MANIFOLDS AND POINCARÉ COMPLEXES

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ABSTRACT. This article contains extended notes of the lectures presented by the first author during the conference "Geometry, Topology and Their Interactions" in Morelia, Mexico on January 8th - 13th, 2007.

The article discusses the three types of manifolds: smooth, piecewise-linear, and topological, as well as a homotopy analogue of a manifold, a Poincaré complex. In each case, examples are given of a manifold which does not admit the next strongest structure, and of two distinct manifolds which become isomorphic after weakening the structure.

CONTENTS

1. INTRODUCTION

2. MANIFOLDS, POINCARÉ COMPLEXES, AND BUNDLES

We will draw distinctions between types of manifolds: topological manifolds (TOP), piecewise linear manifolds (PL), smooth manifolds (O), and

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Poincaré complexes (G), which are homotopy analogues of manifolds. The revolutionary idea that manifolds could have different structures was due to Milnor's [Mil56] discovery of exotic spheres: two non-diffeomorphic manifolds, both homeomorphic to S^7.

The easiest notion to define, but the hardest to work with, is a topological manifold. This is a Hausdorff topological space where every point has a neighborhood homeomorphic to $\mathbb{R}^n$. Usually one requires, in addition, that a manifold be second countable, in which case it embeds in $\mathbb{R}^{2n+1}$.

The hardest notion to define, but the easiest to work with, is that of a smooth manifold. We will not spell out the definition. We will next define PL-manifolds and Poincaré complexes.

A *triangulation* of a space is a homeomorphism from a simplicial complex to the space. A PL-manifold is a topological manifold with a special kind of triangulation. For a vertex v of a simplicial complex X, the star(v) is the union of all simplices containing v and the link(v) is the union of all simplicies of star(v) which don't contain v. A *Piecewise Linear* or simply *PL-manifold* of dimension n is a topological manifold equipped with a homeomorphism from a simplicial complex X such that for any vertex v of X, the star(v) is PL-homeomorphic to the standard n-simplex Δ^n; equivalently, the link(v) is PL-homeomorphic to the boundary of Δ^n.

By a theorem of Whitehead (see [Whi40]), any smooth manifold admits a PL-triangulation.

Example 2.1. Here is an example of a simplicial complex whose underlying space is a 5-dimensional topological manifold, but which is not a PL-manifold. Consider the alternating group A_5 as a subgroup of SO(3), namely the symmetries of the icosahedron. Since S^3 is a double cover of SO(3), there is a group $\widetilde{A}_5$ that acts freely on S^3 and maps epimorphically to A_5 with kernel a cyclic group of order two. $\widetilde{A}_5$ is the binary icosahedral group. Define Σ^3 as the orbit space of the action of this group on S^3. Σ^3 is called the *Poincaré homology 3-sphere*. It is a non-simply connected 3-dimensional manifold with the homology of the standard 3-sphere.

The suspension of Σ^3 is not homeomorphic to S^4, because

$$\pi_1(S(\Sigma^3) - \{\text{north pole}, \text{south pole}\}) = \widetilde{A}_5.$$

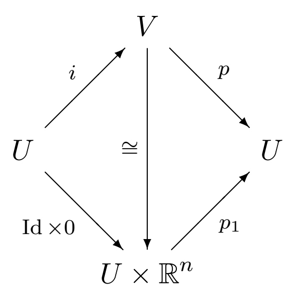
On the other hand, according to the Double Suspension Theorem of Cannon and Edwards (see [Can79]), $S^2(\Sigma^3)$ is homeomorphic to S^5. It follows that this manifold has a simplicial structure induced by that of Σ^3, but this is not a PL-manifold, since the link of a suspension vertex is the suspension of Σ^3, which is not the four sphere.

High-dimensional manifolds are governed by their tangent bundles. The tangent bundle of a smooth n-dimensional manifold M is a vector bundle, classified by a map $M \to BO(n)$. A very pleasing way to see this geometrically is to take the Grassmann manifold $Gr(n, \mathbb{R}^\infty)$ of n-planes in $\mathbb{R}^\infty$ as a model for $BO(n)$, smoothly embed $M \subseteq \mathbb{R}^N$ for some N, and then assign to a point $p \in M$ the tangent space $T_pM \in Gr(n, \mathbb{R}^N) \subset Gr(n, \mathbb{R}^\infty)$. Note that $Gr(n, \mathbb{R}^\infty) = BGL_n(\mathbb{R}) \simeq BO(n)$.

But what is meant by the tangent bundle for a topological or PL-manifold? The answer was provided by Milnor [Mil64] and his notion of a microbundle. We will describe this in the topological category. A n-dimensional microbundle is given by two maps

$$B \xrightarrow{i} E \xrightarrow{p} B$$

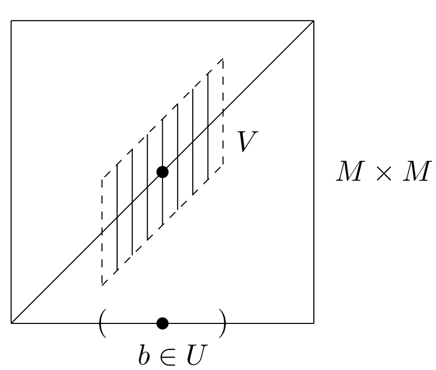
satisfying $p \circ i = \mathrm{Id}_B$ and the following local triviality condition: for every $b \in B$ there exists open neighborhoods U of b and V of $i(b)$ and a homeomorphism $V \to U \times \mathbb{R}^n$ so that the following diagram is defined and commutes



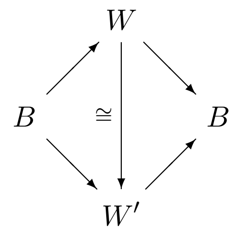
He then shows that

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$$

is a microbundle and calls it the tangent microbundle. Here is a diagram of the tangent microbundle.



Most of the machinery of bundle theory applies to microbundles. For example, two bundles $B \xrightarrow{i} E \xrightarrow{p} B$ and $B \xrightarrow{i'} E' \xrightarrow{p'} B$ are *isomorphic* if there are neighborhoods W and W' of $i(B)$ and $i'(B)$ respectively and a commutative diagram



An $\mathbb{R}^n$-*bundle with a zero section* is a fiber bundle $E \to B$ with fiber $\mathbb{R}^n$ and structure group $TOP(n) = \text{Homeo}\ (\mathbb{R}^n \text{ rel } 0)$. Every $\mathbb{R}^n$-bundle with a zero section determines a microbundle. A theorem of Kister and Mazur [Kis64] gives a one-to-one correspondence between isomorphism classes of $\mathbb{R}^n$-bundles with a zero section and isomorphism class of microbundles.

Let $\text{Topvec}^n(X)$ be the set of isomorphism classes of $\mathbb{R}^n$-bundles with a zero section over a CW-complex X. The standard theory of classifying spaces shows that there is an $\mathbb{R}^n$-bundle with 0-section γ_n over $BTOP(n)$ which is universal in the sense that the map

$$[X, BTOP(n)] \to \text{Topvec}^n(X)$$
$$[f] \to [f^*\gamma_n]$$

is a bijection for any CW-complex X.

In summary, an n-dimensional topological manifold M has a topological tangent bundle. It is classified by a map $M \to BTOP(n)$, which is determined up to homotopy.

The PL theory is parallel to the topological theory; we refer to [HM74] and [KS77] for details and references. There are notions of a PL-microbundle and a PL-$\mathbb{R}^n$ bundle with 0-section over a simplicial complex B; these coincide by an analogue of the Kister-Mazur theorem which was proven by Kuiper and Lashof in [KL66]. There is a classifying space $BPL(n)$ with universal bundle, and, for an n-dimensional PL-manifold M, a PL-tangent (micro)-bundle which is classified by a map $M \to BPL(n)$, uniquely determined up to homotopy. The construction of the space $BPL(n)$ is a bit more difficult, one either uses the Brown Representation Theorem [KS77, Essay IV, Section 8], or constructs a topological group $PL(n)$ as the geometric realization of the simplicial group whose k-simplices are PL-automorphisms of $\Delta^k \times \mathbb{R}^n$ which commute with the projection to and the 0-section from Δ^k (the details have not appeared in print).

For every PL-microbundle, we can forget the PL-structure and consider it as a topological microbundle. This gives us the forgetful map $BPL(n) \to BTOP(n)$.

Next, we define a Poincaré complex. The approximate definition is a space which satisfies Poincaré duality. Our definition, due to Wall, is a precise one which includes the non-simply-connected and non-orientable cases. Assume that X has the homotopy type of a connected CW-complex. By fixing a base point in the universal cover $\widetilde{X}$, one specifies the action of the fundamental group $\pi_1 X$ on $\widetilde{X}$ as deck transformations. For a $\mathbb{Z}\pi_1 X$-module M, define homology and cohomology with local coefficients (see also [DK01]).

$$H_i(X; M) = H_i(S_*(\widetilde{X}) \otimes_{\mathbb{Z}\pi_1 X} M)$$
$$H^i(X; M) = H^i(\mathrm{Hom}_{\mathbb{Z}\pi_1 X}(S_*(\widetilde{X}), M))$$

Here we used the one-to-one correspondence between left and right $\mathbb{Z}\pi_1 X$-modules. Given a $\mathbb{Z}\pi_1 X$-modules M and N, there is a cap product pairing

$$H_n(X; M) \times H^i(X; N) \to H_{n-i}(X; M \otimes_{\mathbb{Z}} N).$$

Definition 2.2. *An n-dimensional Poincaré complex $(X, w, [X])$ is a space X which has the homotopy type of a finite CW-complex, together with a homomorphism $w : \pi_1 X \to \{\pm 1\}$ called the* orientation character*, and a class $[X] \in H_n(X; \mathbb{Z}_w)$ called the* fundamental class *such that the cap product map*

$$- \cap [X] : H^*(X; \mathbb{Z}\pi_1 X) \longrightarrow H_{n-*}(X; \mathbb{Z}\pi_1 X_w)$$

is an isomorphism. For any $\mathbb{Z}\pi_1 X$-module M, define M_w to be the module $M \otimes_{\mathbb{Z}} \mathbb{Z}_w$, where $\mathbb{Z}_w$ is the $\mathbb{Z}\pi_1 X$-module whose underlying abelian group is infinite cyclic, with $\gamma \cdot n = w(\gamma)n$ for all $\gamma \in \pi_1 X$.

Usually we will be sloppy and refer to X as a Poincaré complex, with the fundamental class and the orientation character understood. X is an *orientable* Poincaré complex if its orientation homomorphism is trivial. By Poincaré duality there is an isomorphism of $\mathbb{Z}\pi_1 X$-modules $H^n(X; \mathbb{Z}\pi_1 X) \cong H_0(X; \mathbb{Z}\pi_1 X_w) \cong \mathbb{Z}_w$. Thus, the orientation character $w : \pi_1 X \to \{\pm 1\}$ depends only on the homotopy type of the Poincaré complex X.

Recall that a manifold is closed if it is compact without boundary.

Theorem 2.3. *Let M be an n-dimensional closed manifold. Then M has a structure of a n-dimensional Poincaré Complex.*

For a smooth manifold, this is relatively easy: M has a triangulation and the fundamental class of M is the cycle given by the sum of all n-dimensional simplices, suitably oriented. For a closed topological manifold, Poincaré duality is standard; for example, see [MS74, Appendix A]. The proof that a compact topological manifold has the homotopy type of a finite complex is more difficult; see [KS77, p. 301].

We mention that two key invariants of manifolds, the intersection pairing and the signature, are defined similarly for oriented Poincaré complexes. The notion of Poincaré complex can be extended to pairs (X, Y) and triads $(X; Y, Z)$, where Y and Z are subcomplexes of X. See [Wal99] for details.

A Poincaré complex does not have a tangent bundle, but it has a stable normal bundle, called the Spivak normal bundle. We will discuss this later in this paper, but for now, we will review the notion of a stable normal bundle for a manifold and discuss the bundle theory revelant for Poincaré complexes, namely spherical fibrations.

Two bundles ξ and η over the same base space are *stably isomorphic* if $\xi \oplus \varepsilon^k \cong \eta \oplus \varepsilon^l$ for some k and l. A normal bundle for a manifold is a Whitney sum inverse for the tangent bundle. The normal bundle is not uniquely determined, but the stable class is. This is analogous to the fact that the isotopy class of an embedding of a manifold in Euclidean space is not uniquely determined unless the dimension of the Euclidean space is sufficiently large.

Let X be a CW-complex. A *spherical fibration* $p : E \to X$ is a fibration whose fiber is homotopy equivalent to a sphere S^{n-1} for some n. Two spherical fibrations $p : E \to X$ and $p' : E' \to X$ are *fiber homotopy equivalent* if there is homotopy equivalence $h : E \to E'$ in the category of maps over X. We will show next that this is an equivalence relation. The key point is to show symmetry.

There is a notion equivalent to fiber homotopy equivalence which is fiber-preserving homotopy equivalence. This states that $p : E \to X$ and $p' : E' \to X$ are *fiber-preserving homotopy equivalent* if there is a homotopy equivalence $h : E \to E'$ which is fiber preserving in the sense that $p = p' \circ h'$. Clearly, fiber homotopy equivalence implies fiber-preserving homotopy equivalence and, in fact, by a theorem of Dold, the converse also holds (see [Dol63, Th.6.1]). Thus, it suffices to show that fiber-preserving homotopy equivalence is an equivalence relation.

Let $h : E \to E'$ be fiber-preserving homotopy equivalence and let $h' : E' \to E$ be a homotopy inverse. It follows that there is a homotopy $H' : E' \times I \to B$ such that $H'_0 = p \circ h'$ and $H'_1 = p'$. Let $\widetilde{H'} : E' \times I \to E$ be a homotopy lifting of H' such that $\widetilde{H'_0} = h'$. Then $\widetilde{H'_1} : E' \to E$ is also a homotopy inverse of h, but now $p' = p \circ \widetilde{H'_1}$.

check reasoning

Let $\mathrm{SFib}^n(X)$ be the set of equivalence classes of spherical fibrations over X with fiber homotopy equivalent to S^{n-1}. Let $G(n)$ denote the monoid of self-homotopy equivalences of S^{n-1} and $BG(n)$ the classifying space of $G(n)$. There is a spherical fibration γ_n over $BG(n)$ which is universal in the sense that the map

$$[X, BG(n)] \to \mathrm{SFib}^n(X)$$
$$[f] \to [f^*\gamma_n]$$

is a bijection for any CW-complex X.

Let $F(n)$ be the monoid of base point preserving self-homotopy equivalences of S^{n-1}. Then the adjoint correspondence between smash and mapping spaces shows $\pi_k(F(n)) \cong \pi_{k+n}(S^n)$. The fibration $F(n) \to G(n+1) \to S^n$ (see [MM79, Chapter 3]) shows that for n large, $\pi_k(F(n)) \cong \pi_k(G(n+1))$. Setting $G = \operatorname{colim} G(n)$, we deduce that $\pi_k(G) \cong \pi_k^s$, the stable homotopy group of spheres.

We will first review the notion of a stable normal bundle of a manifold before turning to Poincaré complexes.

Any n-manifold M^n embeds in $\mathbb{R}^{2n+1}$ and has a normal bundle. Any two embeddings in $\mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$ and hence have isomorphic normal bundles. Thus there is a well-defined homotopy class $\nu_M : M \to BO$ (or $\nu_M : M \to BPL$ or $\nu_M : M \to BTOP$) called the *stable normal bundle*.

add Spivak normal structure

2.1. **Spivak Normal Fibration.** Let X be a n-dimensional Poincaré complex. By the Simplicial Approximation Theorem, X is homotopy equivalent to an n-dimensional finite simplicial complex K. Then K embeds in Euclidean space $\mathbb{R}^{n+k}$ for any $k \geq n+1$. Let $N(K)$ be a regular neighborhood of K with boundary $\partial N(K)$. Let $d : N(K) \to K$ be a strong deformation retraction and $\phi : K \to X$ a homotopy equivalence. By a standard method, we can convert the composition $\partial N(K) \xrightarrow{i} N(K) \xrightarrow{d} K \xrightarrow{\phi} X$ into a fibration $\eta = \{p : E \to X\}$. It follows from the Poincaré duality of X that the fiber of p is homotopy equivalent to S^{k-1}. (In fact, Poincaré duality is equivalent to the fiber of p being spherical!) The homotopy class of the corresponding classifying map $\nu_X : X \to BG$ is the *Spivak normal bundle of* X. We will state a powerful uniqueness theorem below to show that ν_X is independent of the choice of embedding. But first we indicate that the Spivak normal bundle is equipped with more structure.

Consider the mapping cylinder $Dp : DE \to X$ of $p : E \to X$. The *Thom space* $\mathrm{Th}(\eta)$ of η is defined by DE/E, the mapping cone of p. Note that $\mathrm{Th}(\eta) \simeq N(K)/\partial N(K)$. We define the *collapse map* $c : S^{n+k} = \mathbb{R}^{n+k} \amalg \{\infty\} \to N(K)/\partial N(K)$ to be the identity in the interior of $N(K)$ and the constant map to the base point on the complement. The corresponding class $[c] \in \pi_{n+k}(\mathrm{Th}(\eta))$ is called the *Spivak class*. There is also a *Thom class* $U \in H^k(\mathrm{Th}(\eta); \mathbb{Z}_w)$ which restricts to a generator of $H^k((Dp)^{-1}(x), p^{-1}(x))$ for all $x \in X$. The Thom isomorphism theorem gives an isomorphism

$$\widetilde{H}_{i+k}(\mathrm{Th}(\eta); \mathbb{Z}) \xrightarrow{\cong} H_i(X; \mathbb{Z}_w)$$

which is basically given by capping with the Thom class. The Spivak class, the fundamental class, and the Thom class are all related; the Thom isomorphism composed with the Hurewicz homomorphism applied to the Spivak class gives the fundamental class. In more detail: the Hurewicz homomorphism $h : \pi_{n+k}(\mathrm{Th}(\eta)) \to H_{n+k}(\mathrm{Th}(\eta); \mathbb{Z}) \cong H_{n+k}(N(K), \partial N(K)) \cong \mathbb{Z}$ sends $[c]$ to a generator. Then, after possibly replacing U by $-U$, we have $U \cap h[c] = [X]$.

Abstracting a bit, a *Spivak normal fibration* over an n-dimensional Poincaré complex X is a pair (ζ, c) consisting of a spherical fibration $\zeta = \{p : E \to X\}$ with fiber of dimension $k-1$ and an element $[c] \in \pi_{n+k}(\mathrm{Th}(p))$ so that $U \cap h[c] = [X]$. There is an analogue of Whitney sum, and if (ζ, c) is a Spivak normal fibration then so is $(\zeta \oplus \varepsilon^l, S^l c)$. The Spivak Uniqueness Theorem says that for k sufficiently large, if $(\eta = \{p : E \to X\}, c)$ and $(\eta' = \{p' : E' \to X\}, c')$ are two Spivak normal fibrations with fiber dimension $k-1$, then there is a fiber homotopy equivalence $E \to E'$ (unique up to fiber homotopy) which maps c to c'. This implies that the Spivak normal bundle $\nu_X : X \to BG$ is well-defined.

Suppose now that there exists a closed, topological manifold M and a homotopy equivalence $f : M^n \to X$, where X is a finite n-dimensional Poincaré complex. Assume M^n is embedded in $\mathbb{R}^{n+k}$ for some sufficiently large k. Let $\nu = \{p : N(M) \to M\}$ be the normal bundle of M and let $g : X \to M$ be a homotopy inverse of f. One easily checks that the Spivak normal fibration of M transports to a Spivak normal fibration $(g^*\nu, f_* c_M))$ of X. This gives us the second obstruction to finding a manifold homotopy equivalent to X. Namely, a Spivak normal fibration of X must reduce to an honest sphere bundle over X.

Phrasing this differently, we see that if X has the homotopy type of a closed topological manifold, there is a map $\widetilde{\nu_X} : X \to BTOP$ so that the following diagram commutes.

$$\begin{array}{ccc} & & BTOP \\ & \overset{\widetilde{\nu_X}}{\nearrow} & \downarrow \\ X & \xrightarrow{\nu_X} & BG \end{array}$$

Here we have already converted the map $BTOP \to BG$ to a fibration. The homotopy fiber of this fibration is denoted by G/TOP. The H-space structure coming from Whitney sum on G/TOP allows us to construct a fibration $BTOP \to BG \to B(G/TOP)$. We can now state that for a given Poincaré complex X, the Spivak fibration lifts to a sphere bundle over X if and only if the composite map $X \xrightarrow{\nu_X} BG \longrightarrow B(G/TOP)$ is null-homotopic.

The above mathematics is due to Spivak [Spi67]; an alternate reference is Browder [Bro72].

Suppose $\xi : E \to X$ is an $\mathbb{R}^n$-bundle and $s : X \to E$ is the zero-section of ξ. If we restrict the bundle map ξ to the space $SE = (E - s(X))$, then the projection $S\xi : SE \to X$ is a fibration whose fiber is homotopy equivalent to S^{n-1}. This argument shows that there is a classifying map $BTOP(n) \to BG(n)$ for the universal principal $TOP(n)$-bundle.

It is a fact that for a smooth manifold M, its tangent bundle TM, cotangent TM^*, k-symmetric product $S^k(M)$, and k-exterior power $\wedge^k(TM^*)$ bundles all carry PL-structure. More generally, any k-vector bundle over M is a PL-$\mathbb{R}^k$-bundle with zero-section after we endow M with a PL-triangulation. Therefore, there exists a forgetful map $BO(n) \to BPL(n)$.

Let $X = \operatorname{colim} X(n)$ where X denotes either of the spaces BG, BPL, $BTOP$, or BO. We summarize our discussion in the following table.

Topological Type	**Bundle Theory**	**Classifying Space**	**Equivalence Relation**
Poincaré Complexes	Spherical Fibrations	BG	Homotopy Equivalence
Topological Manifolds	$\mathbb{R}^n$-bundles with Zero-Section	$BTOP$	Homeomorphism
PL-Manifolds	PL-$\mathbb{R}^n$-bundles with Zero-Section	BPL	PL-homeomorphism
Smooth Manifolds	Vector Bundles	$BO \simeq BGL$	Diffeomorphism

As we already saw, there are forgetful maps $BO \xrightarrow{\phi_1} BPL \xrightarrow{\phi_2} BTOP \xrightarrow{\phi_3} BG$. Since any map is homotopic to a fibration projection (see [Whi78, Th. 7.30]), we can and do replace ϕ_i with a homotopic fibration projection and denote it again by ϕ_i. Now, let us define PL/O, TOP/PL, and G/TOP as the homotopy fibers of ϕ_i for $i = 1, 2, 3$, respectively. These spaces play a crucial role in the reduction of the structure on a high-dimensional closed manifold.

3. Smoothing Theory

Next, we define what it means for a topological manifold to admit a PL-structure and when two such structures are equivalent.

A *PL-structure* on a topological manifold M is a homeomorphism $\varphi : N \to M$ where N is a PL-manifold. Two such PL-structures (φ_i, N_i), $i = 0, 1$, on M are *isotopic* if there exists a level-preserving homeomorphism $\psi : N_0 \times I \to M \times I$ (level-preserving means $\forall n \in N_0, \forall t \in I, \mathrm{pr}_I \psi(n, t) = t$) and a PL-isomorphism $\theta : N_0 \to N_1$ such that $\varphi_0 = \psi_{|N_0 \times \{0\}}$ and $\varphi_1 \circ \theta = \psi_{|N_0 \times \{1\}}$. It is not difficult to see that isotopy is an equivalence relation. Let $\mathcal{S}^{TOP/PL}(M)$ be the set of isotopy classes of PL-structures on M. Note that this set can possibly be empty and when M is a PL-manifold, it has a distinguished element, the isotopy class of (id_M, M).

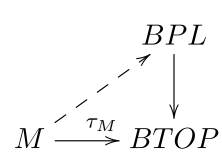
Let $PL(M)$ denote the set of equivalence classes of PL-structures on M, where two PL-structures (φ_i, N_i), $i = 0, 1$, on M are equivalent if there exists a PL-isomorphism $\theta : N_0 \to N_1$. The group $\mathrm{Homeo}(M)$ of self-homeomorphisms of M acts on $\mathcal{S}^{TOP/PL}(M)$ by $h \cdot (N, \varphi) = (N, h \circ \varphi)$. We observe that the normal subgroup $\mathrm{Homeo}_0(M)$ comprised of self-homeomorphisms of M which are isotopic to the identity map acts trivially on $\mathcal{S}^{TOP/PL}(M)$. Thus the mapping class group $\mathrm{MCG}(M) = \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M)$ acts on $\mathcal{S}^{TOP/PL}(M)$ and the quotient is naturally in bijection with $PL(M)$.

On a given PL-manifold M, we can analogously define the notions of a smooth structure, a smooth isotopy of two smooth structures, and the sets of smooth isotopy classes and smooth structures on M, respectively denoted by $\mathcal{S}^{PL/O}(M)$ and $O(M)$.

Example 3.1. Milnor [Mil56], with his discovery of exotic differential structures on spheres, showed that $\mathcal{S}^{PL/O}(S^7)$ has more than one element. Kervaire-Milnor [KM63] showed that $|\mathcal{S}^{PL/O}(S^7)| = 28$ and $|O(S^7)| = 15$.

Theorem 3.2 (Fundamental Theorem of Smoothing)**.**

- *Let $\tau_M : M \to BTOP$ denote a classifying map of the tangent bundle of an n-dimensional topological manifold M. Let $n \geq 5$. M admits a PL-structure if and only if there is a lift $M \to BPL$ such that the following diagram commutes.*



In fact, there is a bijection

$$\mathcal{S}^{TOP/PL}(M) \to \mathrm{Lift}^{TOP/PL}(\tau_M)$$

where $\mathrm{Lift}^{TOP/PL}(\tau_M)$ *denotes vertical homotopy classes of lifts of* τ_M *(vertical homotopy means a homotopy through lifts.) Here* $[\varphi, N] \mapsto [\tau_N \circ \varphi]$ *where* $\tau_N : N \to BPL$ *is a classifying map of the PL tangent bundle of* N*. Moreover, if* M *admits a* PL*-structure, there is a bijection*

$$j^{TOP/PL} : [M, TOP/PL] \to \mathcal{S}^{TOP/PL}(M).$$

- *Let* $\tau_M : M \to BPL$ *denote the classifying map of the tangent bundle of an* n*-dimensional* PL*-manifold* M*. It admits a smooth structure if and only if there exists a lift w In fact, there is a bijection*

  $$\mathcal{S}^{PL/O}(M) \to \mathrm{Lift}^{PL/O}(\tau_M)$$

  where $\mathrm{Lift}^{PL/O}(\tau_M)$ *denotes vertical homotopy classes of lifts of* τ_M*. Here* $[\varphi, N] \mapsto [\tau_N \circ \varphi]$ *where* $\tau_N : N \to BO$ *is a classifying map of the smooth tangent bundle of* N*. Moreover, if* M *admits a smooth structure, there is a bijection*

  $$j^{PL/O} : [M, PL/O] \to \mathcal{S}^{PL/O}(M).$$

Remark 3.3. *This remark describes the bijective correspondences* $j^{TOP/PL}$ *and* $j^{PL/O}$*. These correspondences are not particularly well-documented in the literature, but see* [HM74, Part II] *and* [Bro72, II.4].

We will only describe $j^{PL/O}$*; the case of* $j^{TOP/PL}$ *is completely parallel. The space* PL/O *is an abelian* H*-space, with the* H*-space structure corresponding to direct sum, using an isomorphism* $\mathbb{R}^\infty \cong \mathbb{R}^\infty \oplus \mathbb{R}^\infty$*. The map* $BO \to BPL$ *is a principal* PL/O*-fibration and hence the abelian group* $[M, PL/O]$ *acts freely and transitively on* $\mathrm{Lift}^{PL/O}(\tau_M)$*. Then the map* $j^{PL/O} : \mathcal{S}^{PL/O}(M) \to \mathrm{Lift}^{PL/O}(\tau_M)$ *is the bijection defined by choosing a lift* $\widetilde{\tau}_M : M \to BO$ *as a base point and requiring* $j^{PL/O}[f] = [f].[\widetilde{\tau}_M]$*.*

The books [KS77] and [HM74] are devoted to the proof of the fundamental theorem of smoothing theory. The proof of the first part uses surgery and high-dimensional topology, but the statement and proofs of the second part are dimension independent. An exposition of the framework of the fundamental theorem is given in [Rud].

Often the fundamental theorem is stated using concordance, a notion that is equivalent to isotopy subject to the dimension restrictions in the fundamental theorem. Two PL-structures (φ_0, N_0) and (φ_1, N_1) on a topological manifold M are *concordant* if there exists PL-manifold P with boundary

give precise references (see [HM74, Theorem II.4.1]) ; maybe highlight manifolds with boundary and a homeomorphism $P \to M \times I$ so that the induced map $N_0 \sqcup N_1 \to \partial P$ is a PL-homeomorphism.

We next discuss low dimensional cases. The space PL/O is 6-connected, hence by the fundamental theorem any PL-manifold of dimension seven or less admits a smooth structure, unique up to isotopy if the dimension is less than seven. To show that PL/O is 6-connected it suffices to show that all smoothings of the PL-sphere S^n are isotopic for $n \leq 6$. For $n = 0, 1, 2, 3$ this follows from Moise [Moi77], for $n = 4$ this follows from Cerf [Cer68] and for $n = 5, 6$ this follows from Kervaire-Milnor [KM63].

For manifolds of dimension less than or equal to three, the notions of topological manifolds, PL-manifolds, and smooth manifold coincide, in the sense that a manifold with weaker structure can be given a stronger structure, unique up to isotopy. For manifolds of dimension less than or equal to three, one can prove existence and uniqueness of PL-structures on topological manifolds by direct arguments. In dimensions 0 and 1 this is an exercise. Existence of PL-structures on topological 2-manifolds is due to Rado and on topological 3-manifolds is due to Moise (see [Moi77] for proofs in both dimensions.) A proof for uniqueness of PL-structures on topological 2-manifolds in given by Epstein [Eps66, Theorem A4]. A modern proof of both existence and uniqueness of PL-structures on topological 3-manifolds is given by Hamilton [Ham76]. To summarize, any topological manifold of dimension less than or equal to three admits a smooth structure and any homeomorphism between two smooth manifolds of dimension less than or equal to three is isotopic (homotopic through homeomorphisms) to a diffeomorphism.

Here are some important calculations of homotopy groups of classifying spaces and the associated homotopy fibers of the forgetful maps.

(1) $\pi_n(G) = \pi_n^s = \{\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, 0, \mathbb{Z}/2, \mathbb{Z}/240, \cdots\}$. These are the stable homotopy group of spheres.

(2) $\pi_n(O) = \{\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots\}$. The homotopy groups are 8-periodic.

(3) $\pi_n(\mathbb{Z} \times G/TOP) = \{\mathbb{Z}, 0, \mathbb{Z}/2, 0, \ldots\}$. These homotopy groups are 4-periodic.

(4) $\pi_n(G/PL) = \{\mathbb{Z}, 0, \mathbb{Z}/2, 0, \ldots\}$. These homotopy groups are 4-periodic.

(5) TOP/PL is an Eilenberg-MacLane space $K(\mathbb{Z}/2, 3)$.

(6) $\pi_n(PL/O) = \{0, 0, 0, 0, 0, 0, 0, \mathbb{Z}/28, \mathbb{Z}/2, (\mathbb{Z}/2)^3, \mathbb{Z}/6, \mathbb{Z}/992, 0, \ldots\}$ classifies the exotic n-spheres, which are finite in number in every dimension. For instance, $\pi_7(PL/O) = \mathbb{Z}/28$ means that there

are exactly 28 distinct smooth structures on S^7, up to orientation-preserving diffeomorphism (see [KM63]).

Brieskorn Variety 3.4. Let $f : \mathbb{C}^5 \to \mathbb{C}$ be the map defined by $f(z) = z_0^2 + z_1^2 + z_2^2 + z_3^3 + z_4^5$. Then $V = V(2, 2, 2, 3, 5) := f^{-1}(0)$ is a 4-dimensional complex variety with an isolated singularity at the origin. The link $\Sigma^7 = V \cap S^9$ is smooth 7-dimensional manifold which is homeomorphic, not diffeomorphic, to S^7. That is to say, Σ^7 is an *exotic sphere*. We now outline the ingredients needed to show this.

The following facts are shown in [Bri66], see also [Hir95]. Let $\varepsilon \in \mathbb{C}^5 - \{0\}$ have sufficiently small norm. Then $\Omega^8 = f^{-1}(\varepsilon) \cap D^{10}$ is a smooth compact 8-manifold whose boundary is diffeomorphic to Σ^7. The normal bundle of Ω in D^{10} is orientable since Ω and $\mathbb{C}^5$ are both complex, hence orientable, manifolds. It follows that both Ω and $\partial\Omega$ are stably parallelizable and that $\partial\Omega$ is parallelizable. It is a general fact (independent of the exponents $(2, 2, 2, 3, 5)$) that Ω is 3-connected and $\partial\Omega$ is 2-connected. For these particular exponents, moreover, one can show $H_4(\Omega) \cong \mathbb{Z}^8$ and that the intersection pairing $H_4(\Omega) \times H_4(\Omega) \to \mathbb{Z}$ is unimodular and positive definite. It follows that the signature $\sigma(\Omega) = 8$. (In fact, the intersection matrix is the famous E_8-matrix, a symmetric integral matrix with determinant one and even numbers on the diagonal.) Since the intersection pairing is unimodular, $\partial\Omega \cong \Sigma^7$ is 3-connected, hence 6-connected, hence a homotopy sphere, and hence homeomorphic to a sphere by the h-cobordism theorem (see Example 3.7).

Suppose, by way of contradiction, that Σ^7 is diffeomorphic to S^7. Then $M = \Omega^8 \cup_\Sigma D^8$ is a closed, smooth, stably parallelizable 8-manifold. The intersection form on Ω is dual to the pairing $H^4(\Omega, \Sigma) \times H^4(\Omega, \Sigma) \to \mathbb{Z}$, defined by $(\alpha, \beta) \mapsto \langle \alpha \cup \beta, [\Omega] \rangle$, where $[\Omega] \in H_8(\Omega, \Sigma)$ is the fundamental class. It is then an easy exercise to see that that intersection pairings of Ω and M agree, and hence that M has signature 8.

Now, the Hirzebruch Signature Theorem implies that for a closed, smooth 8-dimensional manifold X, the signature is given by

$$\sigma(X^8) = \frac{1}{45} \langle 7p_2 - p_1^2, [X] \rangle$$

where $p_i \in H^{4i}(X; \mathbb{Z})$ are the Pontryagin classes of X.

Let $i : \Omega \to M$ be the inclusion map. Then $p_1(\Omega) = i^*(p_1(M))$. Since Ω is parallelizable, this shows that $i^*(p_1(M)) = 0$. But $i^* : H^4(M) \to H^4(\Omega)$ is an isomorphism, so $p_1(M) = 0$. The signature theorem gives us $\sigma(M) = \frac{7}{45} \langle p_2, [M] \rangle$. This shows that $45\sigma(M)$ is divisible by 7, which is a contradiction.

An *h-cobordism* $(W; M, N)$ is a smooth manifold W with boundary $\partial W = M \sqcup N$, where the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences. One says that M and N are *h-cobordant.* One defines an h-cobordism in the PL and topological categories is a similar way.

Theorem 3.5 (h-Cobordism Theorem)**.** *Any h-cobordism* $(W; M, N)$ *with* $\dim W \geq 6$ *and* W *simply connected is diffeomorphic relative* M *to the trivial h-cobordism* $(M \times [0,1]; M \times \{0\}, M \times \{1\})$*.*

The h-cobordism Theorem was first proved by Smale in 1960's. It holds in the smooth, PL, and of PL-manifolds. In the topological category it is true for manifolds with $\dim W \geq 5$ by the results of Freedman [Fre82]. Using the h-cobordism Theorem, one can readily prove the generalized Poincaré conjecture in higher dimensions.

Theorem 3.6 (Generalized Poincaré Conjecture)**.** *Let* M *be a closed manifold homotopy equivalent to* S^n*. Then* M *is homeomorphic to* S^n*.*

We point out that the conjecture has now been settled in all dimensions. Most recently, the last unsolved case of dimension 3 follows from Perelman's proof of Thurston's Geometralization Conjecture (see [KL08], [Thu82]).

Proof of the Generalized Poincaré Conjecture for $n \geq 5$. Let K_1 and K_2 be two disjoint submanifolds of M such that each is homeomorphic to the n-disk. Remove the interiors of these manifolds from M and denote the resulting submanifold by W. It is not difficult to observe that the natural inclusion $\partial K_i \hookrightarrow W$ is a homotopy equivalence for $i = 1, 2$. Hence, the triple $(W; \partial K_1, \partial K_2)$ forms an h-cobordism. So, by Theorem 3.5, there is a homeomorphism

$$(\Phi, \phi_1, \phi_2) : (W; \partial K_1, \partial K_2) \to (\partial K_1 \times [0,1]; \partial K_1 \times \{0\}, \partial K_1 \times \{1\})$$

relative ∂K_1. Applying the Alexander's trick (coning), we extend ϕ_2 to a homeomorphism $\bar{\phi_2} : K_2 \to K_1$. Using this map, we can now define a homeomorphism $\overline{\Phi} : M \to K_1 \times \{0\} \cup_{\partial K_1 \times \{0\}} \partial K_1 \times [0,1] \cup_{\partial K_1 \times \{1\}} K_1 \times \{1\}$ by

$$(\Phi, \mathrm{Id}, \bar{\phi_2}) : (W, K_1, K_2) \to (\partial K_1 \times [0,1], K_1 \times \{0\}, K_1 \times \{1\}).$$

Since the manifold $K_1 \times \{0\} \cup_{\partial K_1 \times \{0\}} \partial K_1 \times [0,1] \cup_{\partial K_1 \times \{1\}} K_1 \times \{1\}$ is homeomorphic to S^n, we obtain the desired homemorphism. □

Next, we discuss some uniqueness and existence questions of topological structures on manifolds.

Example 3.7. Suppose S^n, $n \geq 6$, is equipped with a PL-structure. Let us denote this PL-manifold by M and let K_1 and K_2 be two disjoint standard n-simplices of M. Proceeding as in the proof of the Generalized Poincaré Conjecture, we obtain a PL-homeomorphism $\phi : M \to S$, where $S = \partial K_1 \times [0,1] \cup_{\partial K_1 \times \{0\}} K_1 \cup_{\partial K_1 \times \{1\}} K_1$. The simplicial complex S defines the standard PL-structure on S^n.

This argument shows that any PL-structure on S^n, $n \geq 6$, is equivalent to the standard PL-structure. A variant of this proof works for $n = 5$.

One can also arrive to the same conclusion by applying the Fundamental Theorem of Smoothing. This is because the set of isotopy classes of all PL-structures on S^n is in bijection with $\pi_n(TOP/PL)$. But this group is trivial, since TOP/PL is a $K(\mathbb{Z}/2, 3)$-space.

Example 3.8. Consider the example of the exotic sphere Σ^7 defined in 3.1. Recall that Σ^7 is homeomorphic to S^7, but has a PL-structure induced from that of Ω^8. Let $f : \Sigma^7 \to S^7$ be a PL-homeomorphism. We can construct a PL-manifold $\Omega^8 \cup_f D^8$. The same argument that showed Σ^7 is not diffeomorphic to S^7 also shows that $\Omega^8 \cup_f D^8$ has no smooth structure.

Example 3.9. An example of a topological manifold with no PL-structure is Freedman's $|E_8|$-manifold. In fact, two stronger results hold. First, $|E_8|$ does not admit a PL-structure stably: $|E_8| \times K$ does not admit a PL-structure for any PL-manifold K. Second, the $|E_8|$-manifold is not triangulable: it is not homeomorphic to a simplicial complex.

The $|E_8|$-manifold is a simply-connected, closed, topological 4-manifold with intersection form E_8 and signature 8. It was constructed by Freedman [Fre82], who, with the help of Quinn's proof of the annulus conjecture in dimension 4, showed that these properties characterize the manifold up to homeomorphism [FQ90].

If $|E_8|$ admitted a PL-structure, then it would admit a smooth structure, since PL and smooth structures coincide for 4-manifolds. If $|E_8|$ admitted a smooth structure, then it would admit a spin structure, since the intersection form is even and hence by Wu's formula, the second Stiefel-Whitney class vanishes. Rokhlin's theorem (see [Roh52]) asserts that the signature of a smooth, closed, spin 4-manifold is divisible by 16. Thus $|E_8|$ admits no PL-structure.

In fact, $|E_8|$ does not admit a PL-structure stably. Since $TOP/PL = K(\mathbb{Z}/2, 3)$, for any topological manifold M^n, there is a cohomology class $\kappa(M^n) \in H^4(M^n; \mathbb{Z}/2)$ which vanishes if and only if the tangent bundle admits a PL-structure. This cohomology class is called the *Kirby-Siebenmann obstruction* and for a closed 4-manifold with even intersection form is given by the signature divided by 8 considered as an element of $\mathbb{Z}/2$ (see [KS77, p. 329]). This argument shows that $|E_8| \times K$ does not admit a PL-structure for any manifold K.

Finally, it can be shown that $|E_8|$ is not triangulable. This possibility was first observed by Freedman (see [Fre82, Corollary 1.6]). In fact, we show that every triangulable simply-connected 4-manifold M has a PL-structure (which $|E_8|$ does not have). Let v denote any vertex of M^4. Then the pair $(\text{link}(v), \text{star}(v))$ is a PL-manifold with boundary. Since $\text{link}(v)$ is contractible, $\text{star}(v)$ is a homology 3-sphere. By the Seifert-Van Kampen Theorem applied to $M = (M - v) \cup \text{link}(v)$ one sees that $1 = \pi_1(M - v) = \pi_1(\text{link}(v) - v) = \pi_1(\text{star}\, v)$. Thus $\text{star}(v)$ is a PL homotopy 3-sphere and so, by the Poincaré Conjecture, it is a PL 3-sphere. Therefore, M is PL.

It is unknown if there are any nontriangulable manifolds of dimension greater than four.

Example 3.10. Next, we show that there are distinct PL-manifolds which are homeomorphic. For this, we consider the 5-torus T^5 with the standard PL-structure coming from decomposition of the torus as a product of circles. This PL-structure on the torus is not unique and can be altered to obtain a family of 2^{10} distinct PL-isotopy classes.

By the Fundamental Theorem of Smoothing Theory, there is a bijection $\psi : \mathcal{S}^{TOP/PL}(T^5) \to [T^5, TOP/PL]$. Since $TOP/PL = K(\mathbb{Z}/2, 3)$, there is thus a bijection $\mathcal{S}^{TOP/PL}(T^5) \to H^3(T^5; \mathbb{Z}/2)$ which is in turn isomorphic to $(\mathbb{Z}/2)^{10}$. Hence $|\mathcal{S}^{TOP/PL}(T^5)| = 2^{10}$.

Now, we know that the set of PL-structures on T^5 is in bijection with $\mathcal{S}^{TOP/PL}(T^5)/\text{MCG}(T^5)$. Note the mapping class group maps to $\text{Aut}(H_1(T^5)) = GL_5(\mathbb{Z})$ and $GL_5(\mathbb{Z})$ acts on $\mathbb{R}^5/\mathbb{Z}^5 = T^5$. Thus there is a natural action of $\text{MCG}(T^5)$ on $H^3(T^5; \mathbb{Z}/2)$. One can show (using a surgery theoretic argument [Wal99, Chapter 15A]) that the map ψ is equivariant with respect to the action of the mapping class group. This shows that $\text{PL}(T^5) \simeq H^3(T^5; \mathbb{Z}/2)/\text{GL}_5(\mathbb{Z})$ and thus has at least cardinality two, since the zero and a non-zero element of $H^3(T^5; \mathbb{Z}/2)$ are in distinct orbits of $\text{GL}_5(\mathbb{Z})$. We have just proved that there are at least two PL-manifolds homeomorphic to T^5. Wall [Wal99, p. 236] asserts that there are exactly two.

One consequence of smoothing theory is that the number of diffeomorphism classes of smooth structures on any compact topological manifold of dimension greater than four is finite. A different sort of finiteness holds for tori: any smooth structure on a torus becomes standard after passing to a finite cover. Although this is a consequence of smoothing theory, the logical and historical route to this result is quite complicated.

Transversality and handlebody structures on manifolds are relatively easy in the PL and smooth categories. In the PL-category they are a consequence of linear algebra and in the smooth category they are a consequence of the inverse function theorem. However, things are more difficult in the topological category. Transversality, handlebody structures and smoothing theory for high-dimensional manifolds are all due to Kirby-Siebenmann [KS77]. The key ingredient in their work was Kirby's proof of the Annulus Conjecture. Kirby needed the fact that any PL-structure on the tori (dim > 4) became standard after passing to a finite cover. This was accomplished independently by Hsiang-Shaneson and Wall who used PL-surgery theory to show any PL-structure on a homotopy torus (dim > 4) becomes standard after passing to a finite cover.

Recently, for application to a problem in dynamics, the first named author showed [FKS] that any two smooth structures on a nilmanifold (dim > 4) become diffeomorphic after passing to a finite cover.

The Fundamental Theorem of Smoothing Theory fails in dimension four due to the spectacular work of Simon Donaldson. The Kirby-Siebenmann invariant of $|E_8|\#|E_8|$ vanishes, so the topological tangent bundle stably admits a reduction to a vector bundle; also $(|E_8|\#|E_8|) \times \mathbb{R}$ is smoothable. But Donaldson [Don83] showed that if the intersection form of a simply-connected smooth closed 4-manifold was positive definite, then its intersection form is given by the identity matrix. In particular $|E_8|\#|E_8|$ is not smoothable (its intersection form is even). Donaldson also showed that uniqueness fails, by showing that $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ admits a smooth structure not diffeomorphic to the standard one. But since $H^3(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2;\pi_3(PL/TOP)) = 0$, they have the same (stably) tangent bundles and the two smooth structures become diffeomorphic after crossing with $\mathbb{R}$. An even more spectacular example is that $\mathbb{R}^4$ admits an uncountably number of diffeomorphism classes of smooth structures [Tau87].

4. Smoothing Homotopy Equivalences

Now, we consider some uniqueness questions for topological manifolds. Recall the Poincaré Conjecture, which has now been proven in all dimensions due to Smale (dim ≥ 5), Freedman (dim $= 4$), and Perelman (dim $= 3$). It states that any manifold homotopy equivalent to S^n is in fact homeomorphic to S^n. Therefore, one may ask the following question.

Question 1. What classes of closed manifolds have the property that when two such manifolds are homotopy equivalent then they are in fact homeomorphic?

Next, we give two examples that show that not all closed manifolds are this rigid. First, we need to explain what it means for a homotopy equivalence to be simple.

Let R be a commutative ring with unity. Recall that the algebraic K_1-group of R is defined by

$$K_1(R) = GL(R)/[GL(R), GL(R)]$$

Let π be a group, $\mathbb{Z}\pi$ be the associated group ring, and let $g \in K_1(\mathbb{Z}\pi)$ be the element represented by the matrix $g \cdot 1 \in GL_1(\mathbb{Z}\pi)$. The *Whitehead group* of $\mathbb{Z}\pi$ is defined by

$$Wh(\pi) = K_1(\mathbb{Z}\pi)/\langle \pm g | g \in \pi \rangle.$$

We note that the Whitehead group is also a quotient of the reduced K_1-group of R, $\widetilde{K}_1(\mathbb{Z}\pi) = GL(\mathbb{Z}\pi)/[(-1)]$ where $(-1) \in GL(1, \mathbb{Z}\pi)$. Also observe that an isomorphism $f : F \to F'$ of free finite dimensional R-modules with a specified bases defines an element $[f] \in \widetilde{K}_1(\mathbb{Z}\pi)$.

Now, let C_* be a finite based contractible free $\mathbb{Z}\pi$-chain complex with a contraction s. It, in fact, follows that $s+\partial : C_{\text{odd}} \to C_{\text{even}}$ is an isomorphism. Then, we can define $\tau(C) = [s + \partial] \in Wh(\pi)$ called the *torsion* of C (see 11.3 of [DK01] for details). Here $C_{\text{odd}} = \oplus C_{2i+1}$, $C_{\text{even}} = \oplus C_{2i}$, and $[s + \partial]$ is the image under the quotient map of the class in $\widetilde{K}_1(\mathbb{Z}\pi)$ corresponding to the isomorphism $s + \partial$.

Given a weak homotopy equivalence $f : C_1 \to C_2$ between two finite based free $\mathbb{Z}\pi$-chain complexes, we define the *torsion* of f by $\tau(f) = \tau(C(f))$, where $C(f)$ is the algebraic mapping cone of f. This chain complex is finite, contractible, and free with the basis induced from a given basis on C_1 and C_2.

If $f : X \to Y$ is a homotopy equivalence between finite complexes X and Y, we can homotop f to a cellular map $g : X \to Y$. Then, $g_* : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ is a weak chain homotopy equivalence between finite based free $\mathbb{Z}\pi_1(Y)$-complexes. Hence, we can define the *torsion* of the homotopy equivalence f by $\tau(f) = \tau(g_*)$. We say that f is a *simple* homotopy equivalence when $\tau(f) = 0 \in Wh(\pi_1(Y))$.

There is also a geometry definition of Whitehead group defined in terms of elementary expansions and elementary collapses (see 11.2 of [DK01]). If X is a CW-complex, then an *elementary expansion* of X is a complex Y, obtained by gluing D^n to X along $D^{n-1}_- \to X$, where $\partial D^n = S^{n-1} = D^{n-1}_+ \cup_{S^{n-2}} D^{n-1}_-$. That is, Y is obtained by attaching a disk D^n along a part of its boundary D^{n-1} to X. An *elementary collapse* is the complex X in this construction.

Theorem 4.1. *A map $f : X \to Y$ between finite CW-complex is a simple homotopy equivalence if and only if it is homotopic to a finite sequence of elementary expansions and collapses.*

An important theorem of Chapman shows that simple-homotopy type is a homeomorphism invariant.

Theorem 4.2 (Chapman, [Cha74])**.** *If $f : X \to Y$ is a homeomorphism of finite CW-complexes, then f is a simple-homotopy equivalence.*

Now, we are ready to give our first example of homotopy equivalent manifolds which are not homeomorphic.

Example 4.3. Let p and q be relatively prime positive integers. We write the 3-sphere by $S^3 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$. Consider the action of the cyclic group $\mathbb{Z}/p = \langle t | t^p = 1\rangle$ on S^3 defined by

$$t \cdot (z_1, z_2) = (\zeta z_1, \zeta^q z_2)$$

where $\zeta = e^{2\pi i/p}$. This action is free, so the quotient space of the action defines a closed manifold $S^3/\mathbb{Z}/p$. This manifold is called the *lens space* of type (p, q) and is denoted by $L(p, q)$.

Let us consider specifically the lens spaces $L(7, 1)$ and $L(7, 2)$. These manifolds are homotopy equivalent (see 11.35, [DK01]). On the other hand, any homotopy equivalence between $L(7, 1)$ and $L(7, 2)$ is not simple (see 11.36, [DK01]). Therefore, by the theorem of Chapman they are not homeomorphic.

Example 4.4. There exists an oriented sphere bundle $S(\xi) : S^3 \hookrightarrow E \to S^4$ such that E is fiber homotopy trivial and the first Pontrjagin class $p_1(S(\xi))$ is not zero. The first condition means that there exists a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{h} & S^3 \times S^4 \\ \pi \big\downarrow & & \pi_2 \big\downarrow \\ S^4 & \xrightarrow{id} & S^4 \end{array}$$

such that h is a homotopy equivalence. We claim that E is not homeomorphic to $S^3 \times S^4$.

First, let us show the existence of an oriented vector bundle ξ over S^4 whose associated sphere bundle has a total space which is fiber homotopy trivial. By the classifying theory for vector bundles and spherical fibrations, it suffices to show that the homomorphism $\psi : \pi_4(BSO(4)) \to \pi_4(BG(4))$ induced by the forgetful map has nontrivial kernel. Since $\pi_4(BSO(4)) \cong \pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_4(BG(4)) \cong \pi_3(G(4)) = \mathbb{Z}/12 \oplus \mathbb{Z}$, our claim easily follows. Moreover, after identifications, the homomorphism $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/12 \oplus \mathbb{Z}$ is given by $(x, y) \mapsto ([x], y)$. Now, given any $(k, l) \in \mathbb{Z} \oplus \mathbb{Z}$ with $k \equiv 2l \bmod 4$, we can choose ξ so that it first Pontrjagin class is ku and its Euler class is lu where u is a generator of $H^4(S^4)$ (see [MS74, 20.10]). By setting $k = 12$ and $l = 0$, it follows that the associated sphere bundle has a fiber homotopy trivial total space and a nontrivial first Pontrjagin class.

Let $\xi : \mathbb{R}^4 \hookrightarrow \overline{E} \to S^4$ be such a vector bundle and denote $p_*^{\mathbb{Q}}$ its rational Pontrjagin class. By definition, $p_1(S(\xi)) = p_1(\xi)$ and hence, $p_1(\xi) \neq 0$. By a similar argument as in the proof of Lemma 20.8 in [MS74], it follows that $\tau(E) \oplus \nu(E)|_\xi \cong \pi^*\xi \oplus \pi^*\tau(S^4)$. Since $\nu(E)|_\xi$ is a trivial line bundle and $\tau(S^4)$ is a trivial 4-plane bundle, this implies $p_1^{\mathbb{Q}}(E) = p_1^{\mathbb{Q}}(\tau(E) \oplus \varepsilon^1) = p_1^{\mathbb{Q}}(\pi^* S(\xi) \oplus \varepsilon^4) = p_1^{\mathbb{Q}}(\pi^* S(\xi)) = \pi^*(p_1^{\mathbb{Q}}(\xi)) \neq 0$.

On the other hand, $S^3 \times S^4$ is parallelizable. Thus, $p_1^{\mathbb{Q}}(S^3 \times S^4) = 0$. Since rational Pontrjagin classes are invariant under homeomorphisms, this shows that E cannot be homeomorphic to $S^3 \times S^4$.

Note that in the examples we have considered, the manifolds are not aspherical. A topological space is called *aspherical* if its universal cover is contractible. The following conjecture due to Armand Borel gives more insight to the question we posed earlier.

Conjecture 4.5 (Borel)**.** *Let M and N be closed aspherical manifolds. If M is homotopy equivalent to N, then it is homeomorphic to N.*

As it is often the case and this is not an exception that a conjecture is easily stated but turns out to be a very difficult problem. For instance, in dimension 3 it implies the Poincaré Conjecture; if Σ^3 is a homotopy 3-sphere, the Borel conjecture implies that the manifolds $T^3\#\Sigma^3$ and $T^3\#S^3$ are homeomorphic. Milnor's prime decomposition of 3-manifolds (see [Mil62]) shows that $\Sigma^3 \cong S^3$.

The Borel Conjecture is known to hold in many cases. It is true if M is the n-dimensional torus T^n for $n \geq 4$, or if M is a smooth manifold of dimension $n \geq 5$ with a Riemannian metric of sectional curvature $\mathcal{K}_M \leq 0$ by Farrell-Jones (see [FJ93]).

One should note that this question addresses manifolds with torsion free fundamental groups. For if a manifold M^n is aspherical, it is an n-dimensional model for the classifying space $B\pi$, where $\pi = \pi_1(M)$. This implies that the fundamental group π has a finite cohomological dimension bounded by n. In particular, it must be torsion free.

We also remark that the conjecture fails to hold in the smooth category. A similar argument as in Example 3.4 shows that for the exotic sphere Σ^7, manifolds $T^7\#\Sigma^7$ and T^7 are not diffeomorphic.

5. Presurgery

We begin our discussion with the s-Cobordism Theorem which is a generalization of the h-cobordih-cobordismsm Theorem and a fundamental result in Geometric Topology.

Theorem 5.1 (s-Cobordism Theorem)**.** *Let $(W; M, f, N, g)$ be an h-cobordism where M is a connected closed manifold of $dim(M) \geq 5$. Then W is diffeomorphic relative M to the trivial cobordism $(M \times [0,1]; M \times \{0\}, M \times \{1\})$ if and only if its Wheathead torsion $\tau(W, M) = 0 \in Wh(\pi_1(M))$.*

The *Whitehead torsion* of an h-cobordism $(W; M, f, N, g)$ is defined by

$$\tau(W, M) = (i_{\partial_0 W} \circ f)_*^{-1}(\tau(i_{\partial_0 W} \circ f : M \to W)).$$

In summary, the s-Cobordism Theorem tells us that the manifolds M and N must be diffeomorphic provided the Whitehead torsion of the h-cobordism is trivial. Thus, to show that given homotopy equivalent closed manifolds M and N are diffeomorphic, one can try to produce such a cobordism by first constructing a cobordism between M and N and a map

$$(F; f, \mathrm{id}) : (W; M, N) \to (N \times [0,1]; N \times \{0\}, N^- \times \{1\}),$$

where $f : M \to N$ is a homotopy equivalence. Then, one can try to "change" F and W relative the boundary to F' and W' such that $F' : W' \to N \times [0,1]$ becomes a homotopy equivalence and $\tau(W, M) = 0$.

More generally, we can construct a cobordism F where instead of the target manifold N, we consider a CW-complex X and a map $f : M \to X$. The goal will then be to alter F and W relative M in a similar way, to obtain a cobordism

$$(F'; f, g') : (W'; M, N') \to (X \times [0,1]; X \times \{0\}, X^- \times \{1\})$$

such that g' becomes a homotopy equivalence. This naturally leads us to the following:

Question 2. Let X be a CW-complex. When is X homotopy equivalent to a closed manifold?

Now, if a CW-complex X is homotopy equivalent to a closed manifold, then it must satisfy Poincaré duality. This is the first obstruction to obtaining a positive answer to this question. To make further progress, one must, as usual, proceed via bundle theory.

Exotic Poincaré Complex 5.2. We give an example [GS65], [MM79, 2.5] due to Browder of a simply-connected five-dimensional Poincaré complex X which does not have the homotopy type of a closed manifold. This construction is motivated by the fact that the sequence $\pi_2(O) \to \pi_2(PL) \to$

$\pi_2(TOP) \to \pi_2(G)$ is isomorphic to $0 \to 0 \to 0 \to \mathbb{Z}/2$. The proof we give is a simplification due to Diarmuid Crowley.

We first give some geometric motivation. Suppose $\xi = \{\mathbb{R}^n \to E \xrightarrow{\pi} B\}$ is a smooth vector bundle over a smooth manifold. Consider $B \subset E$ as a submanifold given by the zero section. It is not difficult to see that $TE|_B = \xi \oplus TB$. It follows that $TE = \pi^*\xi \oplus \pi^*TB$. A similar analysis can be made of the tangent bundle of the sphere bundle, since the sphere bundle is codimension one in E. One special case would be where B is a sphere, in which case the bundle is constructed by clutching and the tangent bundle TB is stably trivial.

Now we turn to our example, which is a homotopy theoretic version of the above. Note that $\pi_4(S^2) \cong \mathbb{Z}/2$ generated by η^2, the composition of the suspension of the Hopf map with the Hopf map. Thus there is a nontrivial element $[\widehat{\eta^2}] \in [S^2, \mathrm{Map}(S^2, S^2)]_* \cong [S^2 \wedge S^2, S^2]_*$. Consider the corresponding spherical fibration $\xi = \{S^2 \to X \xrightarrow{p} S^3\}$ given by clutching:

$$X = \frac{D^3 \times S^2 \cup D^3 \times S^2}{(x,y) \sim (x, \widehat{\eta^2}(x)(y))} \qquad \text{for } x \in \partial D^3.$$

One can show that X is a simply-connected five-dimensional Poincaré complex either by showing the X is a cohomology $S^2 \times S^3$ by a spectral sequence computation or by using the main result of Gottlieb [Got79]: this says that if $F \to E \to B$ is a fibration of finite CW complexes with F and B Poincaré, then so is E. The next assertion is that the Spivak normal bundle of X is $p^*\xi$. This is motivated by the geometric considerations above (note $\xi \oplus \xi$ is trivial) and is shown in [GS65] by constructing a Spivak class. It would be interesting to attempt to use [Got79] to give a "geometric" proof that the Spivak bundle is $p^*\xi$.

The Euler class $e(\xi) \in H^3(S^3)$ is 2-torsion since S^2 is even-dimensional (see [MS74, page 99]), hence the Euler class vanishes, hence $p : X \to S^3$ admits a section $s : S^3 \to X$. The bundle ξ is classified by the nontrivial element of $\pi_3(BG) = \pi_2(G) = \pi_2^s = \langle \eta^2 \rangle \cong \mathbb{Z}/2$. Kirby-Siebenmann [KS77, p. 300, 304] show that $\pi_2(TOP) = \pi_2(PL) = \pi_2(O)$. Now, $\pi_2(O) = \pi_2(SO(3)) = 0$. Thus, ξ is not stably fiber homotopy equivalent to a sphere bundle of an $\mathbb{R}^n$-bundle with 0-section. It follows that $p : X \to S^3$ admits a section $s : S^3 \to X$. Since there is no reduction of the stable structure group of $\xi = s^*p^*\xi$ to TOP the same is true for $p^*\xi = \nu_X$. Thus X does not have the homotopy type of a closed topological manifold.

This completes our set examples that distinguish between different types of manifolds. In the table below, we list them with a reference to a particular existence or uniqueness question they address. On the left side of the table, we give references to examples of manifolds which do not admit the next strongest structure. On the right side of the table, we refer to examples of distinct manifolds which become isomorphic after weakening the structure.

Existence	**Example**	**Uniqueness**	**Example**
$PL \supsetneq O$	3.8	$PL \prec O$	3.4+3.7
$TOP \supsetneq PL$	2.1, 3.9	$TOP \prec PL$	3.10
$G \supsetneq TOP$	5.2	$G \prec TOP$	4.3, 4.4

Jim 2.1 is an ex of triang mfld which is not PL. 3.9 is an ex of top mfld which is not triang. Do we not ask uniqueness questions, because PL-mflds are PL homeomorphic iff they admit equiv. PL-triangulations which are simplicially homeomorphic?

5.1. **Pontrjagin-Thom Construction.** Let X be an n-dimensional Poincaré complex. We define a set of equivalence classes of normal invariants $\mathscr{T}(X)$ of X as follows.

A class in $\mathscr{T}(X)$ is represented by a vector bundle $\xi : E \to X$ such that the associated sphere bundle is a Spivak normal fibration. This bundle is called a *normal invariant* of the Poincaré complex X. Two such bundles are said to be equivalent if they are stably isomorphic.

From our discussion in the previous section, we can conclude that for a Poincaré complex X to be homotopy equivalent to a closed manifold it is necessary that the set of normal invariants $\mathscr{T}(X)$ is nonempty. This will become important in the consequent theorem.

Normal invariants of X can be defined differently by using the set of cobordism classes of degree one normal maps $\mathscr{N}(X)$.

A *normal map of degree one* is a collection $(M, i, \bar{f}, \xi)$, where $\xi : E \to X$ is a k-dimension bundle, M is an n-dimensional closed manifold with an embedding $i : M \to \mathbb{R}^{n+k}$, $\bar{f} : \nu(M) \to E$ is a bundle map over a degree one map $f : M \to X$, and $\nu(M)$ is the normal bundle of i.

Let $(M_1, i_1, \bar{f}_1, \xi)$ and $(M_2, i_2, \bar{f}_2, \xi)$ be two degree one normal maps. They are said to be *cobordant* if there exist a cobordism $(W; I, \overline{F}, \xi \times [0,1])$ satisfying the following. W is $(n+1)$-dimension compact manifold together with an embedding $I : W \to \mathbb{R}^{n+k} \times [0,1]$, such that $I^{-1}(\mathbb{R}^{n+k} \times \{0,1\}) = M_1 \amalg M_2$. The boundary of W is $M_1 \amalg M_2$. There is a bundle map $\overline{F} : \nu(W) \to \xi \times [0,1]$ over $F : W \to X \times [0,1]$, where F has degree one. We also require that the composition of the map $\nu(I|_{\partial W}) : \nu(M_1) \amalg \nu(M_2) \to \nu(W)$ with $\overline{F}$ coincides with $\bar{f}_1 \amalg \bar{f}_2$.

The set of cobordism classes of degree one normal maps $(M, i, \bar{f}, \xi)$ for which the bundle ξ has a fixed dimension k is denoted by $\mathscr{N}_k(X)$. We observe that for any positive integer k there exists a canonical map $\mathscr{N}_k(X) \to \mathscr{N}_{k+1}(X)$. This map can easily be defined by a general construction of inflating a k-bundle of X to a $(k+1)$-bundle by summing with a trivial 1-bunble of X. Then, the set of normal invariants $\mathscr{N}(X)$ of X is defined by $\operatorname{colim} \mathscr{N}_k(X)$.

Theorem 5.3 (Pontrjagin-Thom)**.** *Let X be an n-dimensional connected finite CW-complex. There is a bijection $\Phi : \mathscr{N}(X) \to \mathscr{T}(X)$. Moreover, if X is a connected Poincaré complex and $\mathscr{T}(X)$ is nonempty, then there is also a bijection*

$$\Psi : \mathscr{T}(X) \xrightarrow{\approx} [X, G/O].$$

Proof. We give a sketch of the proof that there is a bijection Φ. For a complete proof and for the proof of the second statement of the theorem we refer the reader to [Lüc02] and [MM79].

Define $\Phi : \mathscr{N}(X) \to \mathscr{T}(X)$ to be the map that sends the cobordism class represented by a degree one normal map $(M, i, \bar{f}, \xi)$ to the bundle $\xi : E \to X$. This map is clearly well defined. It also follows straight from the definitions that the associated sphere bundle of ξ is a Spivak normal fibration. Its Spivak class $[c] \in \pi_{n+k}(\mathrm{Th}(\xi))$ can be defined by the composition of the collapse map $c_M : \mathbb{R}^{n+k} \amalg \{\infty\} \to \mathrm{Th}(\nu)$ of M with the map $f_* : \mathrm{Th}(\nu) \to \mathrm{Th}(\xi)$ induced by the bundle map $\bar{f} : \nu(M) \to E$.

Next, we describe the inverse of Φ. Let $\xi : E \to X$ be a k-bundle such that its sphere bundle is a Spivak normal fibration. Let $c : \mathbb{R}^{n+k} \amalg \{\infty\} \cong S^{n+k} \to \mathrm{Th}(\xi)$ be the base point preserving map that represents the Spivak class $[c] \in \pi_{n+k}(\mathrm{Th}(\xi))$ corresponding to ξ. We can homotop c relative to the base point $\{\infty\}$ to a map c_0 so that c_0 is transverse to the 0-section $X \subset \mathrm{Th}(\xi)$. The preimage $M = {c_0}^{-1}(X)$ is then a submanifold of $\mathbb{R}^{n+k}$ of dimension n. Let $f : M \to X$ denote the restriction of the map c_0 to M. Transversality gives us a bundle map $\overline{f} : \nu(M) \to \xi$ covering f. Hence, we obtain a degree one normal map $(M, i, \overline{f}, \xi)$. Since different choices of the map c and the map c_0 produce manifolds cobordant to M, the above construction yields a well-defined inverse of Φ. □

6. Surgery

In this section we give a brief overview of surgery on compact manifolds. A good introduction to the theory is the survey article of Lück [Lüc02] and the book by Ranicki [Ran02]. The reader should consult the original sources such as the books of Browder [Bro72] and Wall [Wal99] for more detailed constructions. For a thorough exposition on classifying spaces and bundle theory see Milgram and Madsen [MM79]. More modern aspects of surgery theory can be found in the books of Ranicki [Ran92] and Weinberger [Wei94].

We have already seen that if an n-dimensional CW-complex X is homotopy equivalent to an n-dimensional closed manifold, then it must be a Poincaré complex with a degree on normal map from a closed n-manifold. Suppose that for a given n-dimensional connected Poincaré complex X we can find a degree one normal map $(M, i, \bar{f}, \xi)$ where $\xi : E \to X$ is a k-dimension vector bundle. When $f : M \to X$ is not a homotopy equivalence, we can ask the following.

Question 3. Is it possible to alter the manifold M and the map f in a sequence of steps, so that the resulting map $f' : M' \to X$ is a homotopy equivalence?

Of course, this question is somewhat ambiguous. The goal here is to to change the manifold M by a certain construction called *surgery*, so that, if possible, the resulting degree one normal $f : M' \to X$ becomes a homotopy equivalence. We explain this next.

6.1. **Surgery up to the Middle Dimension.** Recall that if there is a map $f : M \to X$ which is i-connected for all $i > 0$, then according to a theorem of Whitehead (see [Whi78]) f is a homotopy equivalence. Let $\pi_*(f) = \pi_*(\mathrm{cyl}(f), X)$, then there exist a long exact sequence

$$\cdots \to \pi_{i+1}(M) \to \pi_{i+1}(X) \to \pi_{i+1}(f) \to \pi_i(M) \to \pi_i(X) \to \cdots$$

This shows $f : M \to X$ is a homotopy equivalence if and only if $\pi_i(f) = 0$ for all $i > 0$. In fact, by Poincaré duality we only need that $\pi_{i+1}(f) = 0$ for all $i + 1 \leq [\frac{n}{2}]$. Then, one strategy to change the map $f : M \to X$ to a homotopy equivalence can be to "kill" the elements of the group $\pi_*(f)$. We describe such an operation next.

Let $k < [\frac{n}{2}]$ and suppose $\pi_i(f) = 0$ for all $i \leq k$. Assume there is an embedding $\phi : S^k \to M$ such that the diagram

$$\begin{array}{ccc} S^k & \xrightarrow{j} & D^{k+1} \\ \phi\downarrow & & \downarrow\Phi \\ M & \xrightarrow{f} & X \end{array}$$

represents an element $\alpha \in \pi_{k+1}(f)$. The existence of the map $\bar{f} : \nu(M) \to \xi$ covering f shows that the normal bundle of the embedding $\phi : S^k \to M$ is trivial. Therefore, we can extend ϕ to an embedding $\bar{\phi} : S^k \times D^{n-k} \to M$ that fits into the diagram

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{\bar{j}} & D^{k+1} \times D^{n-k} \\ \bar{\phi}\downarrow & & \downarrow\overline{\Phi} \\ M & \xrightarrow{f} & X \end{array}$$

We can now replace M with a new manifold N that results from removing the interior of the image of $S^k \times D^{n-k}$ in M, then gluing $D^{k+1} \times S^{n-k-1}$ along the boundary, which is the image of $S^k \times S^{n-k-1}$. That is, we set

$$N = D^{k+1} \times S^{n-k-1} \cup_{\bar{\phi}(S^k \times S^{n-k-1})} (M - \bar{\phi}(S^k \times \mathrm{int} D^{n-k})).$$

Let $g : N \to X$ be the map defined by f on $M - \bar{\phi}(S^k \times \mathrm{int} D^{n-k})$ and by the restriction of $\overline{\Phi}$ on $D^{k+1} \times S^{n-k-1}$. It follows that $\pi_i(g) = 0$ for all $i \leq k$ and there is surjective homomorphism $\pi_{k+1}(f) \to \pi_{k+1}(g)$ such that its kernel contains the element α.

The operation we have just described of constructing a new manifold N and a degree one map $g : N \to X$ from the degree one normal data of a given map $f : M \to X$ is called *a surgery step*.

Furthermore, the map $g : N \to X$ obtained after the surgery on M is normally cobordant to $f : M \to X$. This cobordism can be constructed from $M \times [0,1]$ by gluing $D^{k+1} \times D^{n-k}$ along the image of $\bar{\phi}$ to $M \times \{1\}$, i.e.

$$W = D^{k+1} \times D^{n-k} \cup_{\bar{\phi}(S^k \times D^{n-k})} M \times [0,1].$$

The cobordism map $F : W \to X$ is defined in an obvious way. This shows that the resulting degree one normal map represents the same class of normal invariants of X as $(M, i, \bar{f}, \xi)$.

When describing the surgery step, we made an assumption that there is an embedding $\phi : S^k \to M$ such that its extension represents a given element $\alpha \in \pi_{k+1}(f)$. By the Whitney Embedding Theorem (see [Whi43] and [Whi44]) this is always the case provided that $k < [\frac{n+1}{2}]$.

For any $m \geq 0$, let $i_m : \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+m}$ be the canonical inclusion. We have just outlined the proof of the following theorem.

Theorem 6.1 (Surgery Step). *Let $(M, i, \bar{f}, \xi)$ be a degree one normal map. If $k < [\frac{n}{2}]$, then one can apply the surgery step based at any element $\alpha \in \pi_{k+1}(f)$. Moreover, a finite number of surgery steps can be applied to construct a degree one normal map $(N, j, \bar{g}, \xi \oplus \varepsilon^m)$, such that g is k-connected and $(N, j, \bar{g}, \xi \oplus \varepsilon^m)$ is normally cobordant to $(M, i_m \circ i, \bar{f} \oplus id_{\varepsilon^m}, \xi \oplus \varepsilon^m)$.*

6.2. **Surgery Exact Sequence.** Let X be a finite connected Poincaré complex of dimension n. The previous theorem tells us that given a degree one normal map $f : M \to X$, we can perform a finite number of surgery steps to get a k-connected degree one normal map $g : N \to X$ when $n = 2k$ or $n = 2k+1$. If we could do further surgery to kill the elements of $\pi_{k+1}(g)$ without introducing new elements in the previous homotopy groups, Poincaré duality would imply that the resulting map is a homotopy equivalence. It turns out that this is not always possible.

In the even dimensional case $n = 2k$, the weak Whitney Embedding Theorem does not apply. Any element $\alpha \in \pi_{k+1}(f)$ can be represented by a pair of maps $\Phi : D^{k+1} \to X$, $\phi : S^k \cong \partial D^{k+1} \to M$ such that $f \circ \phi = \Phi \circ j$ (see diagram in 4.5). By Hirsch's Immersion Theorem (see [Hir59]), the map $\phi : S^k \to M$ can be chosen to be an immersion. In fact, the regular homotopy type of such a map ϕ depends only on the class α. Then, to do surgery based at $\alpha \in \pi_{k+1}(f)$ we need that ϕ is regularly homotopic to an embedding. It is here that we encounter a restriction on the dimension $k \geq 3$ and an obstruction called the *selfintersection number* of an immersion.

In the case when $n = 2k + 1$, the embedding theorem does apply. Hence, we can choose an embedding $\phi : S^k \to M$ associated to the diagram representing a given element $\alpha \in \pi_{k+1}(f)$. The difficulty here is that now $k + 1 > [\frac{n}{2}]$. So, when we perform surgery to kill the element α, we may produce a nontrivial element in $\pi_k(g)$. One can think of this as an effect of Poincaré duality.

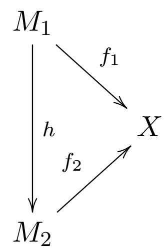
To summarize, given a degree one normal map $(\bar{f}, f) : M^n \to X$ with $n \geq 5$, there exists an algebraic obstruction $\theta(\bar{f})$ to doing surgery on M such that the resulting map $g : N \to X$ is a homotopy equivalence. This obstruction lives in a certain abelian group called the *L-group* of $\pi = \pi_1(X)$, denoted by $L_n(\mathbb{Z}\pi, w)$. This group depends only on the fundamental group π and the orientation homomorphism $w : \pi \to \{\pm 1\}$. It can be defined by using quadratic forms and formations on the finitely generated free modules of $\mathbb{Z}\pi$ (see [Ran92]). A more geometric definition of L-groups which describes them as a certain bordism group of normal invariants of Poincaré pairs, is given by Wall in [Wal99, §9].

Next, we answer the question that motivated our discussion of whether there exists a manifold homotopy equivalent to a given Poincaré complex.

The *structure set* $\mathscr{S}_n(X)$ of X is defined as the set of equivalence classes of orientation preserving homotopy equivalences $f : M \to X$ from n-dimensional oriented closed manifolds to X. Such homotopy equivalences $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are said to be in the same equivalence class if there exists an oriented compact manifold W with $\partial W = M_1 \sqcup M_2$ and $M_1 \cap M_2 = \emptyset$ and a orientation preserving homotopy equivalence of triads

$$(F; f_1, f_2) : (W; M_1, M_2) \to (X \times [0,1]; X \times \{0\}, X \times \{1\}).$$

The *simple structure set* $\mathscr{S}^s_n(X)$ of X is the set of equivalence classes of pairs consisting of an oriented n-dimensional closed manifold M and a simple homotopy equivalence $f : M \to X$. Two such pairs $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are defined to be equivalent if there is an orientation preserving homeomorphism $h : M_1 \to M_2$ such that the diagram



is simple homotopy commutative.

Note that by the s-Cobordism Theorem the simple structure set can be defined by replacing homotopy equivalences in the definition of $\mathscr{S}(X)$ by simple homotopy equivalences.

Now, several of the examples we have considered can be translated into the language of structure sets. For instance, according to Example 4.4 it follows that the structure set $\mathscr{S}_7(S^3 \times S^4) \neq *$. Yet, by the Poincaré Conjecture, both sets $\mathscr{S}_3(S^3)$ and $\mathscr{S}_4(S^4)$ are singletons represented by the maps $\mathrm{id}_{S^3} : S^3 \to S^3$ and $\mathrm{id}_{S^4} : S^4 \to S^4$, respectively. This naturally leads one to ponder the relations between the structure set of a product and the structure sets of the individual factors.

Another example of this reformulation is the Borel Conjecture. It states that for a closed aspherical n-manifold M we have $\mathscr{S}_n(M) = \mathscr{S}_n^s(M) = *$.

To present the main theorem of this section we need to generalize several previous constructions. First, we define a relative version of the structure set $\mathscr{S}(X, Y)$. Suppose (X, Y) is a Poincaré pair. Assume there is a orientation preserving homotopy equivalence $h : K \to Y$ from a closed oriented manifold K to Y. An element in $\mathscr{S}(X, Y)$ is an oriented manifold pair (M, K) with an orientation preserving homotopy equivalence of pairs $(f, h) : (M, K) \to (X, Y)$. Two elements (M_1, f_1) and (M_2, f_2) are said to be equivalent if there is an oriented manifold 4-ad $(W; K \times [0, 1], M_1, M_2)$ with $M_1 \cap M_2 = \emptyset$ and a orientation preserving homotopy equivalence

$$(F; h \times \mathrm{id}, f_1, f_2) : (W; K \times [0,1], M_1, M_2) \to (X \times [0,1]; Y \times [0,1], X \times \{0\}, X \times \{1\}).$$

We also need a generalization of the normal invariants. The set of *normal invariants* of the Poincaré pair (X, Y), denoted by $\mathscr{N}(X, Y)$, is the set of equivalence classes of degree one normal maps $(M, K, i, \bar{f}, \xi)$ such that $f|_{\partial M}$ is a fixed orientation preserving homotopy equivalence $h : K \to Y$. Two such elements $(M_j, K, i_j, \bar{f}_j, \xi_j)$, $j = 0, 1$, are said to be equivalent if they are cobordant by a degree one normal cobordism $(W, I, \bar{F}, \xi \times [0, 1])$ with an underling map of 4-ads

$$(F; h \times \mathrm{id}, f_1, f_2) : (W; K \times [0,1], M_1, M_2) \to (X \times [0,1], Y \times [0,1], X \times \{0\}, X \times \{1\}).$$

The Pontrjagin-Thom construction can also be extended to Poincaré pairs (X, Y). It states that there is a bijection $\mathscr{N}(X, Y) \simeq [X/Y, G/O]$ provided that the set of normal invariants of (X, Y) is nonempty. Here we assume that all maps are base point preserving.

Lastly, a key fact is the existence of an action of the L-groups on the structure sets. More precisely, there exist an action

$$L_{n+1}(\mathbb{Z}\pi, w) \times \mathscr{S}_n(X) \to \mathscr{S}_n(X).$$

defined as follows. Given an $\alpha \in L_{n+1}(\mathbb{Z}\pi, w)$ and $[f : M \to X] \in \mathscr{S}_n(X)$, by the realizability of the surgery obstruction (see [Wal99, Th.10.4]) there is a cobordism of triads

$$(F; F|_{M_1}, F|_{M_2}) : (W; M_1, M_2) \to (M \times [0,1]; M^- \times \{0\}, M \times \{1\})$$

covered by a bundle map $\overline{F}$ such that $F|_{M_1}$ is a diffeomorphism, $F|_{M_2}$ is a homotopy equivalence, and $\theta(\overline{F}) = \alpha$. We define $\alpha \cdot [f] = [f \circ F|_{M_2}]$.

Theorem 6.2 (Surgery Exact Sequence I). *Let CAT denote either of TOP, PL, or O. Let $n \geq 5$ and suppose X is an n-dimensional connected finite Poincaré complex. Suppose the structure set $\mathscr{S}_n^{CAT}(X)$ is nonempty and let $[g]$ be the base point. Then there exists a surgery exact sequence*

$$\cdots \to \mathscr{S}_n^{CAT}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\tau} [SX, G/CAT] \xrightarrow{\theta} L_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \mathscr{S}_n^{CAT}(X) \xrightarrow{\tau} [X, G/CAT] \xrightarrow{\theta} L_n(\mathbb{Z}\pi, w).$$

defined as follows.

- *A normal invariant $x \in [X, G/CAT]$ is in the image of τ if and only if $\theta(x) = 0$.*
- *$[f_1], [f_2] \in \mathscr{S}_n^{CAT}(X)$ with $\tau([f_1]) = \tau([f_2])$ if and only if there is $\alpha \in L_{n+1}(\mathbb{Z}\pi, w)$ such that $\alpha \cdot [f_1] = [f_2]$.*
- *$\alpha \in L_{n+1}(\mathbb{Z}\pi, w)$ such that $\partial(\alpha) := \alpha \cdot [g] = [g]$ if and only if there exists $y \in [SX, G/CAT]$ such that $\theta(y) = \alpha$.*

Proof. We only sketch the proof over the smooth category O. The PL case is similar and TOP case requires a little more work.

First, let us define the map τ. Suppose $h : N \to X$ is an orientation preserving homotopy equivalence. Assume N is embedded in $\mathbb{R}^{n+k}$ by a map i for a sufficiently large k. Let $\nu : \nu(N) \to N$ be the normal bundle of N and let $h' : X \to M$ be a homotopy inverse of h. Then, $(N, i, \bar{h}, h'^*\nu)$ is a degree one normal map. We define $\tau((N, h)) = (N, i, \bar{h}, h'^*\nu)$.

Now, suppose $(M, i, \bar{f}, \xi)$ is a degree one normal map where $f : M \to X$. Then, $\theta(\bar{f}) = 0 \in L_n(\mathbb{Z}\pi, w)$ if and only if f is normally cobordant to a homotopy equivalence $h : N \to X$. Therefore, $\theta(\bar{f}) = 0$ if and only if $(M, i, \bar{f}, \xi)$ is in the image of τ.

Next, suppose two given homotopy equivalences $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ have the same image in $[X, G/O]$. This means that there is a degree one normal map $(\overline{F}, F)$ with an underlying map of triads

$$(F; f_1, f_2) : (W; M_1, M_2) \to (X \times [0,1]; X \times \{0\}, X \times \{1\}).$$

Fixing a homotopy inverse f_1^{-1} of f_1, we can easily construct a degree one normal map $(\overline{G}, G)$ where

$$(G; \mathrm{id}, f_1^{-1} \circ f_2) : (W; M_1, M_2) \to (M_1 \times [0,1]; M_1 \times \{0\}, M_1^- \times \{1\}).$$

Then, $\theta(\overline{G}) \in L_{n+1}(\mathbb{Z}\pi, w)$ and $\theta(\overline{G}) \cdot [f_1] = [f_2]$.

Finally, let $g : M \to X$ be a homotopy equivalence representing the base point in $\mathscr{S}_n(X)$. Suppose there exists an element $\alpha \in L_{n+1}(\mathbb{Z}\pi, w)$ such that $\alpha \cdot [g] = [g]$. Let

$$(F; f_1, f_2) : (W; M_1, M_2) \to (M \times [0,1]; M^- \times \{0\}, M \times \{1\})$$

be a cobordism covered by a bundle map $\overline{F}$ such that f_1 is a homeomorphism, f_2 is a homotopy equivalence, and $\theta(\overline{F}) = \alpha$. By composing F with $g \times \mathrm{id}$: $M \times [0,1] \to X \times [0,1]$, we get a degree one normal map $(\overline{G}, G)$ with

$$G : (W; M_1, M_2) \to (X \times [0,1]; X \times \{0\}, X \times \{1\}),$$

which represents a homotopy class in $[SX, G/O]$. It follows that $\theta(\overline{G}) = \theta(\overline{F}) = \alpha$. The converse is immediate. □

There is a similar surgery exact sequence for simple structure sets.

Theorem 6.3 (Surgery Exact Sequence II). *Suppose $n \geq 5$. Let X be an n-dimensional connected finite Poincaré complex. If the simple structure set $\mathscr{S}_n^{CAT,s}(X)$ is nonempty, then there exists a surgery exact sequence*

$$\cdots \to \mathscr{S}_n^{CAT,s}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\tau} [SX, G/CAT] \xrightarrow{\theta} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \mathscr{S}_n^{CAT,s}(X) \xrightarrow{\tau} [X, G/CAT] \xrightarrow{\theta} L_n^s(\mathbb{Z}\pi, w).$$

6.3. **Examples and Applications.** Next, we present several applications which can be derived from surgery theory.

Example 6.4. For $n \geq 5$, the map $\theta : [S^n, G/\mathrm{TOP}] \to L_n(\mathbb{Z})$ is an isomorphism.

This follows immediately from the surgery exact sequence and the generalize Poincaré conjecture; according to which $\mathscr{S}_n^{\mathrm{TOP}}(S^n) = *$.

We can reformulate this isomorphism by:

$$\pi_n(G/\mathrm{TOP}) \cong L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0(4) \\ 0 & \text{if } n \equiv 1(4) \\ \mathbb{Z}/2 & \text{if } n \equiv 2(4) \\ 0 & \text{if } n \equiv 3(4) \end{cases}$$

Example 6.5. We have already mentioned that given a closed aspherical manifold, the Borel conjecture can be restated by saying that $\mathscr{S}_n^{\mathrm{TOP}}(X) = \mathscr{S}_n^{\mathrm{TOP},s}(X) = *$. Many special cases of the conjecture can be proven by showing that the maps θ on the right and left sides of the structure sets in the surgery exact sequence are injective and surjective, respectively. In [FJ93], Farrell and Jones utilized this approach to show that the conjecture holds for non-positively curved manifolds. Recently, Bartels and Lück proved that it holds, more generally, for aspherical manifolds whose fundamental group is word-hyperbolic or CAT(0) (see [BL]).

Example 6.6. The ideas behind surgery have their origins in the work of Kervaire and Milnor on the classification of smooth structures on a sphere (see [KM63]). One of their celebrated theorems asserts that there are 28 distinct smooth structures on S^7. In our present language, this translates to $|\mathscr{S}_7^{\mathrm{O}}(S^7)| = 28$. We will give part of the idea of the proof of this result in the modern terminology which relies on the surgery exact sequence.

Jim this also follows from Concordance Thm. Does the proof of Concordance use surgery?

We already saw in Example 3.7 that any PL-structure on a sphere of dimension larger than 5 is equivalent to the standard one. This is implies that $\mathscr{S}_n^{\mathrm{PL}}(S^n) = 0$ for $n \geq 5$. So, by the surgery exact sequences for the PL-category, we have $[S^n, G/PL] \cong L_n(\mathbb{Z})$ for all $n \geq 5$. Applying again the surgery exact sequence entails the commutative diagram:

$$\begin{array}{ccccccccc}
 & & \pi_{n+1}(G/O) & \xrightarrow{\psi_1} & L_{n+1}(\mathbb{Z}) & \longrightarrow & \mathscr{S}_n^{\mathrm{O}}(S^n) & \longrightarrow & \pi_n(G/O) \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id} & & \downarrow & & \\
0 & \longrightarrow & \pi_{n+1}(G/PL) & \longrightarrow & L_{n+1}(\mathbb{Z}) & \longrightarrow & 0 & &
\end{array}$$

Therefore, we obtain a long exact sequence:

$$\cdots \longrightarrow \pi_{n+1}(G/O) \xrightarrow{\varphi} \pi_{n+1}(G/PL) \longrightarrow \mathscr{S}_n^{\mathrm{O}}(S^n) \longrightarrow \pi_n(G/O) \longrightarrow \cdots$$

It turns out, that this exact sequence coincides with the long exact sequence of homotopy groups associated to the fibration $PL/O \to G/O \to G/PL$ and $\mathscr{S}_n^{\mathrm{O}}(S^n) \cong \pi_n(PL/O)$. For $n < 7$, this homotopy group is trivial and $\pi_7(PL/O) = \mathbb{Z}/28$ (see [Ran02, p.348–354]).

Theorem 6.7. *Any finite group of odd order such that every Sylow subgroup is cyclic acts freely on an odd dimensional sphere.*

Proof. Let G be an odd order group. The condition that all Sylow subgroups of G are cyclic is equivalent to $\mathrm{H}^{n+1}(G, \mathbb{Z}) = \mathbb{Z}/|G|$ for some n (see [CE56, Chapter XII]) and $n+1$ is called a *period* of G. By a theorem of Swan, this in turn is equivalent to the existence of an n-dimensional CW-complex W with $\pi_1(W) = G$ and $\widetilde{W} \simeq S^n$. The complex W is called a *Swan complex.*

Now, every Swan complex is finitely dominated (see [AD02, 3.4]). To every a finitely dominated CW-complex Y, one can associate the *Wall finiteness obstruction* $[Y]$ which is an element of the reduced projective class group $\widetilde{K}_0(\mathbb{Z}\pi_1(Y)$. It turns out that there is a Swan complex Z of dimension $2n+1$, associated to G, such that its finiteness obstruction vanishes (see [Wal79, 12.6]). Hence, Z is homotopy equivalent to a finite CW-complex X. Note that X is a Poincaré complex. Our goal is to show that it has the homotopy type of a manifold.

The second obstruction we have encountered for X to have a homotopy type of a manifold is the existence of a reduction of the Spivak normal fibration of X to a sphere bundle over X. In section 4.1, we saw that this occurred if and only if the corresponding Spivak bundle map $X \to B(G/TOP)$ is null-homotopic. But $B(G/TOP)$ is an infinite loop space and thus the group $[X, B(G/TOP)]$ injects into $\bigoplus_p [X, B(G/TOP)]_{(p)}$. By applying the transfer map, it follows that the second group injects into $\bigoplus_p [\widetilde{X}/G_p, B(G/TOP)]_{(p)}$ where G_p is a Sylow p-subgroup of G. Since G_p is cyclic, $\widetilde{X}/G_p$ is homotopy equivalent to a lens space (reference???). Hence, the Spivak bundle map to $B(G/TOP)$ in null-homotopic.

need a reference!

It is not difficult to see that any multiple of the period is again period of G. So, we can assume that $n \geq 2$ and X has dimension at least 5. Then, we can apply finite number of surgery steps and the obstruction to obtaining a manifold homotopy equivalent to X lies in the group $L_{2n+1}(\mathbb{Z}G, w)$. But any odd degree L-group of an odd order group is trivial (see [Bak75]). So, there exists an $(2n+1)$-dimensional manifold M homotopy equivalent to X. The universal cover of M is then homotopy equivalent to S^{2n+1} and hence, by the generalized Poincaré conjecture, homeomorphic to it admitting a free action of G. □

This result is a special case of a much more general theorem of Madsen, Thomas and Wall.

Let n-be a positive integer. G is said to satisfy the *n-condition* if every subgroup of order n is cyclic.

Theorem 6.8 (Madsen-Thomas-Wall, [MTW76])**.** *A finite group G acts freely on some sphere if and only if G satisfies p^2 and $2p$ conditions for all primes p*

As a corollary, it follows that for $k \geq 2$, every finite $(2k+1)$-dimensional Swan complex associated to a group G satisfying p^2 and $2p$ conditions for all primes p is homotopy equivalent to a closed manifold.

Jim is this right? By above, if G has odd order then this is true.

Theorem 6.9. *Every dihedral group of order $2m$, where m is odd, acts freely on a finite CW-complex homotopy equivalent to S^3 but it cannot act freely on any S^n.*

Proof. Let D be a dihedral group of order $2m$. Since each Sylow subgroup of D is cyclic and D has period 4, by Swan's theorem, there exists a 3-dimensional Swan complex X such that $\pi_1(X) = D$. By a similar argument as in the proof of Theorem 6.7, we can assume X is finite. Hence, D acts freely on $\widetilde{X}$ which is homotopy equivalent to a sphere.

Now, by a result of Milnor (see [Mil57]), if a group acts freely on some S^n then any element of order 2 (there can only be one such element) must be in the center. Thus, D cannot act freely on any S^n. This, for instance, shows that the surgery obstruction $\theta(x) \in L_3(\mathbb{Z}D, w)$ is nonzero for any normal invariant $x \in [X, G/\mathrm{TOP}]$ associated to a reduction of the Spivak normal fibration of X.

We remark that the L-theoretic description of Milnor's theorem was given by the first author in [Dav83]. He showed $\theta(x) \neq 0$, by introducing an invariant called the surgery semicharacteristic associated to an odd dimensional Poincaré complex. □

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