1 On the Maslov Index

Let $(\mathbb{R}^{2n}, \omega)$ be a nonsingular skew-symmetric form. Let L_0, L_1, L_2 be pairwise transverse Lagrangians (i.e. $L_i = L_i^\perp$ and $L_i \oplus L_j = \mathbb{R}^{2n}$ for $i \neq j$). Define a form $q : L_0 \times L_0 \to \mathbb{R}$ by $q(x, y) = \omega(x_1, y_2)$, where $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$.

Claim 1. *$q : L_0 \times L_0 \to \mathbb{R}$ is a non-singular symmetric form.*

Proof.

$$\begin{aligned} 0 &= \omega(x, y) \\ &= \omega(x_1, y_2) + \omega(x_2, y_1) \\ &= \omega(x_1, y_2) - \omega(y_1, x_2) \end{aligned}$$

Thus q is symmetric.

Now suppose $x \in L_0 - 0$. Let $x = x_1 + x_2$ with $x_i \in L_i$. Since L_1 and L_2 are transverse, both x_1 and x_2 are nonzero. Choose $y_1 \in L_1$ and $y_2 \in L_2$ so that $\omega(x_1, y_2) = 1 = -\omega(x_2, y_1)$. Let $y = y_1 + y_2$. Then $y \in L_0^\perp = L_0$ and $q(x, y) = 1$, so q is non-singular. □

Definition 2. The *Maslov index* $\tau(L_0, L_1, L_2) \in \mathbb{Z}$ is the signature of q.

Remark 3. Given a nonsingular skew-hermitian form $(\mathbb{C}^{2n}, \omega)$ and pairwise tranverse Lagrangians L_0, L_1, L_2, one can define a nonsingular hermitian form $q : L_0 \times L_0 \to \mathbb{C}$ as well as the Maslov index $\tau(L_0, L_1, L_2) \in \mathbb{Z}$.

Remark 4. Let $(\mathbb{R}^{2n}, \omega)$ be a nonsingular skew-symmetric form and let L_0, L_1, L_2 be Lagrangians, not necessarily transverse. Then one can define the Maslov Index $\tau(L_0, L_1, L_2) \in \mathbb{Z}$ in this more general case.

2 Wall's non-additivity of the signature

Recall that a manifold triad is a compact manifold whose boundary is expressed as the union of two compact manifolds which intersect in their common boundary.

Suppose we have a $4k$-dimensional triad of compact oriented manifolds and suppose it is expressed as a union of two triads:

$$(Y_+; X_+ \cup X_0) \cup_{X_0} (Y_-; X_0 \cup X_-)$$

Let $Z = \partial X_0$ and note that $Z = \partial X_+$ and $Z = \partial X_-$. The intersection form on $H_{2k-1}Z$ is nonsingular skew-symmetric and has three Lagrangian, coming from the boundary maps in the homology exact sequences of the pairs (X_0, Z), (X_+, Z), and (X_-, Z).

Theorem 5 (Wall, "Non-additivity of the signature," *Invent. Math.* **7** 1969 269–274)**.**

$$\sigma(Y_+ \cup_{X_0} Y_-) = \sigma(Y_+) + \sigma(Y_-) - \tau(\partial(H_{2k}(X_0, Z), \partial(H_{2k}(X_+, Z), \partial(H_{2k}(X_-, Z))$$

A special case, due to Novikov, is when X_+ and X_- are empty, in which case the Maslov index vanishes.