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Manifolds homotopy equivalent to $P^n \# P^n$

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Abstract We classify, up to homeomorphism, all closed manifolds having the homotopy type of a connected sum of two copies of real projective n-space.

1 Statement of results

Let $P^n = P_n(\mathbb{R})$ be real projective n-space. López de Medrano [15] and Wall [19,20] classified, up to PL homeomorphism, all closed PL manifolds homotopy equivalent to P^n when $n > 4$. This was extended to the topological category by Kirby and Siebenmann [14, p. 331]. Four-dimensional surgery [11] extends the homeomorphism classification to dimension 4.

Cappell [3,5,6] discovered that the situation for connected sums is much more complicated. In particular, he showed [4] that there are closed manifolds homotopy equivalent to $P^{4k+1} \# P^{4k+1}$ which are not non-trivial connected sums. Recent computations of the unitary nilpotent group for the integers by Connolly and Ranicki [9], Connolly and Davis [7], and Banagl and Ranicki [1] show that there are similar examples in dimension $4k$ (see [13] for an analysis when $k = 1$).

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In this paper we classify up to homeomorphism all closed manifolds homotopy equivalent to $P^n\#P^n$. Any such manifold has $S^{n-1} \times S^1$ as a twofold cover; equivalently we classify free involutions on $S^{n-1} \times S^1$ inducing a non-trivial map on H_1.

This paper was prompted by a question of Lück [16, Sequence (4.10), Theorem 4.11]—what does the group automorphism $\mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2 * \mathbb{Z}_2$ given by interchanging the $\mathbb{Z}_2$'s induce in L-theory? We give a complete answer to Lück's question and apply the answer to the classify the above manifolds.

By the positive solution to Kneser's conjecture (see [12]) any closed 3-manifold homotopy equivalent to $P^3\#P^3$ is homeomorphic to $Q^3\#R^3$ where Q^3 and R^3 are closed 3-manifolds homotopy equivalent to P^3. The spherical space form conjecture in dimension 3 would imply that Q^3 and R^3 are homeomorphic to P^3. Hence, conjecturally, any closed 3-manifold homotopy equivalent to $P^3\#P^3$ is standard. Henceforth we assume $n > 3$. Note that the fundamental group of $P^4\#P^4$ is small in the sense of Freedman and Quinn [11], so that surgery theory applies.

Let $\bar{I}_n$ (respectively $\bar{J}_n$) be the set of homeomorphism classes of closed manifolds homotopy equivalent to P^n (respectively $P^n\#P^n$). For n even, let $I_n = \bar{I}_n$ and $J_n = \bar{J}_n$. For n odd, let I_n (respectively J_n) be the set of oriented homeomorphism classes of closed oriented manifolds homotopy equivalent to P^n (respectively $P^n\#P^n$). The set I_n was computed in [14, Sect. 16 Annex 3]; we review the computation in Sect. 2.

Let R be a ring with involution. Let A and B be (R, R)-bimodules with involution. (The case of interest is $R = \mathbb{Z}$, and then A and B are just abelian groups with an automorphism of order 2.) Cappell [5] defined unitary nilpotent groups $\mathrm{UNil}_n(R; A, B)$, abelian groups with the following properties.

- They satisfy periodicity $\mathrm{UNil}_n(R; A, B) \cong \mathrm{UNil}_{n+4}(R; A, B)$ and semiperiodicity $\mathrm{UNil}_n(R; A, B) \cong \mathrm{UNil}_{n+2}(R; A^-, B^-)$, where A^- is the original bimodule but with the involution $a \mapsto \bar{a}$ replaced by $a \mapsto -\bar{a}$.
- They obstruct splitting: given a homotopy equivalence $h : X \to P^n\#P^n$ for any $n > 4$, there is an element $\mathrm{split}(h) \in \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon)$ which vanishes if and only if h is splittable.[1] Here $\epsilon = (-1)^{n+1}$. Conversely all obstructions are realized. Note that to define the element $\mathrm{split}(h)$ one needs to choose an orientation for $P^n\#P^n$, that is, a generator for the infinite cyclic group $H_n(P^n\#P^n; \mathbb{Z}^w)$ where w is the orientation character.
- There is a split injection $i : \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon) \to L_n(\mathbb{Z}[\mathbb{Z}_2^\epsilon * \mathbb{Z}_2^\epsilon])$.

The groups $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ have been completely computed (see, for example, [7]). The groups $\mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ and $\mathrm{UNil}_1(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ vanish. As abelian groups $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \oplus_\infty \mathbb{Z}_2$ and $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \oplus_\infty \mathbb{Z}_2 \oplus_\infty \mathbb{Z}_4$.

The "switch map" $\mathrm{sw} : P^n\#P^n \to P^n\#P^n$ interchanges the two summands. We also use the same notation to denote the induced involution on $\mathbb{Z}_2 * \mathbb{Z}_2$, on $L_n(\mathbb{Z}[\mathbb{Z}_2^\epsilon * \mathbb{Z}_2^\epsilon])$, and on $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon)$. The semiperiodicity mentioned above is equivariant with respect to the switch map.

The following theorem, proved in Sects. 3 and 4 is the main technical result of this paper.

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[1] h is splittable if it is homotopic to a map f, transverse to S, which induces a homotopy equivalence $f^{-1}(S) \to S$, where S is the codimension one sphere defining the connected sum.

Theorem 1 1. *The map* sw : $\mathrm{UNil}_2(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \to \mathrm{UNil}_2(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ *is the identity.*
2. *Connolly and Davis* [7] *give a* $\mathbb{Z}$*-module isomorphism*

$$(\, j_1 \; j_2 \,) : \frac{t\mathbb{Z}_4[t]}{\{2p(t^2) - 2p(t) : p(t) \in t\mathbb{Z}_4[t]\}} \times t\mathbb{Z}_2[t] \to \mathrm{UNil}_3(\mathbb{Z};\mathbb{Z},\mathbb{Z}).$$

Let $\pi : \frac{t\mathbb{Z}_4[t]}{\{2p(t^2)-2p(t)\}} \to t\mathbb{Z}_2[t]$ *be the quotient map* $\pi[tp] = [tp]$. *Then using the above coordinates* $(\, j_1 \; j_2 \,)$ *for* $\mathrm{UNil}_3(\mathbb{Z};\mathbb{Z},\mathbb{Z})$, *the switch map is* $\mathrm{sw} = \left(\begin{smallmatrix} 1 & 0 \\ \pi & 1 \end{smallmatrix}\right)$.

Thus the switch map sw : $\mathrm{UNil}_3(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \to \mathrm{UNil}_3(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ is the identity on all elements which are multiples of two, as well as on some elements which are not multiples of two, but is non-trivial on some elements of order two.

Given homotopy equivalences $h_1 : X_1 \to P^n$ and $h_2 : X_2 \to P^n$ and an element $\vartheta \in \mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\epsilon,\mathbb{Z}^\epsilon)$, Wall realization [20, Theorems 10.4 and 10.5] produces a normal bordism

$$(g; h_1\#h_2, h) : (W; X_1\#X_2, X) \to P^n\#P^n \times ([0,1]; \{0\}, \{1\})$$

with $h : X \to P^n\#P^n$ a homotopy equivalence and with the rel ∂ surgery obstruction of g the image of ϑ in the L-group

$$\sigma_*(g) = i(\vartheta) \in i(\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\epsilon,\mathbb{Z}^\epsilon)) \subset L_{n+1}(\mathbb{Z}[\mathbb{Z}_2^\epsilon * \mathbb{Z}_2^\epsilon])$$

Section 2 proves the following theorem.

Theorem 2 *Let* $n > 3$ *and* $\epsilon = (-1)^{n+1}$. *There is a bijection from the set*

$$(\textit{unordered pairs from the set } I_n) \times \left(\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\epsilon,\mathbb{Z}^\epsilon)/\,\mathrm{sw}\right) \to J_n$$

defined by

$$\left(\{[h_1 : X_1 \to P^n], [h_2 : X_2 \to P^n]\}, [\vartheta]\right) \mapsto [X],$$

where the homotopy equivalence $h : X \to P^n\#P^n$ *is produced by Wall realization as above.*

Moreover if $[\vartheta] \neq 0$, *then* $[X]$ *is not represented by the connected sum of manifolds with fundamental group* $\mathbb{Z}_2$.

For $n \not\equiv 3 \pmod 4$, $I_n \to \bar{I}_n$ *and* $J_n \to \bar{J}_n$ *are bijective, while for* $n \equiv 3 \pmod 4$, $I_n \to \bar{I}_n$ *and* $J_n \to \bar{J}_n$ *are at most* 2*-to-*1.

Theorem 1 computes $\mathrm{UNil}_n(\mathbb{Z};\mathbb{Z},\mathbb{Z})/\,\mathrm{sw}$ for $n = 2, 3$; the low-dimensional manifolds to keep in mind are $P^5\#P^5$ and $P^4\#P^4$, respectively. A precise description of the maps $I_n \to \bar{I}_n$ and $J_n \to \bar{J}_n$ is given at the end of Sect. 2.

The computation of the switch map on UNil should be considered as the main result of this paper. We now indicate the difficulty. For a ring with involution R, define

$NL_n(R)$ to be the kernel of the augmentation map $L_n(R[t]) \to L_n(R)$ given by sending t to 0. The papers [1,7,9] proceed to compute $NL_n(R)$, and hence $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ since Connolly and Ranicki [9, Definition 13, Theorem A] define a map

$$r : \mathrm{UNil}_n(R; R, R) \to NL_n(R)$$

and show it is an isomorphism. However the induced switch map on $NL_n(R)$ is not apparent and the inverse map r^{-1} is not explicit.

Here is how to get around the difficulty. Cappell [5] defined a split injection $i : \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to L_n(\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_2])$ which commutes with the switch map induced by the ring automorphism of $\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_2]$ given by switching factors. We give an explicit formula for $F = i \circ r^{-1}$. Then for x in a certain generating set of $NL_n(\mathbb{Z})$, we find y such that $\mathrm{sw}\, F(x) = F(y)$, and thereby deduce that $\mathrm{sw}(r^{-1}(x)) = r^{-1}(y)$.

2 The structure set of $P^n \# P^n$

In this section we compute the structure set of $P^n \# P^n$ and reduce the homeomorphism classification to the structure set of $P^n \# P^n$ modulo the $\mathbb{Z}_2$-action given by the switch map. The material in the section is a standard application of the surgery machine.

The structure set $\mathcal{S}(M)$ of a closed topological manifold M is defined to be the set of equivalence classes of "s-triangulations," simple homotopy equivalences $h : X \to M$ where X is a closed topological manifold. Two such s-triangulations $h_1 : X_1 \to M$ and $h_2 : X_2 \to M$ are equivalent if there is a homeomorphism $f : X_1 \to X_2$ so that $h_2 \circ f$ and h_1 are homotopic.

For a space M having the homotopy type of a finite CW complex, let $\mathrm{hAut}(M)$ be the group of homotopy classes of simple self-homotopy equivalences $M \to M$. The following lemma is standard and its proof is trivial.

Lemma 3 $\mathrm{hAut}(M)$ *acts on* $\mathcal{S}(M)$ *by post-composition. The forgetful map*

$$\begin{aligned} \mathcal{S}(M)/\,\mathrm{hAut}(M) &\to \frac{\textit{closed manifolds simple homotopy equivalent to M}}{\textit{homeomorphism}} \\ [h : X \to M] &\mapsto [X] \end{aligned}$$

is a bijection of sets.

Recall the fundamental group of $P^n \# P^n$ for $n > 2$ is isomorphic to both the infinite dihedral group,

$$D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle t, a \mid ata = t^{-1}, a^2 = 1\rangle$$

and to the free product (letting $t = ba$)

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = 1 = b^2\rangle,$$

which has [18, Theorem 6.4] Whitehead torsion group $\mathrm{Wh}(\mathbb{Z}_2) \oplus \mathrm{Wh}(\mathbb{Z}_2) = 0$. Thus for $M = P^n \# P^n$ or P^n we can drop the word "simple" in the definition of $\mathcal{S}(M)$ and $\mathrm{hAut}(M)$.

We next review the computations of $\mathcal{S}(P^n)$, I_n, and $\bar{I}_n$. Write $n = 4m + \ell > 3$ for unique $m \geq 0$ and $0 < \ell \leq 4$. Recall that [14, p. 331] the topological structure set $\mathcal{S}(P^n)$ is in bijection with the set

$$\bigoplus_{0<k\leq 2m+[\ell/4]} \mathbb{Z}_2 \oplus \begin{cases} \mathbb{Z} & \text{if } \ell = 3 \\ 0 & \text{if } \ell \neq 3. \end{cases}$$

The $\mathbb{Z}_2$ summands (normal invariants) arise as the surgery obstructions of degree one normal maps which are the transverse restrictions to P^{2k} of the given homotopy equivalence to P^n. The map from $\mathbb{Z}$ to the structure set is given by Wall realization $\mathbb{Z} \cong \widetilde{L}_{n+1}(\mathbb{Z}[\mathbb{Z}_2]) \to \mathcal{S}(P^n)$ and the splitting map from the structure set to $\mathbb{Z}$ is the Browder-Livesay desuspension invariant of a free $\mathbb{Z}_2$-action on S^{4m+3} to an action on some embedded S^{4m+2}.

It is (almost) elementary to show that any self-homotopy equivalence of P^n which does not reverse orientation is homotopic to the identity. Thus for n even, $\mathrm{hAut}(P^n) = *$ and for n odd, $\mathrm{hAut}(P^n) = \mathbb{Z}_2$. This acts trivially on the normal invariant, but for $n = 4m + 3$ acts by multiplication by -1 on the $\mathbb{Z}$-summand, since reversing orientation reverses the sign of any signature invariant. The following theorem summarizes the discussion.

Theorem 4 *Let $n = 4m + \ell > 3$ where $m \geq 0$ and $0 < \ell \leq 4$. There are bijections*

$$\begin{aligned} \Phi : \mathcal{S}(P^n) &\longrightarrow \bigoplus_{0<k\leq 2m+[\ell/4]} \mathbb{Z}_2 \oplus \begin{cases} \mathbb{Z} & \text{if } \ell = 3 \\ 0 & \text{else.} \end{cases} \\ I_n &\longrightarrow \mathcal{S}(P^n) \\ \bar{I}_n &\longrightarrow \frac{\mathcal{S}(P^n)}{\Phi^{-1}(z) \sim \Phi^{-1}(-z)} \end{aligned}$$

Identify

$$P^n \# P^n = \frac{S^{n-1} \times S^1}{(w, z) \sim (-w, \bar{z})}$$

The sphere defining the connected sum is $S^{n-1} \times \{\pm i\}/\sim$. Following Cappell [4, Proof 3], define self-homeomorphisms $\gamma_1, \gamma_2, \gamma_3$ of $P^n \# P^n$ by:

- $\gamma_1[w, z] = [w, -z]$, which interchanges the two summands of $P^n \# P^n$; also called the switch map sw,
- $\gamma_2[(w_1, w_2, \ldots, w_n), z] = [(w_1, w_2, \ldots, w_{n-1}, -w_n), z]$, which reflects through $P^{n-1} \# P^{n-1}$, and

- $\gamma_3[w, z] = \begin{cases} [\tau(z^2)(w), z] & \text{if } \operatorname{Im} z \geq 0 \\ [\tau(\bar{z}^2)(w), z] & \text{if } \operatorname{Im} z \leq 0 \end{cases}$,
  which Dehn twists along the connecting cylinder via the isotopy $\tau : S^1 \to SO(n)$ generating $\pi_1(SO(n)) \cong \mathbb{Z}_2$.

When n is even, γ_2 is isotopic to the identity via

$$[(w_1 \cos(\pi t) - w_2 \sin(\pi t), w_1 \sin(\pi t) + w_2 \cos(\pi t), \dots, -w_n), z)].$$

Lemma 5 ([4, Proof 3], see also [13]) $\mathrm{hAut}(P^n \# P^n) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$. *For n even,* $\mathrm{hAut}(P^n \# P^n) = \langle \gamma_1, \gamma_3 \rangle$.

In particular, all elements of $\mathrm{hAut}(P^n \# P^n)$ are splittable.

Corollary 6 ([4, Lemma 2]) *If an s-triangulation* $h : X \to P^n \# P^n$ *is not splittable, then* X *is not a connected sum of manifolds with fundamental group* $\mathbb{Z}_2$.

Lemma 7 *Let* $\mathcal{S}_{\mathrm{split}}(P^n \# P^n)$ *be the subset of* $\mathcal{S}(P^n \# P^n)$ *given by splittable homotopy equivalences. Let* $n > 3$ *and* $\epsilon = (-1)^{n+1}$.

1. *Cappell's nilpotent normal cobordism construction gives a bijection*

$$\begin{aligned} \mathrm{nncc} : \mathcal{S}(P^n \# P^n) &\to \mathcal{S}_{\mathrm{split}}(P^n \# P^n) \times \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon) \\ h &\mapsto (-\operatorname{split}(h) \cdot h, \operatorname{split}(h)), \end{aligned}$$

   where $\cdot$ *refers to the action of the L-group on the structure set.*
2. *The above bijection is equivariant with to the action of* $\mathrm{hAut}(P^n \# P^n)$ *on the structure and split structure sets given by post-composition and on* $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon)$ *given by the switch map if the homotopy automorphism represents the non-trivial element of* $\mathrm{Out}(\mathbb{Z}_2 * \mathbb{Z}_2)$ *and by the identity otherwise.*
3. *Connected sum gives a bijection* $\# : \mathcal{S}(P^n) \times \mathcal{S}(P^n) \to \mathcal{S}_{\mathrm{split}}(P^n \# P^n)$.

Proof 1. The action of the L-group on the structure set is given by Wall realization, see [20, Theorems 10.4 and 10.5].

For $n > 5$, the nilpotent normal cobordism construction is given in [6, Sects. II.1, III.2] (see also [3, Theorem 3]). In [3, Theorem 1] the map split : $\mathcal{S}(P^n \# P^n) \to \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon)$ is denoted by χ^h. By [3, Theorem 2], $\operatorname{split}(-\operatorname{split}(h) \cdot h) = -\operatorname{split}(h) + \operatorname{split}(h) = 0$, so the first component of nncc is indeed splittable.

In dimension $n = 5$, since the submanifold S^4 of $P^5 \# P^5$ defining the connected sum is simply-connected, the bijection nncc exists by the modification in [6, Theorem 5, Sect. V.2] of the 4-manifold stable surgery in [10, Theorems 4.1, 5.1]. Alternatively, proceed analogous to the $n = 4$ case below.

In dimension $n = 4$, we prove the existence of the bijection nncc by the following indirect method. For $h \in \mathcal{S}(P^4 \# P^4)$ one can, following Cappell, define the obstruction $\operatorname{split}(h) \in \mathrm{UNil}_5(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}^-)$. Since the fundamental group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$ is small, by Wall realization and plus construction [11, Theorems 11.3A, 11.1A], there exists a topological normal bordism H from h to another homotopy equivalence $h' = -\operatorname{split}(h) \cdot h$ with surgery obstruction $\sigma_*(H) = i(-\operatorname{split}(h))$, where

$i : \mathrm{UNil}_5(\mathbb{Z};\mathbb{Z}^-,\mathbb{Z}^-) \hookrightarrow L_5(\mathbb{Z}[\mathbb{Z}_2^- * \mathbb{Z}_2^-])$. Furthermore $\mathrm{split}(h') = 0$ as before. It remains to show that h' is splittable. We accomplish this by periodicity. (See [20, Theorem 9.9]).

Let Z be a closed, simply-connected $4k$-manifold with signature $\sigma(Z)$.

$$\mathrm{split}(h' \times \mathrm{id}_Z) = \sigma(Z)\,\mathrm{split}(h') = 0.$$

Taking $Z = P_2(\mathbb{C})$ and using [21, Proof of Theorem 1], one sees that h' is $\mathbb{Z}$-homology splittable, i.e. h' is homotopic to a map whose restriction to the transverse inverse image Σ^3 of the connecting S^3 in $P^4\#P^4$ is a $\mathbb{Z}$-homology equivalence. But then h' is necessarily (topologically) splittable, by the "neck exchange" trick of Jahren and Kwasik [13, Proof of Theorem 2]–essentially due to the fact [11, Corollary 9.3C] that the homology 3-sphere Σ bounds a contractible 4-manifold Δ.

2. Let $h : M \to P^n\#P^n$ represent an element of the structure set.

By Lemma 5, every element of $\mathrm{hAut}(P^n\#P^n)$ can be represented by a map γ which is the identity on the codimension one sphere S defining the connected sum and which sends the complement $P^n\#P^n - S$ to itself (possibly interchanging the two components). It is then clear that $\mathrm{split}(\gamma \circ h) = \gamma_*\,\mathrm{split}(h)$ where γ_* is either the switch map or the identity.

Wall realization provides a normal bordism H from h to $h' \in \mathcal{S}_{\mathrm{split}}(P^n\#P^n)$ with surgery obstruction $\sigma_*(H) = -i(\mathrm{split}(h)) \in L_{n+1}(\mathbb{Z}[\mathbb{Z}_2^\epsilon * \mathbb{Z}_2^\epsilon])$. Then $(\gamma \times \mathrm{id}_{[0,1]}) \circ H$ is a normal bordism from $\gamma \circ h$ to $\gamma \circ h'$ and, since γ is a homotopy equivalence preserving the orientation character, the surgery obstruction satisfies

$$\sigma_*((\gamma \times \mathrm{id}_{[0,1]}) \circ H) = \gamma_*(\sigma_*(H)) = i(-\,\mathrm{split}(\gamma \circ h)).$$

It follows that

$$\begin{aligned}\gamma \circ (-\,\mathrm{split}(h) \cdot h) &= \gamma \circ h' \\ &= -\,\mathrm{split}(\gamma \circ h) \cdot (\gamma \circ h).\end{aligned}$$

In other words, the first component of the map nncc is also equivariant with respect to the action of $\mathrm{hAut}(P^n\#P^n)$.

3. Observe that the connected sum operation

$$\# : \mathcal{S}(P^n) \times \mathcal{S}(P^n) \to \mathcal{S}_{\mathrm{split}}(P^n\#P^n)$$

on topological structure sets is surjective. For $n > 4$ this uses the affirmative solution to the Poincaré conjecture for homotopy spheres of dimension $n - 1 > 3$, and for $n = 4$ this uses the Jahren and Kwasik neck exchange trick, replacing a homotopy sphere Σ^3 by a genuine S^3.

To show injectivity of # for all $n > 3$, suppose that $h_1\#h_2$ is s-bordant to $h_1'\#h_2'$ via H, where each $h_i, h_i' \in \mathcal{S}(P^n)$. By taking the inverse image of the connecting S^{n-1} under a homeomorphism, we may assume that H is a homotopy. Then we have a relative splitting problem of the $(n+1)$-dimensional homotopy equivalence H along

$S^{n-1} \times [0,1]$, which is already split along its boundary $S^{n-1} \times \{0,1\}$. The relative form of the nilpotent normal cobordism construction in particular provides an s-bordism H' from $h_1 \# h_2$ to $h'_1 \# h'_2$, which splits as a boundary connected sum $H' = H'_1 \natural H'_2$ along $S^{n-1} \times [0,1]$. Hence h_i, h'_i represent the same element in $\mathcal{S}(P^n)$ for all $i = 1, 2$. □

Proof of Theorem 2 Lemmas 3 and 5 show that to compute $\bar{J}_n$, the set of homeomorphism classes of closed manifolds homotopy equivalent to $P^n \# P^n$, one needs to compute the action of $\gamma_1, \gamma_2, \gamma_3$ on $\mathcal{S}(P^n \# P^n)$. In terms of the bijection of Lemma 7,

$$\mathcal{S}(P^n \# P^n) \leftrightarrow \mathcal{S}(P^n) \times \mathcal{S}(P^n) \times \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon),$$

$\gamma_1 \cdot (h_1, h_2, x) = (h_2, h_1, \mathrm{sw}\, x)$. (More on $\mathrm{sw}\, x$ later in this paper.)

Note that post-composition with γ_3 acts via the identity on $\mathcal{S}_{\mathrm{split}}(P^n \# P^n)$ since such a structure can be represented by a connected sum $h_1 \# h_2 : M_1 \# M_2 \to P^n \# P^n$ and $\gamma_3 \circ h_1 \# h_2 = h_1 \# h_2 \circ \gamma$ where γ is the homeomorphism given by a Dehn twist in the domain. Also, γ_3 acts trivially on $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\epsilon, \mathbb{Z}^\epsilon)$ since it induces the identity on the fundamental group. Thus γ_3 acts trivially on $\mathcal{S}(P^n \# P^n)$ by Lemma 7, parts (1) and (2).

Now assume that n is odd. Post-composition by reflection through P^{n-1} acts on $\mathcal{S}(P^n)$ by multiplication by $h \mapsto \Phi^{-1}(-\Phi(h))$ according to Theorem 4. Lemma 7 then shows that $\gamma_2 \cdot (h_1, h_2, x) \mapsto (\Phi^{-1}(-\Phi(h_1)), \Phi^{-1}(-\Phi(h_2)), x)$.

Theorem 2 follows. □

The proof shows a bit more. It shows that the quotient function $\mathcal{S}(P^n \# P^n)/\langle \mathrm{sw} \rangle \to J_n$ is a bijection. It allows us to be more precise about the map $J_n \to \bar{J}_n$ when $n \equiv 3 \pmod 4$. Using the notation of Theorems 4 and 2, the elements $(\{\Phi^{-1}(y), \Phi^{-1}(z)\}, \vartheta) \in J_n$ and $(\{\Phi^{-1}(-y), \Phi^{-1}(-z)\}, \vartheta) \in J_n$ map to the same element of $\bar{J}_n$. This is the only way a pair of elements in J_n can have the same image in $\bar{J}_n$.

3 Switch action on $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$

Theorem 8 *The switch map* sw *operates as the identity on* $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$.

Lemma 9 *Let* $r : \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to NL_n(\mathbb{Z})$ *be the Connolly–Ranicki isomorphism and* $i : \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to L_n(\mathbb{Z}[D_\infty])$ *be the injection defined by Cappell. Let* $F : NL_n(\mathbb{Z}) \to L_n(\mathbb{Z}[D_\infty])$ *be the map* $F = i \circ r^{-1}$. *Then* F *is given by the formula:*

$$F[(E, \chi)] = [(\mathbb{Z}[D_\infty] \otimes_{\mathbb{Z}[t]} E, a \otimes \chi)]$$

where "a" is the left $\mathbb{Z}[D_\infty]$*-module endomorphism of* $\mathbb{Z}[D_\infty]$ *defined by right multiplication by* $a \in D_\infty$.

Proof It is easy to verify

$$\begin{aligned} G : NL_n(\mathbb{Z}) &\to L_n(\mathbb{Z}[D_\infty]) \\ [(E, \chi)] &\mapsto [(\mathbb{Z}[D_\infty] \otimes_{\mathbb{Z}[t]} E, a \otimes \chi)] \end{aligned}$$

is a well-defined homomorphism. Here (E, χ) is a quadratic Poincaré complex over $\mathbb{Z}[t]$ in the sense of Ranicki [17].

We need to show $G \circ r = i$. We first check first this on $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$. Let $x = [(P, Q, \rho_1 a, \rho_2 b, \mu_1 a, \mu_2 b)] \in \mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ (see [5] for the definition of the UNil groups). From Connolly–Ranicki,

$$r(x) = \left[\left(P[t] \oplus Q[t], \begin{pmatrix} \rho_1 & 1 \\ -1 & \rho_2 t \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 t \end{pmatrix}\right)\right].$$

Then

$$\begin{aligned} G(r(x)) &= \left[\left(\mathbb{Z}[D_\infty] \otimes_{\mathbb{Z}[t]} (P \oplus Q), \begin{pmatrix} \rho_1 a & a \\ -a & \rho_2 b \end{pmatrix}, \begin{pmatrix} \mu_1 a \\ \mu_2 b \end{pmatrix}\right)\right] \\ &= \left[\left(\mathbb{Z}[D_\infty] \otimes_{\mathbb{Z}[t]} (P \oplus Q), \begin{pmatrix} \rho_1 a & 1 \\ -1 & \rho_2 b \end{pmatrix}, \begin{pmatrix} \mu_1 a \\ \mu_2 b \end{pmatrix}\right)\right] \\ &= i(x). \end{aligned}$$

where the second equality is seen by pulling back along the isomorphism $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and the last equality is the definition of $i(x)$ (see [5]).

To deal with the odd-dimensional case, in order to appeal to Connolly–Ranicki we use the fact that all of the maps G, R, and i can be defined for any ring R, not just $\mathbb{Z}$ and are functorial with respect to maps of rings. Therefore the standard Shaneson splitting argument of Connolly and Ranicki [9, Proposition 19] implies that $G \circ r = i$ on $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ also. □

Remark 10 The map F has a geometric interpretation which led us to the formula of the previous lemma. A homotopy equivalence $X^n \to P^n\#P^n$ can be considered both as a one-sided splitting problem by splitting along $P^{n-1}\#P^{n-1}$ and a two-sided splitting problem by splitting along S^{n-1}. The passage from the one-sided splitting obstruction to the two-sided splitting obstruction coincides with the map F. More precisely F is the composite

$$\begin{aligned} NL_n(\mathbb{Z}) \to L_n(\mathbb{Z}[t]) \to L_n(\mathbb{Z}[t, t^{-1}]) &\xrightarrow{\mathrm{aqk}^{-1}} LN_n(\mathbb{Z} \to \mathbb{Z}_2^- * \mathbb{Z}_2^-) \\ &\xrightarrow{\partial} L_n(\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_2]). \end{aligned}$$

The first map is an inclusion, and the second is induced by an inclusion of rings (note we have the trivial involution $t \mapsto t$).

The group $LN_n(\mathbb{Z} \to \mathbb{Z}_2^- * \mathbb{Z}_2^-)$ is defined in [20, Chap. 11] as the obstruction group for one-sided splitting and sits in the exact sequence

$$\begin{aligned} \cdots \to L_{n+1}(\mathbb{Z}[\mathbb{Z}_2^- * \mathbb{Z}_2^-]) \to L_{n+2}(\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_2]) \\ \to LN_n(\mathbb{Z} \to \mathbb{Z}_2^- * \mathbb{Z}_2^-) \to \cdots \end{aligned}$$

where the first map is a transfer map corresponding to the line bundle of the "twofold cover" $B\mathbb{Z} \to BD_\infty$. This LN-group is identified algebraically in [17, Sect. 7.6, pp. 691–695] as a relative term in the exact sequence. That is, each element of LN_n is represented by a pair consisting of a quadratic Poincaré complex of dimension n over $\mathbb{Z}[\mathbb{Z}_2^- * \mathbb{Z}_2^-]$ and a quadratic pair of dimension $n+2$ over $\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_2]$ with algebraic boundary the transfer pair of the complex.

The map ∂ is defined in [17, Sect. 7.2, p. 565] as the boundary map in the transfer sequence; it forgets the quadratic pair. Geometrically it corresponds to a transversal restriction of a one-sided splitting problem to a degree one normal map.

The map aqk is an isomorphism. It was defined geometrically by Wall (see [20, Theorem 12.9]) as the $\mathbb{Z}_2$-equivariant defect for handle exchanges in the middle dimension of a certain regular twofold cover. The map aqk was dubbed the anti-quadratic kernel by Ranicki [17, Sect. 7.6, pp. 698–699] and was given an algebraic definition.

Actually for us, aqk^{-1} is the relevant map and is somewhat easier to define. However, we will omit the definition here, and refer to [17, Proof 7.6.3, p. 702] for the formula

$$(\partial \circ \mathrm{aqk}^{-1})\,[(E, \chi)] = \left[(\mathbb{Z}[D_\infty] \otimes_{\mathbb{Z}[t]} E, a \otimes \chi)\right].$$

A careful reading of [17, Sect. 7.6, pp. 737–745] shows that ∂ takes the one-sided splitting obstruction to the image of the two-sided splitting obstruction and that the composite map F above satisfies $F \circ r = i$. □

Proof of Theorem 8 Recall [7, Theorem 4.6(2)] (see also [8]) that the map $L_2(\mathbb{Z}[t]) \to L_2(\mathbb{F}_2[t])$ is an isomorphism and the latter is detected by restriction of the Arf invariant for the characteristic two field $\mathbb{F}_2(t)$—first map to $L_2(\mathbb{F}_2(t))$ and then by the Arf invariant to the idempotent quotient $\mathbb{F}_2(t)/\{f^2 - f \mid f \in \mathbb{F}_2(t)\}$. So it suffices to compute the switch map sw on the values of the inverse map

$$\frac{t\mathbb{F}_2[t]}{\{f^2 - f \mid f \in \mathbb{F}_2[t]\}} \to NL_2(\mathbb{Z})$$

defined by

$$[tp] \mapsto [P_{tp,1}] = \left[\mathbb{Z}[t]^2, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} tp \\ 1 \end{smallmatrix}\right)\right].$$

Write $t = ba$, which generates the infinite cyclic subgroup $\mathbb{Z}$ of index two in D_∞. Recall the definition of $F : NL_2(\mathbb{Z}) \to L_2(\mathbb{Z}[D_\infty])$ in Lemma 9.

Then $F[P_{tp,1}]$ is represented by the quadratic form:

$$\left(\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} p(t)b \\ a \end{smallmatrix}\right)\right).$$

Note

$$\begin{aligned} \mathrm{sw}\, F[P_{tp,1}] &= \mathrm{sw}\left[\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} p(t)b \\ a \end{smallmatrix}\right)\right] \\ &= \mathrm{sw}\left\{\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)^* [\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} p(t)b \\ a \end{smallmatrix}\right)]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{sw}[\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} bp(t)bb \\ aaa \end{smallmatrix}\right)] \\
&= \mathrm{sw}\left[\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} bp(t) \\ a \end{smallmatrix}\right)\right] \\
&= \left[\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} ap(t^{-1}) \\ b \end{smallmatrix}\right)\right] \\
&= \left[\mathbb{Z}[D_\infty]^2, \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} p(t)a \\ ta \end{smallmatrix}\right)\right] \\
&= F[P_{p,t}].
\end{aligned}
$$

Therefore $\mathrm{sw}[P_{tp,1}] = [P_{p,t}]$ by injectivity of F. But both $[P_{p,t}]$ and $[P_{tp,1}]$ have the same Arf invariant $[tp]$. □

4 Switch action on $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$

We compute the switch action on $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ in a similar way to what we did for $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$; we use [7] to find generators, use Lemma 9 to compute the switch map on the generators, and use the obstruction theory of [7] to express the result in terms of the original generators.

Connolly and Davis [7] give a $\mathbb{Z}$-module isomorphism

$$
(\, j_1 \; j_2 \,) : \frac{t\mathbb{Z}_4[t]}{\{2p(t^2) - 2p(t) : p(t) \in t\mathbb{Z}_4[t]\}} \times t\mathbb{Z}_2[t] \to \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}).
$$

Theorem 11 *Let* $\pi : \frac{t\mathbb{Z}_4[t]}{\{2p(t^2)-2p(t)\}} \to t\mathbb{Z}_2[t]$ *be the quotient map* $\pi[tp] = [tp]$. *Then using the above coordinates* $(\, j_1 \; j_2 \,)$ *for* $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$*, the switch map is* $\mathrm{sw} = \left(\begin{smallmatrix} 1 & 0 \\ \pi & 1 \end{smallmatrix}\right)$.

Corollary 12 $x \in \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ *is fixed by the switch map if* x *is divisible by two.*

Example 13 Consider the Poincaré (-1)-quadratic complex (C, ψ) of dimension one over $\mathbb{Z}[D_\infty]$ defined by the data:

- $C_1 = \mathbb{Z}[D_\infty] \oplus \mathbb{Z}[D_\infty] = C_0$ and $d := \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$,
- $\psi_0 = \left(\begin{smallmatrix} 2a & 1 \\ 1 & b \end{smallmatrix}\right) : C^0 \to C_1$ and $\psi_1 = \left(\begin{smallmatrix} -2a & 0 \\ -2 & -b \end{smallmatrix}\right) : C^0 \to C_0$.

Observe it is the image of a certain (-1)-quadratic nilcomplex of dimension one over the triple $(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$, see [2, Definition 11.11], under the natural group monomorphism $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to L_3(\mathbb{Z}[D_\infty])$ of cobordism classes. Moreover from the proof of Lemma 17, its cobordism class $[(C, \psi)]$ is also the image of an order 4 element $[\mathcal{N}_{t,1}] = j_1[t]$ of $t\mathbb{Z}_4[t]/\{2p(t^2) - 2p(t)\}$. Then

$$
\mathrm{sw}[\mathcal{N}_{t,1}] = \mathrm{sw}(j_1[t] \oplus 0) = j_1[t] \oplus j_2[t] = [\mathcal{N}_{1,t}].
$$

In particular $\mathrm{sw}[(C, \psi)] \neq \pm[(C, \psi)]$, hence $[(C, \psi)] \in L_3(\mathbb{Z}[D_\infty])$ is the image of a non-zero element of $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ which is not fixed by the switch map.

The actual computation of Connolly–Davis was of the group $\widetilde{\mathcal{L}}(\mathbb{Z}[t], 2)$, the Witt group of quadratic linking forms on $\mathbb{Z}[t]$-modules with exponent 2 which become Witt trivial under the map $t \to 0$. Recall that these are defined to be triples (M, b, q) where

- M is a $\mathbb{Z}[t]$-module isomorphic to $\mathbb{F}_2[t]^k$ for some k;
- $b : M \times M \to \mathbb{Q}[t]/\mathbb{Z}[t]$ is a nonsingular symmetric linking form;
- $q : M \times M \to \mathbb{Q}[t]/2\mathbb{Z}[t]$ is a quadratic refinement of b (so that in particular $[q(x)] = b(x, x) \in \mathbb{Q}[t]/\mathbb{Z}[t]$).

This group was shown to be isomorphic to $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ via the composite isomorphism

$$\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \xrightarrow{\cong} NL_3(\mathbb{Z}) \xleftarrow{\cong} NL_0(\mathbb{Z}, \langle 2\rangle) \xleftarrow{\cong} \widetilde{\mathcal{L}}(\mathbb{Z}[t], 2).$$

Here the first isomorphism was from Connolly and Ranicki [9], the second from a localization exact sequence, and the third by a devissage argument.

The maps j_1 and j_2 were defined in terms of certain basic linking forms, so it is sufficient for us to compute the effect of the switch map on those:

Definition 14 ([7, Definition 1.6]) For $p, g \in \mathbb{Z}[t]$ where either $p(0) = 0$ or $q(0) = 0$, define $[\mathcal{N}_{p,g}] \in \widetilde{\mathcal{L}}(\mathbb{Z}[t], 2)$ by the $(+1)$-quadratic linking form

$$\mathcal{N}_{p,g} = \left(\mathbb{Z}_2[t]^2, \begin{pmatrix} p/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} p/2 \\ g \end{pmatrix}\right).$$

Then $j_1[tp] = \left[\mathcal{N}_{tp,1}\right]$ and $j_2[tp] = \left[\mathcal{N}_{1,tp}\right] - \left[\mathcal{N}_{t,p}\right]$.

Lemma 15 *Let $s : \widetilde{\mathcal{L}}(\mathbb{Z}[t], 2) \to NL_3(\mathbb{Z})$ be the isomorphism mentioned above.*

1. *$s[\mathcal{N}_{tp,g}]$ is represented by the (-1)-quadratic one-dimensional Poincaré complex in $NL_3(\mathbb{Z})$:*

$$\begin{tikzcd}
C^0 = \mathbb{Z}[t]^2 \arrow[r, "2"] \arrow[d, "\begin{pmatrix} p(t)t & 1 \\ 1 & 2g(t) \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} -p(t)t & -1 \\ -1 & -2g(t) \end{pmatrix}" description] & \mathbb{Z}[t]^2 = C^1 \arrow[d, "0"] \\
C_1 = \mathbb{Z}[t]^2 \arrow[r, "2"'] & \mathbb{Z}[t]^2 = C_0
\end{tikzcd}$$

2. *$F(s[\mathcal{N}_{tp,g}])$ is represented by the (-1)-quadratic one-dimensional Poincaré complex in $L_3(\mathbb{Z}[D_\infty])$:*

$$\begin{tikzcd}
C^0 = \mathbb{Z}[D_\infty]^2 \arrow[r, "2"] \arrow[d, "\begin{pmatrix} p(t)b & a \\ a & 2g(t)a \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} -p(t)b & -a \\ -a & -2g(t)a \end{pmatrix}" description] & \mathbb{Z}[D_\infty]^2 = C^1 \arrow[d, "0"] \\
C_1 = \mathbb{Z}[D_\infty]^2 \arrow[r, "2"'] & \mathbb{Z}[D_\infty]^2 = C_0
\end{tikzcd}$$

Proof The isomorphism $\widetilde{\mathcal{L}}(\mathbb{Z}[t],2)\cong NL_0(\mathbb{Z},\langle 2\rangle)$ is the obvious map $[(M,b,q)]\to[(M,b,q)]$.

Ranicki [17, Propostion 3.4.1] showed that $NL_0(\mathbb{Z},\langle 2\rangle)$ can equivalently be described as cobordism classes of $\langle 2\rangle$-acyclic one-dimensional (-1)-quadratic Poincaré complexes over $\mathbb{Z}[t]$. The boundary map $NL_0(\mathbb{Z},\langle 2\rangle)\to NL_3(\mathbb{Z})$ in the localization exact sequence is then given by the map $[(C,\psi)]\to[(C,\psi)]$. The content of the first part of the lemma is thus to find the one-dimensional Poincaré complexes corresponding to the linking forms $\mathcal{N}_{tp,g}$. It follows immediately by applying the correspondence from [17], recalled below in Definition 16, that $\mathcal{N}_{tp,g}$ corresponds to the quadratic Poincaré complex claimed.

The second part is then an immediate consequence of Lemma 9. (Recall that $p(t)ta=p(t)baa=p(t)b$.) □

Definition 16 ([17, Proof 3.4.1]) Let R be a ring with involution, and let S be a central multiplicative subset of R which is invariant under the involution. Let (C,ψ) be a S-acyclic one-dimensional (-1)-quadratic Poincaré complex over R, so that (C,ψ) consists of:

- a monomorphism $d:C_1\to C_0$ of finitely generated free R-modules such that $\mathrm{id}_{S^{-1}R}\otimes_R d$ is an isomorphism,
- a morphism $\psi_0:C^0\to C_1$ such that $\psi_0^*:C^1\to C_0$ induces an isomorphism $\mathrm{Cok}(d^*)\to\mathrm{Cok}(d)$, and
- a morphism $\psi_1:C^0\to C_0$ such that $\psi_1+\psi_1^*=-d\circ\psi_0$.

The nonsingular $(+1)$-*quadratic linking form* (M,b,q) *over* (R,S) *associated to the resolution* (C,ψ) consists of:

- the S-torsion R-module $M:=\mathrm{Cok}(d^*)$ of homological dimension one,
- the sesquilinear map $b:M\times M\to S^{-1}R/R$, with adjoint an isomorphism, well defined for all representatives $x,y\in C^1$ by $b(x,y):=\frac{1}{s}\langle y,\psi_0(z)\rangle$ where $z\in C^0$ is uniquely determined by the formula $d^*(z)=sx$ for any given $s\in S$ such that $s[x]=0\in M$, and
- the quadratic map $q:M\to S^{-1}R/\{r+\bar{r}\mid r\in R\}$, well defined for all x denoted as above by $q(x):=\frac{1}{s}\langle z,\psi_1(z)\rangle$.

We are now in a position to compute the switch involution on the generators of $\widetilde{\mathcal{L}}(\mathbb{Z}[t],2)$.

Lemma 17 *For all polynomials $p,g\in\mathbb{Z}[t]$, the involution* sw *is given by*

$$\mathrm{sw}[\mathcal{N}_{tp,g}]=[\mathcal{N}_{p,tg}].$$

Proof Denote the Poincaré complex over $\mathbb{Z}[D_\infty]$ of Lemma 15(2) by $\begin{bmatrix} p(t)b & a \\ a & 2g(t)a \end{bmatrix}$. Similar to the proof of Theorem 8, note that

$$\begin{aligned}
\mathrm{sw}\begin{bmatrix} p(t)b & a \\ a & 2g(t)a \end{bmatrix} &= \mathrm{sw}\left\{\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}^{*}\begin{bmatrix} p(t)b & a \\ a & 2g(t)a \end{bmatrix}\right\} \\
&= \mathrm{sw}\left\{\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\begin{bmatrix} p(t)b & a \\ a & 2g(t)a \end{bmatrix}\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\right\} \\
&= \mathrm{sw}\begin{bmatrix} bp(t) & b \\ b & 2ag(t) \end{bmatrix} \\
&= \begin{bmatrix} ap(t^{-1}) & a \\ a & 2bg(t^{-1}) \end{bmatrix} \\
&= \begin{bmatrix} p(t)a & a \\ a & 2g(t)baa \end{bmatrix}
\end{aligned}$$

This is precisely $F(s[\mathcal{N}_{p,tg}])$. Hence $\mathrm{sw}\, F(s[\mathcal{N}_{tp,g}]) = F(s[\mathcal{N}_{p,tg}])$, so by injectivity of F and s, $\mathrm{sw}[\mathcal{N}_{tp,g}] = [\mathcal{N}_{p,tg}]$. ☐

Proof of Theorem 11 Let $p, p' \in \mathbb{Z}[t]$ be polynomials. Recall from [7, p. 1072] that $j_1[tp] := [\mathcal{N}_{tp,1}]$, and that $j_2[tp'] := [\mathcal{N}_{1,tp'}] - [\mathcal{N}_{t,p'}]$. Note by Lemma 17 that:

$$\begin{aligned}
\mathrm{sw}(j_1[tp]) &= \mathrm{sw}([\mathcal{N}_{tp,1}]) \\
&= [\mathcal{N}_{p,t}] \\
&= ([\mathcal{N}_{p,t}] - [\mathcal{N}_{1,tp}] + [\mathcal{N}_{t,p}]) \oplus j_2[tp] \\
\mathrm{sw}(j_2[tp']) &= \mathrm{sw}([\mathcal{N}_{1,tp'}] - [\mathcal{N}_{t,p'}]) \\
&= [\mathcal{N}_{t,p'}] - [\mathcal{N}_{1,tp'}] \\
&= 0 \oplus j_2[tp'].
\end{aligned}$$

It remains to show $[\mathcal{N}_{t,p}] + [\mathcal{N}_{p,t}] = [\mathcal{N}_{1,tp}] + [\mathcal{N}_{tp,1}]$.

Recall

$$\begin{aligned}
\mathcal{N}_{t,p} \oplus \mathcal{N}_{p,t} &= \left(\mathbb{F}_2[t]^4, \begin{pmatrix} t/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & p/2 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, \begin{pmatrix} t/2 \\ p \\ p/2 \\ t \end{pmatrix}\right) \\
\mathcal{N}_{1,tp} \oplus \mathcal{N}_{tp,1} &= \left(\mathbb{F}_2[t]^4, \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & tp/2 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ tp \\ tp/2 \\ 1 \end{pmatrix}\right).
\end{aligned}$$

For a polynomial $p \in \mathbb{F}_2[t]$, define polynomials $p_{\mathrm{ev}}, p_{\mathrm{od}} \in \mathbb{F}_2[t]$ by the equation $p = p_{\mathrm{ev}}^2 + tp_{\mathrm{od}}^2$. Label the basis of $\mathbb{F}_2[t]^8$ as $\{e_1, e_2, \dots, e_8\}$ and let

$$\begin{aligned}
v_0 &= p_{\mathrm{ev}} \cdot e_4 + e_6 + tp_{\mathrm{od}} \cdot e_8 \\
v_1 &= e_2 + p_{\mathrm{od}} \cdot e_4 + p_{\mathrm{ev}} \cdot e_8.
\end{aligned}$$

Then $S := \mathrm{span}\,\{v_0, v_1\}$ is a sublagrangian of the $(+1)$-quadratic linking form $(\mathcal{N}_{t,p} \oplus \mathcal{N}_{p,t}) - (\mathcal{N}_{1,tp} \oplus \mathcal{N}_{tp,1})$ of exponent 2 over $(\mathbb{Z}, \langle 2 \rangle)$. Note

$$S^{\perp} = \mathrm{span}\,\{p_{\mathrm{od}} \cdot e_1 + e_3 + p_{\mathrm{ev}} \cdot e_5, e_4, p_{\mathrm{ev}} \cdot e_1 + tp_{\mathrm{od}} \cdot e_5 + e_7, e_8, v_0, v_1\}.$$

Then $(\mathcal{N}_{t,p} \oplus \mathcal{N}_{p,t}) - (\mathcal{N}_{1,tp} \oplus \mathcal{N}_{tp,1})$ is Witt equivalent to the sublagrangian construction:

$$(S^{\perp}/S, \bar{b}, \bar{q}) = \left(\mathbb{F}_2[t]^4, \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, \begin{pmatrix} p \\ t \\ tp \\ 1 \end{pmatrix}\right).$$

Lemma 4.3(2) and (6) of [7] show that this admits a lagrangian. Alternatively, note that the above is an even quadratic linking form so represents an element of $\widetilde{\mathcal{L}}^{\mathrm{ev}}(\mathbb{Z}[t], 2)$, the Witt group of even quadratic linking forms (M, b, q) on $\mathbb{Z}[t]$-modules with exponent 2. Here (M, b, q) is *even* if $b(x, x) \in \mathbb{Z}[t]$ for all $x \in M$. By an easy change of coordinates, one sees that $\widetilde{\mathcal{L}}^{\mathrm{ev}}(\mathbb{Z}[t], 2) \cong NL_0(\mathbb{F}_2)$ which is isomorphic to $t\mathbb{F}_2[t]/\{f^2 - f \mid f \in \mathbb{F}_2[t]\}$ via the Arf invariant of the function field $\mathbb{F}_2(t)$. On the other hand, the Arf invariant of the above form is trivial, hence the form is Witt trivial. □

The above proof is complete, but the last part of the proof was unmotivated. The motivation is the obstruction theory of Connolly–Davis. There is a short exact sequence of $\mathbb{Z}$-modules, which first appeared in [9, Theorem 25],

$$0 \to \widetilde{\mathcal{L}}^{\mathrm{ev}}(\mathbb{Z}[t], 2) \to \widetilde{\mathcal{L}}(\mathbb{Z}[t], 2) \xrightarrow{B} t\mathbb{Z}_2[t] \oplus t\mathbb{Z}_2[t] \to 0.$$

See [7, Definition 6.1] for the definition of B and [7, Lemma 5.7(1), Example 6.2] for exactness.

Consider B as a "characteristic number"—a certain combination of quadratic values of the Wu classes [7, Eq. 5.(4)] v_0, v_1—which is the obstruction [7, Sect. 6, Proof 1.7 Line 3] for a quadratic linking form to Witt equivalent to even form. If the quadratic linking form is even, the Arf invariant over the characteristic two field $\mathbb{F}_2(t)$ can then be applied.

Corollary 18 *The switch map on the invariant B is: $B \circ \mathrm{sw} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ B$.*

Proof Recall from [7, Eqs. (13),(14)] that:

$$\begin{aligned} B_1 \circ j_1 &= \pi, & B_1 \circ j_2 &= 0 \\ B_2 \circ j_1 &= 0, & B_2 \circ j_2 &= \mathrm{id}\,. \end{aligned}$$

Then for any polynomials $p, p' \in \mathbb{Z}[t]$ note

$$(B \circ \mathrm{sw})(j_1[tp] + j_2[tp']) = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \left(j_1[tp] + j_2[tp] + j_2[tp']\right) = \begin{pmatrix} [tp] \\ [tp] + [tp'] \end{pmatrix}.$$

□

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