A source of manifolds over the infinite dihedral group

This follows a conversation with Karsten Grove prompted by a comment of Bruce Hughes.

By a manifold I mean a closed smooth manifold, by a subgroup I mean a closed subgroup, and by an action of a group on a manifold I mean a smooth action.

A manifold admits a group action with orbit space a point if and only if the manifold is diffeomorphic to a homogeneous space G/H. Here is the next level of complexity.

Definition 1. A manifold has cohomogeneity one if it admits a group action with 1-dimensional orbit space. For connected manifolds there are only two possibilities; the orbit space could be an interval or circle.

Given a Lie group G, a closed subgroup H, and an element γ in the normalizer of H, the mapping torus of the G-equivariant map $G/H \to G/H$; $gH \mapsto g\gamma H$ gives a cohomogeneity one manifold over the circle. Conversely all cohomogeneity manifolds over the circle arise this way.

Thus we focus our attention on cohomogeneity one manifolds over the interval.

Definition 2. A *quadruple* $(H \subset H_+, H_- \subset G)$ is a Lie group G with three subgroups H, H_+, H_- where H is a common subgroup of H_+ and H_- so that H_+/H and H_-/H are both diffeomorphic to spheres: $H_+/H \cong S^{\ell_+}$ and $H_-/H \cong S^{\ell_-}$.

Note that by averaging we may assume that the group actions of $H^\pm$ on $S^{\ell_\pm}$ are linear. Also note that

$$\partial(G \times_{H_{\pm 1}} D^{\ell_\pm}) = G \times_{H_\pm} H_\pm/H = G/H$$

Definition 3. The geometric realization of a quadruple is the G-manifold

$$M = \left(G \times_{H_+} D^{\ell_+}\right) \cup_{G/H} \left(G \times_{H_-} D^{\ell_-}\right)$$

Proposition 4. *An manifold has cohomogeneity one over the interval if and only if it is the geometric realization of a quadruple.*

Proof. Suppose G acts on M with orbit space the interval. By averaging, give M a G-invariant Riemannian metric. Choose a point mapping to an end point of the interval and then the minimal geodesic arc in M which maps bijectively to M/G (this geodesic arc intersects all orbits orthogonally). Let H_+ and H_- be the isotropy at the end points of the arc and let H be the isotropy group of an interior point of the arc. It follows from the slice theorem that M is of the desired form. (See Grove-Ziller, Annals of Math.)

Conversely it is easy to see that the geometric realization of a quadruple is cohomogeneity one manifold over the interval. □

Next we switch to the case where $\ell_+ = 0 = \ell_-$, in other words H is of index 2 in both H_+ and H_-.

If $G/H, G/H_+, G/H_-$ are all path-connected (e.g. if G is path-connected) and $\dim G/H \geq 2$, then Seifert-Van Kampen shows

$$\pi_1 M = \pi_1 H/H_+ *_{\pi_1 G/H} \pi_1 H/H_-$$

where the amalgamating subgroup is of index two in the two summands. In other words, $\pi_1 M$ surjects to the infinite dihedral group.

On example is given by $SO(n-1) \subset O(n-1), O(n-1) \subset SO(n)$. Then $M = \mathbb{R}P^n \# \mathbb{R}P^n$.

Here is another example, pointed out by Jan-Li Lin. Let A be a finite group contained as an index 2 subgroups of A_+ and A_-. Embed A in $U(n)$. (The regular representation will work.) Then the quadruple $(A \subset A_+, A_- \subset U(n))$ is a cohomegenity one manifold with $\pi_1 M = A_+ *_A A_-$, which in turn maps to the infinite dihedral group.

Remark 5. Let M be a space whose fundamental group maps the infinite dihedral group. Let $\widehat{M}$ be the corresponding infinite dihedral cover. Then there is an D_∞-equivariant map $\widehat{M} \to E_{\text{fin}} D_\infty = \mathbb{R}$, and hence from $\widehat{M}$ to the interval. In the case where M is a manifold with cohomogeneity one over the interval, then one sees this map geometrically.