

Implications

The majority of proofs are *implications* – you are asked to prove a statement of the form: “If a, then b.” In symbols, “ $a \Rightarrow b$.” (The symbol \rightarrow is not a good substitute for the symbol \Rightarrow .) Statement a is called the *premise* (or synonymously, assumption or hypothesis) and statement b is called the *conclusion*. There are three ways to prove an implication:

- **DIRECT PROOF:** Here one starts from the premise and, step-by-step, deduces the conclusion. The trick here is – when you attack the problem you might start by looking at the conclusion, and then think, what do I need to get here, and work backward to the premise. However, when it comes to the write-up you need to be like Carl Friedrich Gauss and hide all trace of your scratch work, and start with the premise and deduce the conclusion. (Here is an analogue: When planning a trip, you look a map and think about what you need to do to reach your destination. However, to make your trip you start at your home and travel to your destination.)
- **PROOF BY CONTRAPOSITIVE:** The contrapositive of an implication $a \Rightarrow b$ is the implication $\sim b \Rightarrow \sim a$. These statements are logically equivalent. In words – the statement “if a, then b” is logically equivalent to the statement “if not b, then not a.” Thus one way to prove an implication is to start with the negation of the conclusion and deduce the negation of the premise.¹
- **PROOF BY CONTRADICTION:** Here goes, take a deep breath: The only way an implication $a \Rightarrow b$ can be false is if b is false and a is true. In symbols

$$\sim (a \Rightarrow b) \Leftrightarrow (a \ \& \ \sim b)$$

¹One tricky thing could be when the hypotheses have several parts, i.e. you are trying to prove an implication of the form “ $a_1, a_2, a_3 \Rightarrow b$.” This is really an implication of the form $a \Rightarrow b$ where a is the statement $a_1, a_2,$ and a_3 . Then the negation of a is the statement $\sim a_1, \sim a_2,$ or $\sim a_3$. This all makes proofs by contrapositive a bit tricky when there are multiple hypotheses. Here is a way to clarify the logic: Rewrite the implication $a_1, a_2, a_3 \Rightarrow b$ as the statement “Suppose a_1 and a_2 . Then $a_3 \Rightarrow b$.” The contrapositive of this statement is then “Suppose a_1 and a_2 . Then $\sim b \Rightarrow \sim a_3$.”

Thus if you assume the premise (a) is true and the conclusion (b) is false and reach a contradiction, you have shown

$$\sim (a \ \& \ \sim b) \Leftrightarrow \sim \sim (a \Rightarrow b) \Leftrightarrow (a \Rightarrow b)$$

which is what you want to do, right?

Some remarks:

- All three methods, direct proof, proof by contrapostive, and proof by contradiction are rigorous methods of proof and are logically equivalent. But . . . , they are listed in order of elegance and desirability. It can be very difficult to follow a proof by contradiction and really understand why the implication is true. However, with some problems (e.g. to show there are a infinite number of prime numbers) it is the most natural method.² Some students always start by trying a proof by contradiction - that is not good – use it as a last resort.
- Given an implication $a \Rightarrow b$, its converse is the implication $b \Rightarrow a$. These are not logically equivalent. Don't get the premise and conclusion confused.
- Sometimes you are asked to proof an equivalence " $a \Leftrightarrow b$." In words this might be "a if and only if b" or "a iff b" or "the following are equivalent: a and b." A statement like this is really two implications $a \Rightarrow b$ and $b \Rightarrow a$.
- A related thing is when you are asked to show two sets are equal $A = B$. This can be broken into two problems $A \subset B$ and $B \subset A$. Equivalent are the two problems $a \in A \Rightarrow a \in B$ and $b \in B \Rightarrow b \in A$.

²The method of proof by contradiction can apply where even when there is no implication i.e. $a \Leftrightarrow \sim \sim a$.)