# Things I know, but sometimes forget

## 1 Every finite group acts freely on a product of spheres

It is not difficult to show that any finite group acts freely on a product of $S^3$'s. Indeed, $G$ acts freely on $\prod_{g \in G} i_{g!} S^3$, where $S^3$ is given a free $\langle g \rangle$-action and $i_{g!} S^3 = \mathrm{map}_{\langle g \rangle}(G, S^3)$ is the co-induced $G$-space.

## 2 free $\times$ anything = free $\times$ anything else

If $X$ is a $G$-space, let $X_t$ be the same underlying space with the trivial $G$-action. Then there is a bijection of $G$-spaces $G \times X_t \to G \times X$ given by $(g, x) \mapsto (g, gx)$ with inverse map $(g, x) \mapsto (g, g^{-1}x)$. This can be generalized in two ways.

Let $i : H \to G$ be the inclusion of a subgroup. Let $i^* : G$-spaces $\to$ $H$-spaces be the forgetful map given by restricting a $G$-action to an $H$-action and let $i_* : H$-spaces $\to G$-spaces be the induction map $i_* X = G \times_H X$. Then $i_*$ is the right adjoint of $i^*$. Let $X$ be a $G$-space. There is a homeomorphism

$$\begin{aligned} i_* i^* X = G \times_H i^* X &\to G/H \times X \\ [g, x] &\mapsto (gH, gx) \end{aligned}$$

with inverse map $(gH, x) \mapsto [g, g^{-1}x]$.

If $F$ is a free $G$-set, choose a set of orbit representatives $B \subset F$ and define $F \times X_t \to F \times X$ by $(gb, x) \mapsto (gb, gx)$ for $b \in B$.

# 3 Virtually cyclic groups come in three types

A virtually cyclic group is a group with a cyclic subgroup of finite index. They come in three types: finite, groups which surject to $\mathbb{Z}$ ($F \rtimes \mathbb{Z}$ with $F$ finite), and groups which surject to $D_\infty$ ($G_0 *_F G_1$ with $F$ finite and of index two).

**Theorem 1.** *Let $\Gamma$ be an infinite virtually cyclic group.*

1. *If there is a central element of infinite order, then there is an epimorphism $\Gamma \to \mathbb{Z}$.*
2. *If there is not a central element of infinite order, then there is an epimorphism $\Gamma \to D_\infty$.*

*Proof.* By intersecting the conjugates of an infinite cyclic subgroup, we may find an normal infinite cyclic subgroup $C$. Let $G$ be the finite quotient group.

1) In this case $G$ acts trivially on $C$. Embed $C$ as an index $|G|$ subgroup of an infinite cyclic group $C'$. Let $\Gamma' = C' \times_C \Gamma$. The image of the obstruction cocycle under the map $H^2(G; C) \to H^2(G; C')$ is trivial, so there exists a splitting $s : \Gamma' \to C'$ of the inclusion $C' \hookrightarrow \Gamma'$. Then $s|_\Gamma : \Gamma \to s(\Gamma)$ is the desired epimorphism.

2) Let $G_0 = \ker(G \to \mathrm{Aut}\, C)$ (the map is by lifting to $\Gamma$ and using that conjugation preserves the normal subgroup.) Let $\Gamma_0 = \pi^{-1} G_0 < \Gamma$. Then there exists an epimorphism $\phi : \Gamma_0 \to \mathbb{Z}$ by 1). Likewise, $\Phi : \Gamma \to G \to G/G_0 \cong \mathbb{Z}_2$ is an epimorphism. Choose $\gamma \in \Gamma$ so that $\phi(\gamma) = 1$. Then $\Gamma = \Gamma_0 \amalg \Gamma_0\gamma$. Define a epimorphism $\varphi : \Gamma \to \mathbb{Z} \rtimes \mathbb{Z}_2$ by $\varphi(g) = (g, 0)$ and $\varphi(g\gamma) = (g, 1)$ for $g \in \Gamma_0$. □

# 4 RAPL (= Right adjoints preserve limits)

Left adjoints preserve colimits, too! An *adjunction* is a pair of functors $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and a natural isomorphism of functors

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$$
$$\mathcal{D}(d, U(c)) \cong \mathcal{C}(F(d), c).$$

$F$ is the *left adjoint of* $U$ and $U$ is the *right adjoint* of $F$.

Let $\mathcal{I}$ be a category and suppose $\mathcal{C}$ and $\mathcal{D}$ have $\mathcal{I}$-limits. Let $f : \mathcal{I} \to \mathcal{C}$ and $g : \mathcal{I} \to \mathcal{D}$ be functors. Then the maps

$$F(\operatorname*{colim}_{\mathcal{I}} f) \leftarrow \operatorname*{colim}_{\mathcal{I}} F \circ f$$
$$U(\lim_{\mathcal{I}} g) \to \lim_{\mathcal{I}} U \circ g$$

are isomorphisms.

**Example 2.** Consider the adjunction $\text{Set} \underset{U}{\overset{F}{\rightleftarrows}} \text{Group}$ with

$$\text{Group}(F(X), G) \cong \text{Set}(X, U(G))$$

where $F$ takes a set to the free group generated by that set and $U(G)$ is the forgetful functor taking a group to its underlying set. Let $\mathcal{I}$ be the category with two objects and only identity morphisms. Then

$$F(X_1 \amalg X_1) \xrightarrow{\cong} F(X_1) * F(X_2)$$
$$U(G_1 \times G_2) \xleftarrow{\cong} U(G_1) \times U(G_2)$$

**Example 3.** Let $R$ be a ring and $B$ be an $R$-module. Consider the adjunction

$$- \otimes B : R\text{-mod} \to R\text{-mod}$$
$$\text{Hom}(B, -) : R\text{-mod} \to R\text{-mod}$$
$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$$

Let $\mathcal{I}$ be a category with only identity morphisms.

$$(\oplus M_i) \otimes B \xrightarrow{\cong} \oplus(M_i \otimes B)$$
$$\text{Hom}(R, \prod M_i) \xleftarrow{\cong} \prod \text{Hom}(R, M_i)$$

# 5 The degree of a cover equals the degree of a map

Let $M$ be a closed, connected $n$-manifold. Then $H_n M$ is zero or infinite cyclic. If $H_n M$ is infinite cyclic, then we say $M$ is *orientable* in which case $H_n M \to H_n(M, M - \{x\})$ is an isomorphism for all $x \in M$.

There is an obvious local degree equals global degree proof of the following theorem, but this one, based on the transfer, is perhaps easier.

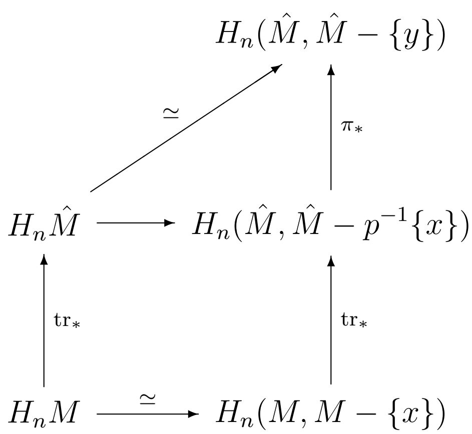
**Theorem 4.** *Let $p : \hat{M} \to M$ be a $k$-fold cover with domain and range closed, connected $n$-manifolds. If $M$ is orientable then so is $\hat{M}$ and $p_* : H_n\hat{M} \to H_nM$ takes a generator to $k$ times a generator.*

*Proof.* We will define the transfer $\text{tr} : H_iM \to H_i\hat{M}$ and show it is an isomorphism for $i = n$. For a singular $i$-simplex $\sigma : \Delta^i \to M$ there are exactly $k$ singular $i$-simplices $\tilde{\sigma}^j : \Delta^j \to \hat{M}$, $j = 1, \ldots, k$ so that $p \circ \tilde{\sigma}^j = \sigma$. Define the chain map

$$tr_\# : S_*M \to S_*\hat{M}$$
$$\sum a_\sigma \sigma \mapsto \sum a_\sigma \sum_{j=1}^{k} \tilde{\sigma}^j.$$

Clearly $p_\# \circ \text{tr}_\# : S_*M \to S_*M$ is multiplication by $k$ and the same is true after passing to homology. It follows that $H_n\hat{M}$ is nonzero, hence $\hat{M}$ is orientable.

Note that for a subset $A$ of $M$, the transfer map is also defined on relative homology $\text{tr}_* : H_i(M, A) \to H_i(\hat{M}, p^{-1}A)$. Choose $x \in M$ and $y \in \hat{M}$ so that $p(y) = x$. Consider the commutative diagram




If $\sigma : \Delta^n \to M$ is an embedding with $x \in \sigma(\text{int } \Delta^n)$, then by excision $\sigma$ represents a generator of $H_n(M, M - \{x\})$. If, in addition, the image of $\sigma$ is contained in a evenly covered neighborhood, then the images of the lifts $\tilde{\sigma}^1, \ldots, \tilde{\sigma}^k$ are all disjoint, so $y$ is contained in the image of exactly one of the lifts, say $\tilde{\sigma}^1$. Then

$$\pi_*(\text{tr}_*[\sigma]) = \pi_*[\tilde{\sigma}^1 + \cdots \tilde{\sigma}^k] = [\tilde{\sigma}^1]$$

Hence the composite of the vertical maps on the right are isomorphisms, thus the transfer map on the left is an isomorphism. Since $p_* \circ \mathrm{tr}_* = k \cdot \mathrm{Id}$, the result follows. $\square$

# 6 Inner automorphisms often induce identities

## 6.1 Groups

Recall a group is a category with one object.

**Lemma 5.** *Let $F : \mathrm{Group} \to \mathrm{Ab}$ be a functor. Suppose $F(f) = F(g)$ for any natural transformation $T : f \to g$ of morphisms of groups. Then for an inner automorphism $c_\gamma : G \to G$ of a group, $F(c_\gamma) = \mathrm{Id}_{F(G)}$.*

*Proof.* There is an natural transformation $T : \mathrm{Id}_G \to c_\gamma$ given by the morphism $\gamma$. $\square$

**Corollary 6.** *An inner automorphism induces the identity on the homology of a group.*

*Proof.* Let $(0 \to 1)$ be the category with two objects and three morphisms, including a morphism from 0 to 1. A natural transformation $T$ of functors $F, F' : \mathcal{C} \to \mathcal{D}$ induces a functor $(0 \to 1) \times \mathcal{C} \to \mathcal{D}$ and conversely.

Let $T : f \to g$ be a natural transformation of morphisms of groups $f, g : G \to G'$. This induces a functor $(0 \to 1) \times G \to G'$ and hence a homotopy $B(0 \to 1) \times BG \to BG'$ from $Bf$ to $Bg$.

Thus we can apply the Lemma above with $F(G) = H_n(BG)$. $\square$

## 6.2 Rings

**Proposition 7.** *An inner automorphism of a ring $R$ induces the identity on $K_nR$.*

*Proof.* Let $\gamma \in R^\times$. Consider the functor $c_{\gamma *} : \mathcal{P}(R) \to \mathcal{P}(R)$ given by $c_{\gamma *}(P) = R \otimes_{c_\gamma} P$. There is an exact natural transformation $T : \mathrm{Id} \to c_{\gamma *} : \mathcal{P}(R) \to \mathcal{P}(R)$ given by $P \to c_{\gamma *}P \quad x \mapsto \gamma^{-1}x$. It induces a functor

$$(0 \to 1) \times Q(\mathcal{P}(R)) \to Q(\mathcal{P}(R))$$

and hence a homotopy between the identity and $BQ(c_{\gamma *})$. $\square$

## 7 A souped-up Hurewicz Theorem

A space $X$ is *$n$-connected* if every map $S^i \to X$ for $i \leq n$ is null-homotopic. The classical Hurewicz Theorem say that for an $n$-connected space, $\pi_i X \xrightarrow{\sim} H_i X$ for $i \leq n+1$.

**Theorem 8.** *Let $k > 1$. If $X$ is $(k-1)$-connected, the Hurewicz map $\pi_{k+1} X \to H_{k+1} X$ is onto.*

The theorem is not true when $k = 1$. A counterexample is given by $S^1 \times S^1$.

*Proof.* First assume $X$ is an Eilenberg-MacLane space $K(G,k)$ with $G$ and abelian group and $k > 1$. There is a short exact sequence of abelian groups

$$0 \to F' \to F \to G \to 0$$

where $F$ and $F'$ are free abelian groups. (Indeed, find a surjection $\phi : F \to G$ with $F$ a free abelian group and note that the subgroup $\ker \phi < F$ is itself free abelian.) By choosing bases for $F$ and $F'$, build a CW complex $Y$ with only a 0-cell, $k$-cells, and $(k+1)$-cells, with $\pi_k Y = G$, and with $H_{k+1} Y = 0$. Build a $K(G,k)$ by adding on cells of dimension $k+2$ and higher. Then $H_{k+1} K(G,k)$ is a quotient of $\ker(\partial : C_{k+1} Y \to C_k Y) = \ker(F' \to F) = 0$.

Now we prove the theorem for a general $(k-1)$-connected space $X$ where $k > 1$. Let $G = \pi_k X$. Choose a map $X \to K(G,k)$ which is the identity on $\pi_k$. Let $F$ be the homotopy fiber. Then the Serre exact sequence (which follows from the Serre spectral sequence) gives a long exact sequence

$$\begin{aligned} &H_{2k} F \to H_{2k} X \to H_{2k} K(G,k) \to \cdots \\ &\qquad\qquad \to H_{k+1} F \to H_{k+1} X \to H_{k+1} K(G,k) \to \cdots \end{aligned}$$

There is a commutative diagram

$$\begin{array}{ccc} \pi_{k+1} F & \xrightarrow{\sim} & H_{k+1} F \\ \simeq \downarrow & & \downarrow \\ \pi_{k+1} X & \longrightarrow & H_{k+1} X \end{array}$$

The left map is an isomorphism because of the homotopy exact sequence and the top map is an isomorphism by the Hurewicz Theorem. Since $H_{k+1} K(G,k) = 0$ by our above arguments, the Serre exact sequence shows the right hand map is onto. Thus the bottom map is onto as desired. $\square$

## 8 Poincaré duality and local coefficients

Let $X$ be a connected CW-complex with fundamental group $\pi$ and universal cover $\widetilde{X}$. If $A$ is a left (right) $\mathbb{Z}\pi$-module, let $\overline{A}$ be the the right (left) $\mathbb{Z}\pi$-module defined by $a\lambda = \overline{\lambda}a$ ($\lambda a = a\overline{\lambda}$) where $\overline{\sum a_g g} = \sum a_g g^{-1}$. Let $\widetilde{X} \curvearrowleft \pi$ be the right action by deck transformations. For a left $\mathbb{Z}\pi$-module $A$, let $H_*(X;A)$ be the homology of the chain complex $C_*(X;A) = C_*\widetilde{X} \otimes_{\mathbb{Z}\pi} A$. For a right $\mathbb{Z}\pi$-module $A$, let $H^*(X;A)$ be the cohomology of the cochain complex $C^*(X;A) = \mathrm{Hom}_{\mathbb{Z}\pi}(C_*\widetilde{X}, A)$. Note that $H^0(X;A) = A^G$ (invariants) and that $H_0(X;A) = A_G = A \otimes_{\mathbb{Z}G} \mathbb{Z}$ (coinvariants). Cup and cap products with local coefficients work as expected:

$$H^i(X;A) \times H^j(X;B) \to H^{i+j}(X; A \otimes_{\mathbb{Z}} B)$$
$$H^i(X;A) \times H_j(X;B) \to H_{i-j}(X; \overline{A} \otimes_{\mathbb{Z}} B)$$

where we take the diagonal right $\pi$-action on $A \otimes_{\mathbb{Z}} B$ and the diagonal left $\pi$-action on $\overline{A} \otimes B$.

Then Poincaré duality states:

**Theorem 9.** *Let $X$ be a closed, connected, oriented $n$-manifold, $[X] \in H_n(X;\mathbb{Z})$ the fundamental class and $A$ any right $\mathbb{Z}\pi$-module. Then*

$$\cap[X] : H^{n-i}(X;A) \xrightarrow{\cong} H_i(X;\overline{A})$$

*The same sort of thing is true for nonorientable manifolds; let $[X] \in H_n(X;\mathbb{Z}_w)$ be a generator, then*

$$\cap[X] : H^{n-i}(X;A) \xrightarrow{\cong} H_i(X;\overline{A}_w)$$

**Remark 10.** Poincaré duality for $A$ is a formal consequence of Poincaré duality with $\mathbb{Z}\pi$-coefficients. Indeed, $\cap[X] : \overline{C^{n-*}(X;\mathbb{Z}\pi)} \to C_*(X;\mathbb{Z}\pi)$ is a chain homotopy equivalence, hence so is $\cap[X] : \overline{C^{n-*}(X;\mathbb{Z}\pi)} \otimes_{\mathbb{Z}\pi} \overline{A} \to C_*(X;\mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} \overline{A}$. But this can be identified with $\cap[X] : C^{n-*}(X;A) \to C_*(X;\overline{A})$.

**Remark 11.** This gives intersection pairings. Indeed, let $A$ and $B$ be right $\mathbb{Z}\pi$-modules. Note that $(A \otimes_{\mathbb{Z}} B)_\pi = A \otimes_{\mathbb{Z}\pi} \overline{B}$ with $a \otimes b \leftrightarrow a \otimes \overline{b}$. Then by applying the cup product, then Poincaré duality, the computation of $H_0$, and the above identification, one has a bilinear pairing for $i + j = n$,

$$H^i(X;A) \times H^j(X;B) \to A \otimes_{\mathbb{Z}\pi} \overline{B}$$

If $\mathbb{Z}\pi \to R$ is an epimorphism of rings with involution (e.g. the identity), and if $A = B = R$, there are isomorphisms of $\mathbb{Z}\pi - \mathbb{Z}\pi$ bimodues

$$\begin{aligned} R \otimes_{\mathbb{Z}\pi} \overline{R} = R \otimes_R \overline{R} &\to R \\ \alpha \otimes \beta &\mapsto \alpha\overline{\beta} \end{aligned}$$

This gives an $R$-valued intersection pairing.

# 9 The Alexander invariants are $S$-torsion

A chain complex is *acyclic* if $H_*C = 0$, *contractible* if there are homomorphisms $s_i : C_i \to C_{i+1}$ so that $\partial_{i+1}s_i + s_{i-1}\partial_i = \mathrm{Id}_{C_i}$ for all $i$, and *finite* if $\oplus_i C_i$ is finitely generated free. An inductive proof shows that a finite chain complex is acyclic if and only if it is contractible.

Let $S = \{p \in \mathbb{Z}[t, t^{-1}] \mid p(1) = 1\}$. A *chain complex* $C$ of $\mathbb{Z}[t, t^{-1}]$-modules is $S$*-acyclic* if for every $\alpha \in H_iC$, there is an $s \in S$ with $s\alpha = 0$. Let $\mathbb{Z}$ be the trivial $\mathbb{Z}[t, t^{-1}]$-module with $p(t)n = p(1)n$. Let $\overline{C} = C \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}$.

**Theorem 12.** *Let $C$ be a finite chain complex of $\mathbb{Z}[t, t^{-1}]$-modules. Then $C$ is $S$-acyclic if and only if $\overline{C}$ is acyclic.*

Before proving the theorem, recall some notation. $S$ is a *multiplicative subset* of $\mathbb{Z}[t, t^{-1}]$, i.e. $1 \in S$ and $p, q \in S \Longrightarrow p, q \in S$. The localization $S^{-1}\mathbb{Z}[t, t^{-1}] = \{a/p \mid a \in \mathbb{Z}[t, t^{-1}], p \in S\}$ is a superring of $\mathbb{Z}[t, t^{-1}]$ and a subring of the quotient field $\mathbb{Q}(t)$. The *localization of* $\mathbb{Z}[t, t^{-1}]$*-module* $M$ is the $S^{-1}\mathbb{Z}[t, t^{-1}]$-module $S^{-1}M = S^{-1}\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t,t^{-1}]} M$. Localization is *flat*, i.e. the map $S^{-1}H_*C \to H_*(S^{-1}C)$ is an isomorphism.

*Proof of Theorem.* If $C$ is $S$-acyclic, then $S^{-1}C$ is contractible. Thus $S^{-1}C \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z} = \overline{C}$ is contractible.

Suppose $\overline{C}$ is acyclic. Then $\overline{C}$ is contractible; let $s_i : \overline{C}_i \to \overline{C}_{i+1}$ be a contracting chain homotopy. $\pi_{i+1} : C_{i+1} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t, t^{-1}] = C_{i+1} \to C_{i+1} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z} = \overline{C}_{i+1}$ is an epimorphism, hence there is a homomorphism $\widetilde{s}_i : C_i \to C_{i+1}$ so that $\pi_{i+1}\widetilde{s}_i = s_i\pi_i$. Let $\phi_i = \partial_{i+1}\widetilde{s}_i + \widetilde{s}_{i-1}\partial_i : C_i \to C_i$. Choose a basis for $C_i$, and consider $\phi_i$ as a square matrix with coefficients in $\mathbb{Z}[t, t^{-1}]$. Then $\phi_i(1)$ is the identity, so that $(\det \phi_i)(1) = 1$, thus $\det \phi_i \in S$. Thus $S^{-1}\phi : S^{-1}C \to S^{-1}C$ is chain isomorphism which is chain homotopic to zero, so that $H_*(S^{-1}\phi)$ is both an isomorphism and the zero map. Thus $S^{-1}H_*C = 0$ and $C$ is $S$-acyclic. □

There is also a highly connected version of the theorem: Let $C$ be a finite chain complex of $\mathbb{Z}[t, t^{-1}]$-modules and $k$ an integer. Then $H_i(S^{-1}C) = 0$ for $i \leq k$ if and only if $H_i\overline{C} = 0$ for $i \leq k$.

Alternatively, one could give an inductive proof of the theorem using the following lemma.

**Lemma 13.** *Let $f : A \to B$ be a map of $\mathbb{Z}[t, t^{-1}]$-modules with $B$ finitely generated free. Then $f$ is $S$-surjective if and only if $\overline{f}$ is surjective.*

*Proof.* If $f$ is $S$-surjective, then $A \to B \to \operatorname{cok} f \to 0$ is exact, with $\operatorname{cok} f$ $S$-acyclic. Right exactness of $S^{-1}\mathbb{Z}[t, t^{-1}] \otimes -$ shows that $\overline{f}$ is surjective.

Suppose $\overline{f}$ is surjective. Let $s : \overline{B} \to \overline{A}$ be a splitting. Lift it to a map $\tilde{s} : B \to A$. Then $S^{-1}f \circ S^{-1}\tilde{s} : S^{-1}B \to S^{-1}B$ is an isomorphism since $\det(f \circ \tilde{s})(1) = \det(f \circ s) = \pm 1$, so $f : A \to B$ is $S$-surjective. ☐

**Corollary 14.** *The Alexander module $H_1(X_K; \mathbb{Z}[t, t^{-1}])$ is acyclic where $X_K$ is a knot exterior.*

*Proof.* Let $f : X_K \to S^1 = K(\mathbb{Z}, 1)$ be a map inducing an isomorphism on homology and let $\widetilde{f} : \widetilde{X}_K \to \mathbb{R}$ be the induced map on infinite cyclic covers. Then the algebraic mapping cone $C(f)$ is acyclic so $C(\widetilde{f})$ is $S$-acyclic. Thus $H_1(X_K; \mathbb{Z}[t, t^{-1}]) = H_1(C(\widetilde{f})$ is $S$-acyclic. ☐

# 10 Some unstable J-homomorphisms

The commutative diagram:

$$\begin{array}{ccccc} \pi_3(SO(3)) & \longrightarrow & \pi_3(SO(4)) & \longrightarrow & \pi_3(SO(5)) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J} \\ \pi_6(S^3) & \longrightarrow & \pi_7(S^4) & \longrightarrow & \pi_8(S^5) \end{array}$$

is isomorphic to the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}/12 & \longrightarrow & \mathbb{Z}/12 \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}/24 \end{array}$$

To give the isomorphism we specify bases for the upper row. Let $\rho : S^3 \to SO(3)$ be given by $\rho(u)v = uvu^{-1}$ for an imaginary quaternion $v$ and let $\sigma : S^3 \to SO(4)$ be given by $\sigma(u)v = uv$ for a quaternion $v$. Let $i_3 : SO(3) \to SO(4)$ and $i_4 : SO(4) \to SO(5)$ be the stabilization defined by sending the extra basis element to itself. Bases for the top horizontal line are given by

$$\{\rho\} \qquad \{i_3\rho, \sigma\} \qquad \{i_4\sigma\}.$$

Here $i_4 i_3 \rho = 2 i_4 \sigma$.

Here is more discussion. We also let $\rho$ and $\sigma$ be the corresponding 3-plane and 4-plane bundles over $S^4$. $\rho$ is also called the quaternionic Hopf bundle. Then $p_1(\rho) = 4, p_1(\sigma) = 2, e(\rho) = 0, e(\sigma) = 1$, identifying $H^4(S^4)$ with $\mathbb{Z}$. Note that for any 4-plane bundle $\xi$ over $S^4$, $p_1(\xi) \equiv 2e(\xi) \pmod 4$. Since $p_1(TS^4) = 0$ and $e(TS^4) = 2$, $TS^4 = 2\sigma - i_3\rho \in \pi_3(SO(4))$. Thus $\rho$ and $TS^4$ do not generate $\pi_3(SO(4))$.

# 11 Three proofs of the Wang Sequence

Let $\overline{X} \to X$ be an infinite cyclic cover with covering translation $t$. The *Wang Sequence* is the LES

$$\cdots \to H_n\overline{X} \xrightarrow{1-t} H_n\overline{X} \to H_n X \to H_{n-1}\overline{X} \to \cdots$$

## 11.1 Cellular proof

Assume $X$ is a CW-complex; give $\overline{X}$ the induced CW-structure. The SES

$$0 \to \mathbb{Z}[t, t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \to 0$$

gives, for all $n$, a SES of abelian groups

$$0 \to C_n\overline{X} \xrightarrow{1-t} C_n\overline{X} \to C_n X \to 0$$

and a SES of chain complexes

$$0 \to C_\bullet\overline{X} \xrightarrow{1-t} C_\bullet\overline{X} \to C_\bullet X \to 0$$

The Zig-Zag lemma gives the result.

The next two proofs rely on the observation that the map $\overline{X} \times_{\mathbb{Z}} \mathbb{R} \to X$ is a homotopy equivalence. This follows from changing the fiber in the principal $\mathbb{Z}$-fibration sequence $\mathbb{Z} \to \overline{X} \to X$ giving the fibration $\mathbb{R} \to \overline{X} \times_{\mathbb{Z}} \mathbb{R} \to X$.

### 11.2 Spectral sequence proof

One changes the fiber on the principal $\mathbb{Z}$-fibration $\mathbb{Z} \to \mathbb{R} \to S^1$ to obtain the fibration sequence

$$X \to \mathbb{R} \times_{\mathbb{Z}} \overline{X} \to S^1$$

One applies the Serre spectral sequence (using local coefficients) to this fibration.

### 11.3 Meyer-Vietoris proof

Note that $\overline{X} \times_{\mathbb{Z}} \mathbb{R}$ is homeomorphic to the mapping torus $T(t) = \overline{X} \times I/(x,0) \sim (tx,1)$. Decompose the mapping torus into a neighborhood of $\overline{X} \times 0$ and its complement. One obtains a LES

$$\cdots \to H_n\overline{X} \oplus H_n\overline{X} \xrightarrow{\begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix}} H_n\overline{X} \oplus H_n\overline{X} \to H_nT(t) \to \dots$$

Cancelling terms after a change of coordinates leads to the Wang sequence.

## 12 The Serre Spectral Sequence with twisted coefficients

Let $(F, e_0) \xrightarrow{i} (E, e_0) \xrightarrow{\pi} (B, b_0)$ be a fiber sequence – i.e. $\pi : E \to B$ is a fibration, $e_0 \in E$ is a base point, $b_0 = \pi^{-1}(e_0)$, and $F = \pi^{-1}(b_0)$. Assume that $F, E, B$ are path-connected and let $M$ be a $\mathbb{Z}\pi_1 E$-module. Let $i^*M$ be the corresponding $\mathbb{Z}\pi_1 F$-module. Then there is a spectral sequence with

$$E^2_{p,q} = H_p(B; H_q(F; i^*M)) \Longrightarrow H_{p+q}(E; M)$$

I have not seen an account of this spectral sequence, except in two special cases. If $M$ is a trivial $\mathbb{Z}\pi_1 E$-module, then this is the usual Serre spectral sequence with twisted coefficients, where $\pi_1 B$ acts on $H_q(F; M)$. If the fibration is a $K(\pi, 1)$-fibration coming from a short exact sequence of groups, this spectral sequence is discussed in Brown, *Cohomology of Groups*, also Weibel, *Introduction to Homological Algebra*.

There is a Serre Spectral Sequence converging to $H_*(E; M)$ by the usual method of assuming that $B$ is a CW-complex (after, perhaps, replacing the base space by a weakly homotopy equivalent CW-complex) and then filtering $E$ by the inverse image of the skeleta of $B$. The problem is to make sense of the $E_2$-term.

It is not clear why $H_q(F; i^*M)$ is a $\pi_1 B$-module. I will indicate the technology required to make sense of this using local coefficients and will redefine $H_q(F; i^*M)$ to be a total space of a local coefficient system. According to Davis-Kirk *Lecture Notes in Algebraic Topology*, a *local coefficient system* on $B$ is an abelian group $A$ and a fiber bundle $X \to B$ with fiber $A$ and structure group $\mathrm{Aut}(A)$. Furthermore, they define an abelian group $H_q(X; A)$ without using a base point. There is a correspondence between a local coefficient system on $B$ and twisted coefficient system on $B$ (i.e. $\mathbb{Z}\pi_1 B$-module) so that homology with local coefficients is isomorphic to homology with twisted coefficients. Briefly, if $A$ is a $\mathbb{Z}\pi_1 B$-module, then the corresponding local coefficient system is $\widetilde{B} \times_{\pi_1 B} A \to B$ and if $X \to B$ with is a local coefficient system with base points $x_0 \in X$ and $b_0 \in B$, the the corresponding twisted coefficient system is the fiber above $b_0$.

An equivalent definition of a local coefficient system is a map $X \to B$ so that every fiber has the structure of a discrete abelian group satisfying the local triviality condition as in Milnor-Stasheff's definition of a vector bundle.

Our assumption is that we have a fibration

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

and (after replacing twisted by local), a local coefficient system $X \to E$ with abelian group $M$. I now define a local coefficient system

$$H_q(F; i^*M) := \coprod_{b \in B} H_q(F_b; M|_{F_b}) \to B$$

I will leave the definition of the topology on the space $H_q(F; i^*M)$ and the proof of the local triviality condition to the diligent reader. With this technology it is not difficult to deduce the spectral sequence

$$E^2_{p,q} = H_p(B; H_q(F; i^*M)) \Longrightarrow H_{p+q}(E; M)$$