Elliptic Regularity and Hodge Theory

Throughout we assume all vector bundles are smooth bundles with metrics over a Riemannian manifold X^n.

1 Review of Hodge Theory

In this note I outline the proof of the following Fundamental Theorem.

Theorem 1 (Fundamental Theorem)**.** *Let $\Delta : C^\infty(E) \to C^\infty(E)$ be a self-adjoint elliptic differential operator. Then*

- $\dim \ker \Delta < \infty$
- $C^\infty E = \ker \Delta \oplus \operatorname{im} \Delta$

There are many, many corollaries to this. However, they can all be subsumed into the following.

Corollary 2. *Let $(C^\infty(E^\bullet), D))$ be an elliptic complex. Let $\Delta = \Delta^p = DD^* + D^*D : C^\infty(E^p) \to C^\infty(E^p)$. Then $(C^\infty(E^\bullet), D)) =$*

$$\begin{array}{ccccccc}
 & \ker\Delta & & \ker\Delta & & \ker\Delta & \\
 & \oplus & & \oplus & & \oplus & \\
\cdots & \operatorname{im} D^* & \xrightarrow{D} & \operatorname{im} D & & \operatorname{im} D^* & \xrightarrow{D} \cdots \\
 & \oplus & & \oplus & & \oplus & \\
\xrightarrow{D} & \operatorname{im} D & & \operatorname{im} D^* & \xrightarrow{D} & \operatorname{im} D &
\end{array}$$

where the maps labeled D are isomorphisms.

Corollary 3 (Hodge Theorem)**.** $\ker \Delta^p \cong H^p(C^\infty(E^\bullet), D)$

Corollary 4. *For an elliptic differential operator* $\ker D^* \to \operatorname{cok} D$ *is an isomorphism.*

For more details see the algebraic Hodge theory post at wordpress.indextheory.com

Classically, one considers the exterior derivative. Then an *harmonic form* is a form in the kernel of the Laplacian. The Hodge theorem says that every DeRham cohomology class has a unique harmonic representative. Furthermore this harmonic representative minimizes the L^2-norm within the cohomology class (since $C^\infty E^p = \ker \Delta \oplus \operatorname{im} D \oplus \operatorname{im} D^*$).

2 Weak Solutions

Here are some ideas behind the proof of Theorem 1. Let $\mathcal{H} \subset C^\infty E$ be the kernel of the Laplacian Δ. Note

$$\operatorname{im} \Delta \subset \mathcal{H}^\perp$$

One needs to show the opposite inclusion. So in other words for any $t_0 \in \mathcal{H}^\perp$, one needs to find a solution to the equation $\Delta(-) = t_0$. This leads one to think about Hilbert space methods and the Riesz Representation Theorem.

Often our discussion will be clearer if we consider a non-necessarily self-adjoint differential operator $D : C^\infty(E) \to C^\infty(F)$.

Given $t_0 \in C^\infty(F)$, a *weak solution to* $D(-) = t_0$ is a bounded linear functional $l : C^\infty(E) \to \mathbb{C}$ so that

$$l(D^* t) = \langle t, t_0 \rangle \qquad \forall t \in C^\infty(F)$$

The following theorem is proved in Warner:

Theorem 5. *Let* $D : C^\infty(E) \to C^\infty(F)$ *be an elliptic differential operator. Then any weak solution* l *to* $D(-) = t_0$ *is represented by a smooth solution* $s_0 \in C^\infty(E)$*, i.e.* $l(-) = \langle -, s_0 \rangle$ *which implies* $Ds_0 = t_0$*.*

Note that $C^\infty(E)$ and $C^\infty(F)$ are inner product spaces so can be completed to Hilbert spaces $L^2(E)$ and $L^2(F)$. But one cannot extend D to these Hilbert spaces since one cannot differentiate measurable sections. This leads us to:

3 Sobolev Spaces

Throughout we will assume our vector bundles have covariant derivatives.

Definition 6. For an integer $k \geq 0$, define an inner product on $C^\infty(E)$ by

$$\|s\|_k = \sqrt{\sum_{j=0}^{k} \int_X \|\nabla^j s\|^2}$$

where ∇^2 is, for instance, the composite

$$C^\infty(E) \xrightarrow{\nabla^E} C^\infty(T^*X \otimes E) \xrightarrow{\nabla^{T^*X \otimes E}} C^\infty(T^*X \otimes T^*X \otimes E)$$

The *Sobelev space* $L_k(E)$ is the completion of $C^\infty(E)$ under this norm. Define $L_{-k}(E) = L_k(E)^*$.

Proposition 7. *A differential operator of order m induces a bounded linear map*

$$D_k : L_k(E) \to L_{k-m}(F).$$

Recall bounded means there is a C so that $\|Ds\| \leq C\|s\|$ (i.e. the image of the unit ball is bounded). This is equivalent to continuous.

Let $C^k(E)$ be set of sections all of whose local k-fold partial derivatives are continuous. Define the *uniform C^k-norm*

$$||s||_{C^k} = \sqrt{\sup_X \sum_{j=0}^{k} \|\nabla^j s\|^2}$$

Lemma 8 (Sobolev Embedding Lemma)**.** *For every $l > (n/2) + k$, there is a constant K so that*

$$\|s\|_{C^k} \leq K\|s\|_l$$

Hence the inclusion $C^\infty(E) \subset C^l(E)$ extends to a continuous embedding

$$L_l(E) \subset C^k(E)$$

Thus $C^\infty(E) = \cap_l L_l(E)$.

4 Fredholm and compact operators

Let V and W be Hilbert spaces. Assume all maps below are bounded linear operators.

Definition 9.

- $F : V \to W$ is *Fredholm* if the kernel and cokernel are finite-dimensional.
- $C : V \to W$ is *compact* if the closure of the image of the unit ball is compact.

A compact operator is a limit of finite rank operators.

Intuitively a Fredholm operator is almost an isomorphism and a compact operator is almost trivial. This is made precise in the following theorem.

Theorem 10. *$T : V \to W$ is Fredholm iff it is invertible modulo compact operators. I.e. $T : V \to W$ is Fredholm iff there is $S : W \to V$ so that $\mathrm{Id}_V - ST$ and $\mathrm{Id}_W - TS$ are compact operators.*

As a special case, if $C : W \to W$ is compact then $\mathrm{Id}_W - C$ is Fredholm.

Lemma 11 (Rellich's Lemma)**.** *For every k, the map $L_{k+1}(E) \to L_k(E)$ is a compact embedding.*

5 Pseudodifferential Operators

Let $\Psi DO_m(E, F) \subset \mathrm{Hom}_{\mathbb{C}}(C^\infty(E), C^\infty(F)$ be the vector space of pseudodifferential operators of order $m \in \mathbb{R}$. They have the following properties:

1. For an integer m, $DO_m(E, F) \subset \Psi DO_m(E, F)$.
2. *Symbol surjectivity:* Recall $\mathrm{Sym}_m(E, F) = \{\sigma \in C^\infty \mathrm{Hom}(\pi^* E, \pi^* F) \mid \sigma(\rho\xi) = \rho^m \sigma(\xi), \forall \rho > 0\}$. Here $\pi : T^*X \to X$. There is an exact sequence
$$\Psi DO_m(E, F) \xrightarrow{\sigma} \mathrm{Sym}_m(E, F) \to 0$$
whose kernel is contained in $\Psi DO_{m-1}(E, F)$. *This surjectivity is very important!*

3. ΨDO's have formal adjoints and the symbol of the adjoint is the adjoint of the symbol.

4. *Composition:* Composition gives a map $\Psi DO_r(E,F) \times \Psi DO_s(F,G) \to \Psi DO_{r+s}(E,G)$ with $\sigma_{P\circ Q} = \sigma_P \circ \sigma_Q$ (I leave it to the reader to interpret the meaning of $\sigma_P \circ \sigma_Q$ – hint, work fiberwise!).

5. *Sobolev extension:* Sobolev spaces $L_k(E)$ can be defined for all $k \in \mathbb{R}$. Each $P \in \Psi DO_m(E,F)$ extends to a bounded linear map $P_k : L_k(E) \to L_{k-m}(F)$.

6. *Smoothing*: If $S \in \Psi DO_{-1}(E,F)$, then for each k the composite $L_k(E) \xrightarrow{S_k} L_{k+1}(F) \to L_k(F)$ is a compact operator by Rellich's Lemma.

Definition 12. A pseudodifferential operator is *elliptic* if $\sigma_P(\xi) : E_x \to F_x$ is an isomorphism for all $\xi \in T^*_x X - 0$.

Here are some corollaries:

Corollary 13. *If $P \in \Psi DO_m(E,F)$, then $P_k C^\infty(E) \subset C^\infty(F)$.*

Proof. This uses Sololev extension and the Sobolev embedding theorem.

$$P_k C^\infty(E) = P_k\left(\cap L_l(E)\right) \subset \cap P_k L_l(E) \subset \cap L_{l-m}(F) = C^\infty(F)$$

□

Corollary 14 (Existence of a parametrix)**.** *If $P \in \Psi DO_m(E,F)$ is elliptic, then there exists an elliptic $Q \in \Psi DO_{-m}(F,E)$ so that*

$$\mathrm{Id} - PQ, \ \ \mathrm{Id} - QP \in \Psi DO_{-1}.$$

Proof. This uses symbol surjectivity and composition. Choose Q so that $\sigma_Q = {\sigma_P}^{-1}$. □

Corollary 15 (Elliptic Regularity)**.** *If P is elliptic and $P_k(s_0) = t_0 \in C^\infty(F)$ for some k, then $s_0 \in C^\infty(E)$.*

Proof. This uses the existence of a parametrix, Sololev extension, and the Sobolev embedding theorem. Let Q be a parametrix for P. Let m be the order of P.

$s_0 = (\mathrm{Id} - Q_{k-m}P_k)s_0 + Q_{k-m}t_0 \in L_{k+1}(E) + C^\infty(E) = L_{k+1}(E)$ *"bootstraping"*

Thus $s_0 \in \cap_l L_{k+l}(E) = C^\infty(E)$. □

6 Proof of the Fundamental Theorem

Lemma 16. *Let $F : V \to W$ be a Fredholm operator. Then there is an orthogonal decomposition $W = \ker F^* \oplus \operatorname{im} F$.*

Proof. $W = \overline{\operatorname{im} F} \oplus (\operatorname{im} F)^\perp$ and $\ker F^* = (\operatorname{im} F)^\perp$. But $\operatorname{im} F$ is closed since $\operatorname{cok} F$ is finite dimensional.

$$\begin{aligned} W &= \operatorname{im} F \oplus (\operatorname{im} F)^\perp \\ &= \operatorname{im} F \oplus \ker F^* \end{aligned}$$

□

Corollary 17. $\ker F^* \xrightarrow{\cong} \operatorname{cok} F$.

Theorem 18. *Let $P \in \Psi DO_m(E, F)$ be elliptic.*

1. *Each P_k is Fredholm*
2. $\ker P \hookrightarrow \ker P_k$ *is an isomorphism.*
3. $\operatorname{cok} P_{k+1} \to \operatorname{cok} P_k$ *is an isomorphism.*
4. $C^\infty F = \ker P^* \oplus \operatorname{im} P$.
5. $\operatorname{cok} P \to \operatorname{cok} P_k$ *is an isomorphism.*

Proof. 1. Choose a parametrix Q for P. Then P_k is invertible modulo compacts by smoothing, so is Fredholm.

2. Elliptic regularity.

3.

$$\begin{array}{ccc} \ker P_{k+1}^* & \xleftarrow{\cong} & \ker P_k^* \\ \cong\downarrow & & \cong\downarrow \\ \operatorname{cok} P_{k+1} & \longrightarrow & \operatorname{cok} P_k \end{array}$$

4.

- $\ker P^*$ is f.d. by 1 and 2 and the fact that P^* is elliptic.
- $C^\infty F = \ker P^* \oplus (\ker P^*)^\perp$
- $\operatorname{im} P \subset (\ker P^*)^\perp$

- $(\ker P^*)^\perp \subset (\ker P_k^*)^\perp = \operatorname{im} P_k$ by 2. and the Lemma.
- $(\ker P^*)^\perp = (\ker P^*)^\perp \cap \operatorname{im} P_k \subset C^\infty F \cap \operatorname{im} P_k \subset \operatorname{im} P$ by elliptic regularity.

5.

$$\begin{array}{ccc} \ker P^* & \xleftarrow{\cong} & \ker P_k^* \\ \cong\downarrow & & \cong\downarrow \\ \operatorname{cok} P & \longrightarrow & \operatorname{cok} P_k \end{array}$$

where the vertical left arrow follows from the algebraic Hodge theorem.

□

7 Pseudodifferential operators; definition of the local form

Here is the local form.

Let $s \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$. Define the *Fourier transform*

$$\hat{s}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{i\langle x,\xi\rangle} s(x) dx$$

Two basic properties:

- $\widehat{D^\alpha s} = \xi^\alpha \hat{s}$ (Fourier transform converts differentiation to multiplication). Here $D^\alpha = i^{-|\alpha|} \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$.
- $\hat{\hat{s}} = s$ (Fourier inversion) after generalizing from compactly supported functions to Schwarz class

$$\mathcal{S} = \{s \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mid \sup_{x \in \mathbb{R}^n, \alpha, \beta} |x^\alpha D^\beta s(x)| < \infty\}$$

Then $\hat{}\colon \mathcal{S} \to \mathcal{S}$ is an isometry.

Let $D = \sum A^\alpha(x) D^\alpha$ be a differential operator of order m. Fourier inversion says

$$s(x) = \frac{1}{(2\pi)^{n/2}} \int e^{i\langle x,\xi\rangle} \hat{s}(\xi) d\xi$$

Then

$$Ds(x) = \frac{1}{(2\pi)^{n/2}} \int e^{i\langle x,\xi\rangle} p(x,\xi)\hat{s}(\xi)d\xi \tag{1}$$

where

$$p(x,\xi) = \sum A^{\alpha}(x)\xi^{\alpha}$$

be the *total symbol* of D. Note that $p(x,\xi)$ is a polynomial of degree m in ξ.

Definition 19. Fix $m \in \mathbb{R}$. A function $p(x,\xi)$ is a *total symbol of order m* if for all α, β, there is a C so that for all $x, \xi \in \mathbb{R}^n$,

$$|D_x^{\alpha} D_{\xi}^{\beta} p(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|}$$

Definition 20. If $p(x,\xi)$ is a total symbol of degree m, then the corresponding *pseudodifferential operator* $P : \mathcal{S} \to \mathcal{S}$ is defined by equation (1).