circ \Delta_0 = 1$.

(iii) For all i, for any $c \in C_i$, there is an $a \in C_i \otimes C_i$ such that $\Delta_i(c) - c \otimes c = a + (-1)^i Ta$.

These conditions are geometrically inspired. Condition (ii) corresponds to the fact that for any cohomology class α, $\alpha \cup 1 = 1 \cup \alpha = \alpha$. Note that (ii) is satisfied for the Alexander-Whitney diagonal approximation. Condition (iii) is related to the identity $Sq^0(\alpha) = \alpha$.

Proposition 1.4.

On the category of topological spaces there exist functorial $\mathbb{Z}[\mathbb{Z}/2]$ - module chain maps

$$\Delta: \ W \otimes S(X) \longrightarrow S(X) \otimes S(X)$$

satisfying (i), (ii), (iii).

Proof.

Condition (i) will hold for any functorial map. Let $C(\Delta^n)$ be the simplicial complex of the standard n-simplex. Consider $C(\Delta^n)$ as a subcomplex of $S(\Delta^n)$. By acyclic model theory there exists a functorial Δ such that $\Delta(W \otimes C(\Delta^n)) \subset C(\Delta^n) \otimes C(\Delta^n)$ for all n. Induction on n shows that $(\varepsilon \otimes 1)(\Delta_0(\Delta^n)) = \Delta^n$, $(1 \otimes \varepsilon)(\Delta_0(\Delta^n)) = \Delta^n$. The proof that $Sq^0 = Id$ (see [6]) shows that condition (iii) holds for $c = \Delta^n$. Then linearity and functorality shows that (ii) and (iii) always hold. □

2. The Main Theorem.

Theorem 2.1.

Let $C = \{C_i, \partial\}_{0 \le i \le n}$ be a chain complex of free $\mathbb{Z}\pi$-modules such that $H_0(C) = Z$ and $H_i(C) = 0$ for $0 < i < n$. Then there exist $\mathbb{Z}[\mathbb{Z}/2 \times \pi]$ -module chain maps

$$\Delta: \ W \otimes C \longrightarrow C \otimes C$$

satisfying conditions (i), (ii), and (iii). Given two such maps they are chain homotopic.

Here the action of π on $C \otimes C$ is given by $g(x \otimes y) = gx \otimes gy$.

Corollary 2.2.

If X is a skeleton of a $K(\pi,1)$ and $C = C(\tilde{X})$ then any map satisfying (i), (ii), and (iii) is a geometric π-higher diagonal approximation.

Before we embark on the proof of 2.1, we need a lemma.

Lemma 2.3.

Let $\varepsilon = \pm 1$. If $b \in (C \otimes C)_{2i+1}$ is ε-symmetric $(Tb = \varepsilon b)$ and a boundary, then it is the boundary of an ε-symmetric chain. If $b \in (C \otimes C)_{2i}$ is ε-even ($b = a + \varepsilon Ta$ for some a) and a boundary, then it is the boundary of an ε-even chain.

Proof.

Suppose D and E are chain complexes such that $D \otimes D$ and

$E \otimes E$ satisfy the conclusion of lemma 2.3. Then $(D\oplus E) \otimes (D\oplus E) = D \otimes D \oplus (D\otimes E \oplus E\otimes D) \oplus E \otimes E$ also satisfies the conclusion.

Now as a $\mathbb{Z}$-module chain complex C is isomorphic to a direct sum of $H_n(C)$, $H_0(C)$ and elementary chain complexes of the form $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$. If D is any one of these chain complexes it is easily verified that $D \otimes D$ satisfies the conclusion. The lemma then follows by induction. □

Proof of 2.1.

To construct the $\mathbb{Z}[\mathbb{Z}/2 \times \pi]$-module chain map $\Delta: W \otimes C \longrightarrow C \otimes C$ it suffices to construct of sequence $\Delta_i: C \longrightarrow C \otimes C$ of $\mathbb{Z}\pi$-maps satisfying the relations 1.1 as well as conditions (i), (ii), and (iii). Choose a $\mathbb{Z}\pi$ basis of C_0 of the form $\Lambda \cup \{\rho_0\}$ where $\varepsilon(\rho_0) = 1$ and $\varepsilon(f) = 0$ for $f \in \Lambda$. Define $\Delta_0(\rho_0) = \rho_0 \otimes \rho_0$. For $f \in \Lambda$, define $\Delta_0(f) = \rho_0 \otimes f + f \otimes f + f \otimes \rho_0$. Extend to a map of C_0 by linearity.

Fix $k > 0$ and $\ell < k-1$. Assume now that Δ has been defined on $W \otimes C_j$ for $j < k$ and that Δ_i has been defined on C_k for $i < \ell$. Let x be a basis element of C_k. We first consider the case $\ell=0$. Let $Z_i = \ker(\partial: C_i \longrightarrow C_{i-1})$. By the Kunneth theorem $H_i((C\otimes C)^i) = Z_i \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes Z_i$. In particular if b is a i-cycle in $(C\otimes C)^i$ with $(\varepsilon\otimes 1)b = 0 = (1\otimes\varepsilon)b$ then b is a boundary in $(C\otimes C)^i$. So $\Delta_0(\partial x) - \partial x \otimes \rho_0 - \rho_0 \otimes \partial x$ is a boundary in $(C\otimes C)^{k-1}$. Say it is ∂a. Then define $\Delta_0(x) = x \otimes \rho_0 + \rho_0 \otimes x + a$. For $\ell > 0$, $(-1)^\ell \Delta_\ell(\partial x) + \Delta_{\ell-1}(x) + (-1)^\ell \Delta_{\ell-1}(x)$ is a boundary in $(C\otimes C)^k$ say ∂a. Define $\Delta_\ell(x) = a$. Extend to a map $\Delta_\ell: C_k \longrightarrow C \otimes C$ by linearity.

Now fix $k > 0$ and assume that Δ has been defined on $W \otimes C_j$ for $j < k$ and that Δ_i has been defined on C_k for $i < k-1$. Let x be a basis element of C_k. Then $(-1)^{k-1}\Delta_{k-1}(\partial x) + \Delta_{k-2}(x) + (-1)^{k-1}T\Delta_{k-2}(x) - (-1)^{k-1} \partial x \otimes \partial x$ is a $(-1)^{k-1}$-even boundary in $(C\otimes C)^k \cap (C\otimes C)_{2k-2}$ and hence by lemma 2.3 lifts to $a + (-1)^{k-1}Ta$ with $a \in (C\otimes C)^k \cap (C\otimes C)_{2k-1}$. We define $\Delta_{k-1}(x) = a + (-1)^{k-1}Ta + \partial x \otimes x$. (For the case $k = 1$ we also have to guarantee that $(\varepsilon\otimes 1)\Delta_0(x) = x$ and that $(1\otimes\varepsilon)\Delta_0(x) = x$, but this can be done by the proof of 2.3.). Extend Δ_{k-1} to a map of C_k by linearity. Now

$$(-1)^k \Delta_k(\partial x) + \Delta_{k-1}(x) + (-1)^k T\Delta_{k-1}(x) = \partial x \otimes x + (-1)^k x \otimes \partial x .$$

So define $\Delta_k(x) = x \otimes x$ for the basis element x. Extend by linearity. This completes the existence part of theorem 2.1.

For the uniqueness part of 2.1 consider $\mathbb{Z}[\mathbb{Z}/2]$-module chain maps $\Delta\colon W \otimes C \longrightarrow C \otimes C$ satisfying

(i') $\Delta(W\otimes C_i) \subset (C\otimes C)^i$ for all i.

(ii') $(\varepsilon\otimes 1) \circ \Delta_0 = 0$

$(1\otimes\varepsilon) \circ \Delta_0 = 0$.

(iii') For all i, $\Delta_i(C_i) \subset \operatorname{im}(1+(-1)^i T)$.

<u>Lemma 2.4</u>.

A $\mathbb{Z}[\mathbb{Z}/2 \times \pi]$-module chain map satisfying (i'), (ii'), and (iii') is of the form $\Delta = \partial\chi + \chi\partial$ for some degree one map χ.

<u>Proof</u>.

Define $\chi_i(c) = \chi(e_i \otimes c)$. Then $\Delta = \partial\chi + \chi\partial$ is equivalent to

2.5 $$\Delta_i = \partial\chi_i + (-1)^i\chi_i\partial + \chi_{i-1} + (-1)^i T\chi_{i-1} .$$

Let x be a basis element of C_0. Then $\Delta_0(x)$ is even and a boundary. Thus there is an $a \in (C\otimes C)_1$ such that $\partial(a+Ta) = \Delta_0(x)$ and $(\varepsilon\otimes 1)a = 0 = (1\otimes\varepsilon)a$. Define $\chi_0(x) = a + Ta$. Extend to a map of C_0 by linearity. Replace Δ by $\Delta - \partial\chi_0 - \chi_0\partial$.

We now assume Δ satisfies (i'), (ii'), and (iii') and that $\Delta_0(C_0) = 0$. We will now only consider χ such that $\chi(C^j) \subset (C\otimes C)^j$.

Fix $k < 0$ and $\ell < k - 1$. Assume that χ has been defined on $W \otimes C_j$ for $j < k$ and that χ_i has been defined on C_k for $i < \ell$. Furthermore assume

$$\Delta_i(c) = \partial\chi_i(c) + (-1)^i\chi_i(\partial c) + \chi_{i-1}(c) + (-1)^i T\chi_{i-1}(c)$$

for $c \in C_j$, $j < k$ and for $c \in C_k$, $i < \ell$. Let x be a basis element of C_k. Choose an element $a \in (C\otimes C)^k$ such that

$$\partial a = \Delta_\ell(x) - (-1)^\ell\chi_\ell(\partial x) - \chi_{\ell-1}(x) - (-1)^\ell T\chi_{\ell-1}(x) .$$

Define $\chi_\ell(x) = a$. Extend to a map of C_k by linearity.

Now fix $k > 0$ and assume that χ has been defined on $W \otimes C_j$ for $j < k$ and that χ_i has been defined on C_k for $i < k - 1$. Furthermore assume that the relation 2.5 is satisfied for $c \in C_j$,

$j < k$ and for $c \in C_k$, $i < k - 1$. Choose $b \in (C \otimes C)^k$ such that

$$\partial b = \Delta_{k-1}(x) - \chi_{k-2}(x) - (-1)^{k-1} T \chi_{k-2}(x) .$$

Now $\Delta_k(x) - (b + (-1)^k Tb)$ is a $(-1)^k$-even cycle so by lemma 2.3,

$$\Delta_k(x) = b + (-1)^k Tb + c + (-1)^k Tc$$

for some cycle $c \in (C \otimes C)^k$. Define $\chi_{k-1}(x) = b + c$. Extend by linearity. This completes the proof of 2.1. □

There are two cases where one can avoid some of the above homological algebra to calculate the geometric higher diagonal approximations. First, if X is a simplicial complex, one can apply acyclic model theory in the simplicial category to X directly. Second, if X is actually a $K(\pi,1)$, the construction of Δ follows from the "fundamental lemma" of homological algebra from a projective complex to an acyclic one. Indeed, if C is acyclic, any two $\mathbb{Z}[\mathbb{Z}/2 \times \pi]$ - module maps

$$W \otimes C \longrightarrow C \otimes C$$

commuting with the augmentation are chain homotopic.

If X is a n-skeleton of a $K(\pi,1)$, then the homotopy type of X is determined by $\pi_1 X$, $\pi_n X$, and the first Eilenberg-MacLane k-invariant $k^{n+1}(x) \in H^{n+1}(\pi_1 X; \pi_n X)$. (See, for example, Olum [4]). Now $k^{n+1}(x)$ can be defined algebraically as follows: Let $D = \{D_i, \partial\}_{i \geq 0}$ be a projective $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$. Choose a chain map

$$\{D_i, \partial\}_{i \leq n} \longrightarrow C_*(\tilde{X})$$

commuting with augmentation. Induced is a map

$$D_{n+1} \longrightarrow \ker(C_n(\tilde{X}) \longrightarrow C_{n-1}(\tilde{X})) = \pi_n X .$$

This cocycle gives $k^{n+1}(X)$. Hence the homotopy type of X is determined by the chain homotopy type of $C(\tilde{X})$. Thus every homotopy invariant of X should be computable algebraically. This gives a philosophical justification for Corollary 2.2.

3. Product Formulae

Given a product map

$$Id \times f\colon\ N^m \times M^n \longrightarrow N^m \times X^n$$

with N^m a closed manifold and $f\colon M^n \longrightarrow X^n$ a degree one normal map, the surgery obstruction $\sigma(\mathrm{Id} \times f) \in L_{m+n}(\mathbb{Z}[\pi_1(X)])$ is determined by the symmetric signature $\sigma(N) \in L^m(\mathbb{Z}[\pi_1(N)])$ and the surgery obstruction $\sigma(f) \in L_n(\mathbb{Z}[\pi_1(X)])$. Indeed $\sigma(N)$ can be represented by a symmetric Poincare complex (C,ϕ) with $C = C_*(N)$ and $\sigma(f)$ by a quadratic Poincare complex (D,ψ) with $H_*(D) = K_*(M)$. Then according to Ranicki's product formula [5,II.8.1]

$$\sigma(\mathrm{Id} \times f) = (C \otimes D,\ \phi \otimes \psi)\ ,$$

using the algebraically defined pairing

$$L^m(\mathbb{Z}[\pi_1 N]) \otimes L_n(\mathbb{Z}[\pi_1 X]) \longrightarrow L_{n+m}(\mathbb{Z}[\pi_1 N] \otimes \mathbb{Z}[\pi_1 X])\ .$$

$$(C,\phi) \otimes (D,\psi) \longmapsto (C \otimes D,\ \phi \times \psi)\ ,$$

and the identification $\mathbb{Z}[\pi_1 N] \otimes \mathbb{Z}[\pi_1 X] = \mathbb{Z}[\pi_1(N \times X)]$.

Here ϕ and ψ are represented by a sequence of maps

$$\{\phi_i \in \mathrm{Hom}_{\mathbb{Z}[\pi_1 N]}(C^{n-r+i}, C_r) \mid r \in \mathbb{Z}\ ,\ i \geq 0\}$$

$$\{\psi_i \in \mathrm{Hom}_{\mathbb{Z}[\pi_1 X]}(D^{n-r-i}, D_r) \mid r \in \mathbb{Z}\ ,\ i \geq 0\}$$

and $(\phi \otimes \psi)_i = \phi_i \otimes \psi_i$.

A geometric $\pi_1 N$-higher diagonal approximation

$$\Delta:\ W \otimes C(\tilde{N}) \longrightarrow C(\tilde{N}) \otimes C(\tilde{N})$$

determines the symmetric signature $\sigma(N) = (C,\phi)$ as follows:

Choose a representative $[N] \in C_m(N;\mathbb{Z}^t)$ for the fundamental class of N . Let $\pi = \pi_1 X$. Apply $\mathbb{Z}^t \otimes_{\mathbb{Z}\pi}$ to the Δ_i associated to Δ to obtain

$$\Delta_i' : C(N;\mathbb{Z}^t) \longrightarrow C(\tilde{N}) \otimes_{\mathbb{Z}\pi} C(\tilde{N})\ .$$

Then the ϕ_i are defined via the slant product

$$\phi_i : C^{n-r+i} \longrightarrow C_r$$

$$\beta \longmapsto \Delta_i'([N])/\beta\ .$$

<u>Lemma 3.1</u>: The class of a quadratic Poincare complex (C',ψ') in $L_i(A)$ depends only on ψ_0 . □

Proof: This is an immediate consequence of the algebraic theory of surgery [5,I.4.3] .

<u>Corollary 3.2</u>: The class of $(C \otimes D,\ \phi \otimes \psi)$ in the product formula depends only on ϕ_0 and ψ_0 .

We now restrict our attention to ϕ_0 . As a corollary of the proof of 2.1 we have:

<u>Corollary 3.3</u>: Let $C = \{C_i, \partial\}_{0 \le i \le n}$ be a chain complex of free $\mathbb{Z}\pi$-module such that $H_0(C) = \mathbb{Z}$ and $H_i(C) = 0$ for $0 < i < n$. Then there exists a $\mathbb{Z}[\pi]$ - module chain map

$$\Delta_0 : C \longrightarrow C \otimes C$$

satisfying conditions (i), (ii), and (iii), for $i = 0$.

Because of its importance in the product formula we make explicit on algorithm for computing ϕ_0. A contracting chain homotopy for C is given by $\mathbb{Z}$-module maps $\{s, \delta\}$

$$\delta : \mathbb{Z} \longrightarrow C_0$$

$$s : C_{i-1} \longrightarrow C_i$$

satisfying $\qquad \partial s + \delta\varepsilon = \mathrm{Id}$ on C_0

and $\qquad \partial s + s\partial + \mathrm{Id}$ on C_i for $0 < i < n$.

Choose a $\mathbb{Z}\pi$-basis of C_0 of the form $\Lambda \cup \{\rho_0\}$ where $\rho_0 = \delta(1)$ and $\varepsilon(f) = 0$ for $f \in \Lambda$.

Define

$$\Delta_0(\rho_0) = \rho_0 \otimes \rho_0 ,$$

$$\Delta_0(f) = \rho_0 \otimes f + f \otimes f + f \otimes \rho_0 \quad \text{for } f \in \Lambda .$$

Extend to a map on C_0 by linearity. Now assume Δ_0 has been defined on C_j for $j < k$. For a $\mathbb{Z}\pi$-basis element x of C_k define

$$\Delta_0(x) = x \otimes \rho_0 + \rho_0 \otimes x + (s \otimes 1 + \delta\varepsilon \otimes s)(\Delta_0(\partial x) - \partial x \otimes \rho_0 - \rho_0 \otimes \partial x) .$$

Extend to a map of C_k by linearity. This Δ_0 satisfies the desired properties.

Let X be an n-skeleton of a $K(\pi,1)$. Let $Y = K(\pi,1)$. Naively, one might try to avoid the algebra in 2.1 by constructing a geometric π-diagonal approximation

$$\Delta_0 : \; C(\tilde{Y}) \longrightarrow C(\tilde{Y}) \otimes C(\tilde{Y})$$

(Using, for example, the contracting chain homotopy $\{s \otimes 1 + \delta\,\varepsilon \otimes s, \delta \otimes \delta\}$ on $C(\tilde{Y}) \otimes C(\tilde{Y}))$, and then restricting the map

$$\Delta_0\big|_{C(\tilde{X})} : C(\tilde{X}) \longrightarrow C(\tilde{X}) \otimes C(\tilde{X})$$

to obtain a π-diagonal approximation for X. However, this Δ_0 need not satisfy the hypothes of 3.3, so there is no guarantee that $\Delta_0\big|_{C(\tilde{X})}$ is the correct chain homotopy class. In fact, unpublished computations of Jones and Milgram show that the above procedure can lead to a geometrically incorrect diagonal approximation for X. We now describe this example in some detail as it was the motivation of this paper.

A fundamental problem in surgery theory is the oozing problem, the problem of determining which elements of $L_*(\mathbb{Z}\pi)$ arise from surgery problems over closed manifolds. A critical example is the Cappell-Shaneson example

$$\mathrm{Id} \times f\colon S^3/\pi \times T^2 \longrightarrow S^3/\pi \times S^2$$

where $\pi = Q(2^n)$ is the generalized quaternion group and $f\colon T^2 \longrightarrow S^2$ is the Kervaire problem" representing the non-trivial element of $L_2(\mathbb{Z}) = \mathbb{Z}/2$. Here $\sigma(f)$ is represented by $(D,\psi) \in L_0(\mathbb{Z},-1) = L_2(\mathbb{Z})$ where

$$\psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = D^0 = \mathbb{Z} \oplus \mathbb{Z} \longrightarrow D_0 = \mathbb{Z} \oplus \mathbb{Z} .$$

Using geometric reasoning, S. Cappell and J. Shaneson showed $\sigma(\mathrm{Id} \times f) \neq 0 \in L_1^h(\mathbb{Z}\pi)$. However, the product formula hints at an algebraic derivation of this result. In a preliminary attempt at this problem, Jones and Milgram constructed a map

$$\Delta_0\colon\ C(K(\pi,1)) \longrightarrow C(K(\pi,1)) \otimes C(K(\pi,1))$$

and restricted to the 3-skeleton to obtain

$$\Delta_0\colon\ C \longrightarrow C \otimes C$$

where $C = C(S^3/\pi)$. (Cartan and Eilenberg [2] give an explicit periodic $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$ corresponding to a cell decomposition of S^3/π). The above Δ_0 lead to a chain homotopy equivalence

$$\phi_0\colon\ C^{3-*} \longrightarrow C_* .$$

Applying the product formula $(C \otimes D,\ \phi \otimes \psi)$, they obtained a trivial element of $L_1^h(\mathbb{Z}\pi)$, seemingly contradicting the Cappell-Shaneson example.

The resolution of this dilemma is that the naive approach does not lead to a geometrically correct result. Unpublished computations of the author show that the methods of this paper give a formation representing $\sigma(\mathrm{Id} \times f) \in L_1^h(\mathbb{Z}\pi)$, and prove algebraically the Cappell-Shaneson result that $\sigma(\mathrm{Id} \times f) \neq 0 \in L_1^h(\mathbb{Z}\pi)$, and $\mathrm{im}(\sigma(\mathrm{Id} \times f)) = 0 \in L_1^p(\mathbb{Z}\pi)$.

For an alternate algebraic approach to this result, see the paper of R. James Milgram, "The Cappell-Shaneson example," appearing in these proceedings.

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