Equivariant Rigidity

For any discrete group G, *a model* $E_{\text{fin}}G$ *for the classifying space for* G*-CW-complexes with finite isotropy* is a space which is G-homotopy equivalent to G-CW-complex so that for all subgroups H of G

$$(E_{\text{fin}}G)^H = \begin{cases} \emptyset & \text{if } |H| = \infty \\ \simeq * & \text{if } |H| < \infty \end{cases}.$$

Given any G-CW-complex X with finite isotopy groups, there is an equivariant map $X \to E_{\text{fin}}G$ and this map is unique up to equivariant homotopy. It follows that any two models are G-homotopy equivalent. Furthermore, a model $E_{\text{fin}}G$ exists for any group G. If G is torsion-free, then a model for $E_{\text{fin}}G$ is a model for EG, the classifying space for CW-complexes equipped with a free, cellular G-action.

A *cocompact* $E_{\text{fin}}G$*-manifold* is a topological manifold M with a properly discontinuous G-action so that M/G is compact, and for all infinite subgroups H of G, M^H is empty and, for all finite subgroups H of G, M^H is a contractible submanifold of M. We say M is a *cocompact manifold model for* $E_{\text{fin}}G$ if, in addition, M is homotopy equivalent to a G-CW-complex. In this case M is a model for $E_{\text{fin}}G$ and hence is uniquely determined up to equivariant homotopy type. Not all groups G admit a cocompact $E_{\text{fin}}G$-manifold. There is no example known of a group G which admits a cocompact $E_{\text{fin}}G$-manifold, but no cocompact manifold model for $E_{\text{fin}}G$. The key geometric example of a cocompact manifold model for $E_{\text{fin}}G$ is given by a discrete, cocompact subgroup of the isometry group of a complete simply-connected Riemannian manifold of nonpositive sectional curvature.

A space whose universal cover is contractible is called *aspherical*. For G torsion-free, a cocompact $E_{\text{fin}}G$-manifold is the universal cover of a closed aspherical manifold with fundamental group G and conversely. Since a manifold has the homotopy type of a CW-complex, any cocompact $E_{\text{fin}}G$-manifold

manifold is a cocompact manifold model for $E_{\text{fin}}G$. The Borel Conjecture for a group G states that any homotopy equivalence between two closed aspherical manifolds with fundamental group G is homotopic to a homeomorphism, or equivalently a G-homotopy equivalence between two cocompact $E_{\text{fin}}G$-manifold is equivariantly homotopic to a homeomorphism. Of course, this is vacuously true if G is not the fundamental group of a closed aspherical manifold.

We say *equivariant rigidity holds for G* if every *cocompact $E_{\text{fin}}G$-manifold* admits a G-CW-structure and if every G-homotopy equivalence between cocompact manifold models for $E_{\text{fin}}G$ is equivariantly homotopic to a homeomorphism. Equivariant rigidity may or may not hold for a given group G (although for "most" groups it holds vacuously.) When G is the fundamental group of a closed aspherical manifold, equivariant rigidity is equivalent to the Borel Conjecture. There are also versions of equivariant rigidity which would apply to models with boundary or noncocompact models, but we won't state these.

There are a sequence of weaker versions of equivariant rigidity:

- Every G-homotopy equivalence between cocompact manifold models for $E_{\text{fin}}G$ is equivariantly homotopic to a homeomorphism.
- Every isovariant G-homotopy equivalence between cocompact manifold models for $E_{\text{fin}}G$ is isovariantly homotopic to a homeomorphism.
- Every isovariant simple G-homotopy equivalence between cocompact manifold models for $E_{\text{fin}}G$ is isovariantly homotopic to a homeomorphism.

In each of the above items one could ask that the actions are locally linear, or one could require successive strata in the singular set to have large codimension.